SUMS PROBLEM COMPETITION, 1999

SOLUTIONS

1. Imagine the possible configurations of coins as infinite sequences $(p_1, p_2, \ldots)$, where $p_1 \ge p_2 \ge \cdots$, and where $p_i = 0$ for i sufficiently large. If $p_i = 0$ for all $i > r$, we shall simply write the configuration as $(p_1, \ldots, p_r)$. If $p_1 = a$, then a "step" consists of replacing $(p_1, p_2, \ldots)$ by $(q_1, q_2, \ldots)$, where $q_i = p_{i+1} + 1$ for $i = 1, \ldots, a$, and $q_i = p_{i+1}$ for $i > a$.

For $n = 1, 2, \ldots$, let $t_n = 1 + 2 + \cdots + n = n(n+1)/2$. Suppose that we have N coins. We shall show that if $N = t_n$, then we eventually obtain a stable configuration with piles of size $n, n-1, \ldots, 2, 1$. More generally, if $t_n \le N < t_{n+1}$, then we shall show that after sufficiently many steps, we obtain a configuration of the form

$$(n + c_1, n - 1 + c_2, \ldots, n + 1 - i + c_i, \ldots, 1 + c_n, 0 + c_{n+1}), \tag{1}$$

where $c_1, \ldots, c_{n+1} \in \{0, 1\}$, ($N - t_n$ of them 1's). Notice that from (1), one step yields the configuration $(n + c_2, n - 1 + c_3, \ldots, 1 + c_{n+1}, 0 + c_1)$, and so the c_i's cycle around, returning to (1) after $n + 1$ steps. We prove this by induction on N. If $N = 1$, we are in a stable configuration. If $N = 2$, the possible configurations are (2) and $(1, 1)$, and these alternate, and are of the form (1), with $(c_1, c_2) = (1, 0)$ and $(0, 1)$, respectively. If $N = 3$ $(= t_2)$, then the possible configurations are (3), $(1, 1, 1)$ and $(2, 1)$. The last is stable, while the first two get to $(2, 1)$ after 2 and 1 steps, respectively.

Suppose that $N > 3$ and that any configuration of $N - 1$ coins leads to a configuration of the form (1). Suppose that we have N coins, and imagine that $N - 1$ of them are silver and one of them is gold, initially in the smallest non-empty pile. In carrying out a step when the gold coin starts in the first pile, distribute the coins of this pile on the other piles so that the gold coin is used *last*. Then if the gold coin is in the i_0-th pile at some stage, we must have $p_{i_0} > p_{i_0+1}$, and so we get a valid configuration $(p'_1, p'_2, \ldots)$ of $N - 1$ coins by removing the gold coin; explicitly, $p'_i = p_i$ if $i \ne i_0$, and $p'_{i_0} = p_{i_0} - 1$. If we perform a step on the N coins (using the gold one last if it is in the first pile) and then remove the gold coin, it is routine to check that we get the same result as if we first remove the gold coin and then perform a step on the silver ones. So as we go through the steps on the N coins, the configurations of the $N - 1$ silver coins are changed as if the gold coin were not there. By the induction hypothesis, after sufficiently many steps the $N - 1$ coins are in a configuration (1). The n there must be the largest one such that $t_n \le N - 1$. Now suppose that the gold coin is in the i-th pile. The N coins are then in the configuration

$$(n + c_1, \ldots, n + 1 - i + c_i + 1, \ldots, 1 + c_n, 0 + c_{n+1}). \tag{2}$$

If $c_i = 0$, we are done, as this is of the form (1), where $(c_1, \ldots, c_{n+1})$ there is replaced by $(c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_{n+1})$. Note that if all the c_j's, $j \ne i$, are 1, then the configuration (2) is in the form (1), but with n replaced by $n + 1$ and all the c_j's replaced by 0; we must have $N = t_{n+1}$ in this case. We deal with the case $c_i = 1$ as follows: When the N coins are as in (2), with the gold coin in the i-th pile, we say that the gold coin is *associated to* c_i. If we perform $i - 1$ steps on the configuration (2), we get

$$(n + c_i + 1, n - 1 + c_{i+1}, \ldots, 1 + c_{i-2}, 0 + c_{i-1}) = (n + 2, n - 1 + c_{i+1}, \ldots, 1 + c_{i-2}, 0 + c_{i-1}).$$

So the gold coin is still associated with c_i. Now performing two more steps, we first get the $n+2$-tuple

$$(n + c_{i+1}, n - 1 + c_{i+2}, \dots, 2 + c_{i-2}, 1 + c_{i-1}, 1, 1),$$

then the $n+1$-tuple

$$(n + c_{i+2}, n - 1 + c_{i+3}, \dots, 3 + c_{i-2}, 2 + c_{i-1}, 2, c_{i+1} + 1), \tag{3}$$

the gold coin being in the $n+1$-st pile. The configuration (3) has the same form as (2), but $(c_1, \dots, c_{n+1})$ there is replaced by $(c_{i+2}, c_{i+3}, \dots, c_{i-1}, c_i, c_{i+1})$, and the gold coin is now associated with c_{i+1}. At least one of the c_j's is 0, since $t_n \le N - 1 < t_{n+1}$, and so repeating the above steps sufficiently often, we can arrange that the gold coin is associated with a c_j which is 0, and then the configuration is in the form (1), as explained above.

2. More generally, suppose that we have $N = m + n$ stars $A_1, \dots, A_m$, $B_1, \dots, B_n$, and that the cloud obscures $B_1, \dots, B_n$. The sum of the distances between the stars is initially

$$S = \sum_{\substack{i,j=1:\\ i<j}}^{m} d(A_i, A_j) + \sum_{\substack{i,j=1:\\ i<j}}^{n} d(B_i, B_j) + \sum_{i=1}^{m}\sum_{j=1}^{n} d(A_i, B_j).$$

By the triangle inequality, we have $d(A_i, A_j) \le d(A_i, B_k) + d(B_k, A_j)$ for all i, j, k. Summing over k, we get $nd(A_i, A_j) \le \sum_{k=1}^{n} d(A_i, B_k) + \sum_{k=1}^{n} d(B_k, A_j)$. Now for fixed i we sum over $j \ne i$. This gives

$$\begin{aligned} n \sum_{\substack{j=1:\\ j\ne i}}^{m} d(A_i, A_j) &\le (m-1) \sum_{k=1}^{n} d(A_i, B_k) + \sum_{\substack{j=1:\\ j\ne i}}^{m} \sum_{k=1}^{n} d(B_k, A_j) \\ &= (m-2) \sum_{k=1}^{n} d(A_i, B_k) + \sum_{j=1}^{m} \sum_{k=1}^{n} d(B_k, A_j). \end{aligned}$$

Finally, we sum over i, and get

$$\begin{aligned} n \sum_{\substack{i,j=1:\\ i\ne j}}^{m} d(A_i, A_j) &\le (m-2) \sum_{i=1}^{m}\sum_{k=1}^{n} d(A_i, B_k) + m \sum_{j=1}^{m}\sum_{k=1}^{n} d(B_k, A_j) \\ &= 2(m-1) \sum_{i=1}^{m}\sum_{k=1}^{n} d(A_i, B_k). \end{aligned}$$

Hence

$$\begin{aligned} n \sum_{\substack{i,j=1:\\ i<j}}^{m} d(A_i, A_j) &\le (m-1) \sum_{i=1}^{m}\sum_{k=1}^{n} d(A_i, B_k) \\ &= (m-1)\Big[S - \sum_{\substack{i,j=1:\\ i<j}}^{m} d(A_i, A_j) - \sum_{\substack{i,j=1:\\ i<j}}^{n} d(B_i, B_j)\Big] \\ &\le (m-1)\Big[S - \sum_{\substack{i,j=1:\\ i<j}}^{m} d(A_i, A_j)\Big], \end{aligned}$$

and therefore

$$\sum_{\substack{i,j=1:\\ i<j}}^{m} d(A_i, A_j) \le \frac{m-1}{m+n-1} S = \frac{m-1}{N-1} S.$$

In particular, if $m = n = 25$, then $(m-1)/(N-1) = (m-1)/(2m-1) = 24/49 < 1/2$.

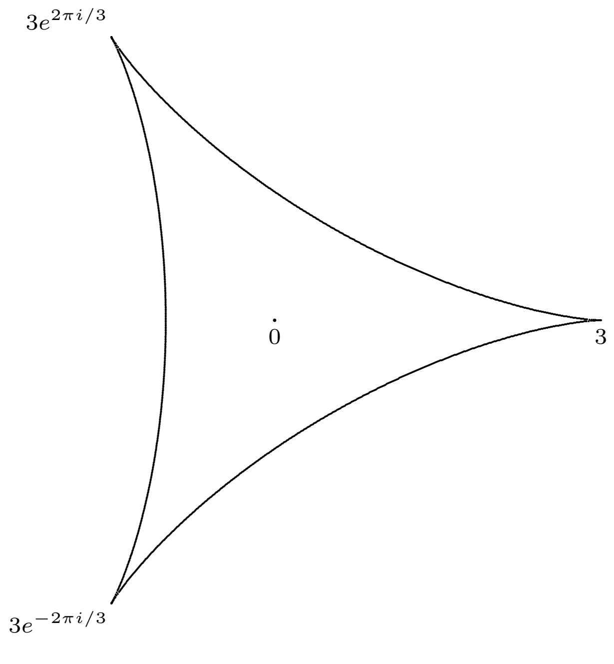
3. Suppose that $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$, where $\theta_1, \theta_2 \in \mathbb{R}$. Then if $z_1 z_2 z_3 = 1$, we have $z_3 = e^{i\theta_3}$ for $\theta_3 = -(\theta_1 + \theta_2)$. Also, $z_1 + z_2 + z_3 = x + iy$ for

$$\begin{aligned} x &= \cos(\theta_1) + \cos(\theta_2) + \cos(\theta_1 + \theta_2) \\ y &= \sin(\theta_1) + \sin(\theta_2) - \sin(\theta_1 + \theta_2). \end{aligned}$$

These equations define a mapping or transformation f from the (θ_1, θ_2)-plane to the complex, or (x, y)-plane. It is easy to calculate that the Jacobian of this transformation is

$$J(\theta_1, \theta_2) = \frac{1}{2i}(e^{i\theta_1} - e^{i\theta_2})(e^{i\theta_1} - e^{i\theta_3})(e^{i\theta_2} - e^{i\theta_3}),$$

remembering that $\theta_3 = -(\theta_1 + \theta_2)$. The open mapping theorem tells us that for any point (θ_1, θ_2) such that $J(\theta_1, \theta_2) \neq 0$, the point $f(\theta_1, \theta_2)$ is an interior point of the image $f(\mathbb{R}^2)$ of f. So if $f(\theta_1, \theta_2)$ is a boundary point of $f(\mathbb{R}^2)$, then $J(\theta_1, \theta_2) = 0$ must hold. That is, the three numbers $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ and $z_3 = e^{i\theta_3}$ cannot be distinct. By symmetry, we may suppose that $z_1 = z_2 = e^{i\theta}$, say. Then $z_3 = e^{-2i\theta}$. Hence every boundary point of $f(\mathbb{R}^2)$ equals $f(e^{i\theta}, e^{i\theta}) = 2e^{i\theta} + e^{-2i\theta}$ for some θ. Now as θ varies (and we may assume that $0 \le \theta \le 2\pi$), $2e^{i\theta} + e^{-2i\theta}$ traverses the following curve in the complex plane:



This is a "hypocycloid". It is now clear that the set S of the question consists of this hypocycloid and its interior. The three cusps are at the points $z_1 + z_2 + z_3$, where $z_1 = z_2 = z_3 = e^{2\pi i j/3}$, $j = 0, 1, 2$.

The question may be generalized to the case of n complex numbers z_j of modulus 1 and product 1. The set of numbers of the form $z_1+\cdots+z_n$ is the set bounded by the curve obtained by taking $z_1=\cdots=z_{n-1}=e^{i\theta}$ and $z_n=e^{-i(n-1)\theta}$. This curve has n cusps.

4. Let us call an n-tuple $(m_1,\ldots,m_n)$ of non-negative integers *good* if the product $M=(m_1+1)(m_2+1)\cdots(m_n+1)$ divides $(m_1+p)(m_2+p)\cdots(m_n+p)$ for all but a finite number of primes p. Using the fact that binomial coefficients are integers, it is easy to see that $(0,1,\ldots,n-1)$ and $(1,3,\ldots,2n-1)$ are good for any n. We now re-write the condition of being good in terms of M so that primes are not mentioned. For this we need *Dirichlet's Theorem*, which states that if a,b are integers with no common factors, then there are infinitely many prime numbers p such that $p\equiv a \bmod b$. In the discussion below, we exclude the good n-tuple $(0,\ldots,0)$, whose M is 1.

Lemma 1. *Given an n-tuple $(m_1,\ldots,m_n)$, let $M=(m_1+1)(m_2+1)\cdots(m_n+1)$. Then $(m_1,\ldots,m_n)$ is good if and only if for each integer $x\in\{1,\ldots,M\}$ which is co-prime to M, the product $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by M.*

Proof. Suppose that $(m_1,\ldots,m_n)$ is good. Let $x\in\{1,\ldots,M\}$ be co-prime to M. By Dirichlet's Theorem, there are infinitely many primes p such that $p\equiv x \bmod M$. Choose any one of these p for which M divides $(m_1+p)(m_2+p)\cdots(m_n+p)$. Then

$$(m_1+x)(m_2+x)\cdots(m_n+x)\equiv(m_1+p)(m_2+p)\cdots(m_n+p)\equiv 0 \mod M,$$

which means that $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by M.

Conversely, suppose that $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by M for all $x\in\{1,\ldots,M\}$ which are co-prime to M. Let $p>M$ be prime. Write $p=aM+x$, where a,x are integers and $0\le x<M$. Then $x\ge 1$, $p\equiv x \bmod M$, and x and M are co-prime because p cannot divide M. So

$$(m_1+p)(m_2+p)\cdots(m_n+p)\equiv(m_1+x)(m_2+x)\cdots(m_n+x)\equiv 0 \mod M,$$

which means that $(m_1+p)(m_2+p)\cdots(m_n+p)$ is divisible by M. □

We now simplify the condition of the first lemma by reducing the condition to be checked to one involving each prime factor of M:

Lemma 2. *Given an n-tuple $(m_1,\ldots,m_n)$, write $M=q_1^{c_1}\cdots q_r^{c_r}$, where $q_1,\ldots,q_r$ are distinct primes. Then $(m_1,\ldots,m_n)$ is good if and only if, whenever q^c is one of the $q_k^{c_k}$'s, for each $x\in\{1,\ldots,q^c\}$ which is not divisible by q, the product $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by q^c.*

Proof. Suppose that $(m_1,\ldots,m_n)$ is good, and let q^c be one of the $q_k^{c_k}$'s. Suppose $x\in\{1,\ldots,q^c\}$ is not divisible by q. By Dirichlet's theorem again, there are infinitely many primes p such that $p\equiv x \bmod q^c$. Choose any one of these p for which M divides $(m_1+p)(m_2+p)\cdots(m_n+p)$. Then

$$(m_1+x)(m_2+x)\cdots(m_n+x)\equiv(m_1+p)(m_2+p)\cdots(m_n+p)\equiv 0 \mod q^c,$$

which means that $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by q^c.

Conversely, suppose that $(m_1,\ldots,m_n)$ satisfies the condition in this lemma. Let $x\in\{1,\ldots,M\}$ be co-prime to M, and let q^c be one of the $q_k^{c_k}$'s. Then $x\equiv x' \mod q^c$ for some $x'\in\{1,\ldots,q^c\}$ which is not divisible by q. Hence

$$(m_1+x)(m_2+x)\cdots(m_n+x)\equiv(m_1+x')(m_2+x')\cdots(m_n+x')\equiv 0 \mod q^c,$$

Thus $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by each $q_k^{c_k}$, and hence by M. So $(m_1,\ldots,m_n)$ is good, by Lemma 1. □

Lemma 3. *Let $(m_1,\ldots,m_n)$ be good. Let q be a prime divisor of M. Then $q\le n+1$.*

Proof. For $j=0,\ldots,q-1$, let $S_j=\{i:1\le i\le n \text{ and } m_i\equiv j \mod q\}$. Then $\{1,\ldots,n\}$ is the union of the disjoint sets S_j. Suppose that there is a $j\neq 0$ such that $S_j=\emptyset$. Let $x=q-j\in\{1,\ldots,q-1\}$. Then $m_i+x\equiv m_i-j\neq 0 \mod q$ for each i. Hence q does not divide $\prod_{i=1}^n(m_i+x)$. This contradicts Lemma 2. So $S_j\neq\emptyset$ must hold for $j=1,\ldots,q-1$, and so $n=\sum_{j=0}^{q-1}|S_j|\ge q-1$. □

Lemma 4. *Let $(m_1,\ldots,m_n)$ be good, and, in the notation of Lemma 2, let q^c be one of the $q_k^{c_k}$'s. Then, writing $\lfloor x\rfloor$ for the largest integer $m\le x$,*

$$c\le\sum_{k=0}^{\infty}\left\lfloor\frac{\lfloor\frac{n}{q-1}\rfloor}{q^k}\right\rfloor. \tag{1}$$

Proof. Fix q. Let $C_q(n)$ denote the largest integer $c\ge 0$ such that for some n-tuple $(m_1,\ldots,m_n)$ of non-negative integers, q^c divides $(m_1+x)(m_2+x)\cdots(m_n+x)$ for each $x\in\{1,\ldots,q^c\}$ which is not divisible by q. Let $D_q(n)$ denote the largest integer $d\ge 0$ such that for some n-tuple $(m_1,\ldots,m_n)$ of non-negative integers, $(m_1+x)(m_2+x)\cdots(m_n+x)$ is divisible by q^d for each integer $x\ge 0$. It is evident that if $n\le n'$ then $C_q(n)\le C_q(n')$ and $D_q(n)\le D_q(n')$. We first show that

$$C_q(n)\le\lfloor\frac{n}{q-1}\rfloor+D\big(\lfloor\frac{n}{q-1}\rfloor\big). \tag{2}$$

To see this, let $c=C_q(n)$, and let $(m_1,\ldots,m_n)$ be an n-tuple of non-negative integers such that q^c divides $(m_1+x)(m_2+x)\cdots(m_n+x)$ for each $x\in\{1,\ldots,q^c\}$ which is not divisible by q. For $j=1,\ldots,q-1$, let T_j denote the set of $i=\{1,\ldots,n\}$ such that $m_j\equiv -j \mod q$. Then the sets T_j are disjoint, and so $n\ge\sum_{j=1}^{q-1}|T_j|$. So we can choose a j such that $|T_j|\le n/(q-1)$. If $z\ge 0$ is an integer, let $x\in\{1,\ldots,q^c\}$ satisfy $x\equiv j+zq \mod q^c$. Then q does not divide x, and q divides m_i+x if and only if $i\in T_j$. For each $i\in T_j$, write $m_i=-j+m_i'q$, where $m_i'\ge 1$ is an integer. Then

$$q^c\;\Big|\;\prod_{i=1}^n(m_i+x)=\prod_{i\in T_j}(m_i+x)\times\prod_{i\notin T_j}(m_i+x)=\prod_{i\in T_j}\big(q(m_i'+z)\big)\times\prod_{i\notin T_j}(m_i+x)$$

Therefore

$$q^{c-|T_j|}\;\Big|\;\prod_{i\in T_j}(m_i'+z),$$

and so $c - |T_j| \le D_q(|T_j|)$. Thus $c \le |T_j| + D_q(|T_j|) \le \lfloor \frac{n}{q-1} \rfloor + D(\lfloor \frac{n}{q-1} \rfloor)$.

We next show that

$$D_q(n) \le \lfloor \frac{n}{q} \rfloor + D(\lfloor \frac{n}{q} \rfloor). \tag{3}$$

The proof of this is similar to that of (2), except that now the sets T_j are defined for each $j \in \{0, 1, \ldots, q-1\}$, and so at least one of them satisfies $|T_j| \le n/q$.

It is easy to see that $D_q(1) = 0$. So using the elementary fact

$$\left\lfloor \frac{\lfloor \frac{m}{q} \rfloor}{q^k} \right\rfloor = \left\lfloor \frac{m}{q^{k+1}} \right\rfloor,$$

we get (1) by repeated use of (2) and (3). □

We are now nearly ready to list the good n-tuples for n small. We make a few observations in the next lemma which shortens the job.

Lemma 5.

(a) If $(m_1, \ldots, m_n)$ is good, then so is any permutation of $(m_1, \ldots, m_n)$.

(b) $(m_1, \ldots, m_n)$ is a good n-tuple if and only if $(0, m_1, \ldots, m_n)$ is a good $n+1$-tuple.

(c) If $(m_1, \ldots, m_n)$ is good, then $m_i \in \{0, 1\}$ for some i.

Proof. (a) is obvious. In (b), notice that the M's of $(m_1, \ldots, m_n)$ and of $(0, m_1, \ldots, m_n)$ are the same. If M divides $(m_1 + x) \cdots (m_n + x)$ for all $x \in \{1, \ldots, M\}$ coprime to M, then it is certainly divides $x(m_1 + x) \cdots (m_n + x)$ for all such x. Conversely, if M divides $x(m_1 + x) \cdots (m_n + x)$, and $x \in \{1, \ldots, M\}$ is coprime to M, then clearly M divides $(m_1 + x) \cdots (m_n + x)$. To prove (c), suppose that $m_i \ge 2$ for all i, and let $x = M - 1$. Then $x \in \{1, \ldots, M\}$ is coprime to M and by Lemma 1,

$$(m_1 - 1) \cdots (m_n - 1) \equiv (m_1 + x) \cdots (m_n + x) \equiv 0 \mod M,$$

and so M divides $(m_1 - 1) \cdots (m_n - 1)$. But $1 \le (m_1 - 1) \cdots (m_n - 1) < M$, and so this is impossible. □

Let us now list the good n-tuples for $n \le 4$. By Lemma 5(a), it is enough to list the good n-tuples $(m_1, \ldots, m_n)$ for which $m_1 \le \cdots \le m_n$.

The case $n = 1$. By Lemma 5(c), the only possibilities are $(m_1) = (0)$ and $(m_1) = (1)$. These are indeed good. In fact, $(0, \ldots, 0)$ and $(1, \ldots, 1)$ are good n-tuples for any n, the second, by Lemma 2, because 2^n divides $(1 + x)^n$ for any odd x.

The case $n = 2$. The pairs $(m_1, m_2) = (0, 0)$ and $(0, 1)$ are the only good pairs, one of whose terms is 0, by Lemma 5(b) and the list of the case $n = 1$. Let (m_1, m_2) be good, with $1 \le m_1 \le m_2$. By Lemma 5(c), $m_1 = 1$. By Lemma 3, the possible prime factors of $M = (m_1 + 1)(m_2 + 1)$ are $q = 2$ and $q = 3$. The sums in (1) of Lemma 4 when $(n, q) = (2, 2)$ and $(n, q) = (2, 3)$ are 3 and 1, respectively. So M must divide $2^3 3^1 = 24$. For $M = 1, 2, 3, 4, 6, 8, 12$ and 24, we list the possible factorizations $M = (m_1 + 1)(m_2 + 1)$ for which $1 = m_1 \le m_2$. The corresponding (m_1, m_2) are $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 5)$ and

$(1,11)$. We check that these are all good by using Lemma 2. For instance, $(1,11)$ is good because 8 divides $(1+x)(11+x)$ for all $x \in \{1,3,5,7\}$ and because 3 divides $(1+x)(11+x)$ for all $x \in \{1,2\}$.

The case $n=3$. The triples $(m_1,m_2,m_3)=(0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(0,1,2)$, $(0,1,3)$, $(0,1,5)$ and $(0,1,11)$ are the only good triples, one of whose terms is 0, by Lemma 5(b) and the list of the cases $n=1,2$. Let (m_1,m_2,m_3) be good, with $1 \le m_1 \le m_2 \le m_3$. By Lemma 5(c), $m_1=1$. The possible prime factors of $M=(m_1+1)(m_2+1)(m_3+1)$ are $q=2$ and $q=3$, by Lemma 3. The sums in (1) of Lemma 4 when $(n,q)=(3,2)$ and $(n,q)=(3,3)$ are 4 and 1, respectively, and so M must divide $2^4 3^1 = 48$. For $M = 1,2,3,4,6,8,12,16,24$ and 48, we list the possible factorizations $M=(m_1+1)(m_2+1)(m_3+1)$ for which $1=m_1 \le m_2 \le m_3$. The corresponding (m_1,m_2,m_3) are $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,2,3)$, $(1,1,5)$, $(1,3,5)$, $(1,2,7)$ and $(1,1,11)$. We check that using Lemma 2 that these are all good except $(1,2,7)$, for which 16 does not divide $(1+3)(2+3)(7+3)$.

The case $n=4$. The 4-tuples $(m_1,m_2,m_3,m_4)=(0,0,0,0)$, $(0,0,0,1)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,0,1,3)$, $(0,0,1,5)$, $(0,0,1,11)$, $(0,1,1,1)$, $(0,1,1,2)$, $(0,1,1,3)$, $(0,1,2,3)$, $(0,1,1,5)$, $(0,1,3,5)$, and $(0,1,1,11)$ are the only good 4-tuples, one of whose terms is 0, by Lemma 5(b) and the list of the cases $n=1,2,3$. Let (m_1,m_2,m_3,m_4) be good, with $1 \le m_1 \le m_2 \le m_3 \le m_4$. By Lemma 5(c), $m_1=1$. The possible prime factors of $M=(m_1+1)(m_2+1)(m_3+1)(m_4+1)$ are $q=2$, $q=3$ and $q=5$, by Lemma 3. The sums in (1) of Lemma 4 when $(n,q)=(4,2)$, $(4,3)$ and $(4,5)$ are 7, 2 and 1, respectively, and so M must divide $2^7 3^2 5^1 = 5760$. It is rather tedious to follow that procedure of the $n=3$ case, but it is very easy to write a short computer program based on Lemma 1 and Lemma 5(c), which quickly finds that the following are the only good 4-tuples (m_1,m_2,m_3,m_4) with $1 \le m_1 \le m_2 \le m_3 \le m_4$: $(1,1,1,1)$, $(1,1,1,2)$, $(1,1,1,3)$, $(1,1,1,5)$, $(1,1,1,11)$, $(1,1,2,2)$, $(1,1,2,3)$, $(1,1,2,5)$, $(1,1,2,11)$, $(1,1,3,3)$, $(1,1,3,5)$, $(1,1,3,11)$, $(1,1,5,5)$, $(1,1,5,11)$, $(1,1,11,11)$, $(1,2,3,4)$, $(1,2,3,9)$, $(1,3,3,5)$, $(1,3,5,7)$, $(1,3,7,29)$ and $(1,5,7,11)$.

5. First suppose that E is the interval $[0,M]$. Let $a_j = jM/n$ for $j=0,1,\ldots,n$. Then

$$\prod_{\substack{j=0\\ j\neq \ell}}^{n} |a_j - a_\ell| = \left(\frac{M}{n}\right)^n \ell!\,(n-\ell)!\,.$$

So to prove the statement in this case, is it enough to show that

$$\ell!\,(n-\ell)! > \left(\frac{n}{2e}\right)^n \quad \text{for } \ell = 0,1,\ldots,n.$$

We prove this by induction. It is certainly true for $n=1$. Suppose that we have proved it for n. Let $\ell \in \{0,\ldots,n+1\}$. If $\ell \le (n+1)/2$, then $n+1-\ell \ge (n+1)/2$, and, by the induction hypothesis,

$$\ell!\,(n+1-\ell)! \ge \frac{n+1}{2}\ell!\,(n-\ell)! > \frac{n+1}{2}\left(\frac{n}{2e}\right)^n.$$

If $\ell > (n+1)/2$, then, by the induction hypothesis again,

$$\ell!\,(n+1-\ell)! = \ell\,(\ell-1)!\,(n-(\ell-1))! > \frac{n+1}{2}\left(\frac{n}{2e}\right)^n.$$

So it is enough to show that

$$\frac{n+1}{2}\Big(\frac{n}{2e}\Big)^n > \Big(\frac{n+1}{2e}\Big)^{n+1}.$$

This is immediate from the well-known and easily derived inequality $\big(1+\frac{1}{n}\big)^n < e$.

Now suppose that E is bounded set which is a disjoint union of intervals of total length M. More generally, E can be any bounded Lebesgue measurable set of measure M. Translating the set to the right or left if necessary, we may suppose that $E \subset [0,\infty)$, and that $\inf E = 0$. For $x \geq 0$, let $f(x) = m([0,x] \cap E)$. Here $m(S)$ denotes the Lebesgue measure of a subset S of $\mathbb{R}$, which, when S is a disjoint union of intervals, is simply the sum of the lengths of these intervals. Then $f(0) = 0$, and $f(x) = M$ for x sufficiently large. Also, if $x \leq y$, then $f(x) \leq f(y)$, and

$$f(y) - f(x) = m([0,y] \cap E) - m([0,x] \cap E) = m((x,y] \cap E) \leq m((x,y]) = y - x. \qquad (1)$$

The function $f(x)$ is therefore continuous, and has image $[0,M]$. For each $j \in \{0,1,\ldots,n\}$, choose x_j such that $f(x_j) = jM/n$. If $j \leq k$, then $(k-j)M/n = f(x_k) - f(x_j) \leq x_k - x_j$ by (1). Hence

$$\prod_{\substack{j=0\\ j\neq \ell}}^{n} |x_j - x_\ell| \geq \prod_{\substack{j=0\\ j\neq \ell}}^{n} \Big(|j-\ell|\frac{M}{n}\Big),$$

which we have seen is greater than $(M/2e)^n$. It remains to show that we can choose the x_j's so that, in addition, they are in E. The sets $S_j = \{x \geq 0 : f(x) = jM/n\}$ are closed and bounded below by 0, because $f(x)$ is continuous. So each S_j has a least element and we first choose x_j to be this least element. We claim that then each x_j is in the closure $\bar{E}$ of E. If $j = 0$, then $x_j = 0$, which is in $\bar{E}$ because $\inf E = 0$. If $j > 0$, then $f(x) < jM/n$ for $x < x_j$, and so $f((x,x_j] \cap E) = f(x_j) - f(x) > 0$, so that in particular $(x,x_j] \cap E \neq \emptyset$ for any $x < x_j$. Hence $x_j \in \bar{E}$.

Finally, because we have shown that $\prod_{j=0:\, j\neq\ell}^{n} |x_j - x_\ell|$ is strictly greater than $(M/2e)^n$, if we replace each x_j by a point y_j sufficiently near x_j, then $\prod_{j=0:\, j\neq\ell}^{n} |y_j - y_\ell|$ is still greater than $(M/2e)^n$. Since $x_j \in \bar{E}$, we can choose y_j to be in E.

6. Let $A_i = (x_i, y_i)$ and $B_i = (u_i, v_i)$ for $i = 1,2,3$. Form the matrices

$$M = \begin{pmatrix} x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 1 & 0 \\ x_3 & y_3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -2u_1 & -2v_1 & 0 & 1 \\ -2u_2 & -2v_2 & 0 & 1 \\ -2u_3 & -2v_3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

An easy calculation shows that

$$MN^T = \begin{pmatrix} -2(u_1x_1 + v_1y_1) & -2(u_2x_1 + v_2y_1) & -2(u_3x_1 + v_3y_1) & 1 \\ -2(u_1x_2 + v_1y_2) & -2(u_2x_2 + v_2y_2) & -2(u_3x_2 + v_3y_2) & 1 \\ -2(u_1x_3 + v_1y_3) & -2(u_2x_3 + v_2y_3) & -2(u_3x_3 + v_3y_3) & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

In this matrix, add $x_i^2+y_i^2$ times row 4 to row i, $i=1,2,3$, and then $u_j^2+v_j^2$ times column 4 to column j, $j=1,2,3$. This of course does not change the determinant, and the matrix becomes

$$\begin{pmatrix} (x_1-u_1)^2+(y_1-v_1)^2 & (x_1-u_2)^2+(y_1-v_2)^2 & (x_1-u_3)^2+(y_1-v_2)^2 & 1 \\ (x_2-u_1)^2+(y_2-v_1)^2 & (x_2-u_2)^2+(y_2-v_2)^2 & (x_2-u_3)^2+(y_2-v_2)^2 & 1 \\ (x_3-u_1)^2+(y_3-v_1)^2 & (x_3-u_2)^2+(y_3-v_2)^2 & (x_3-u_3)^2+(y_3-v_2)^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

which is the matrix

$$\begin{pmatrix} d_{1,1}^2 & d_{1,2}^2 & d_{1,3}^2 & 1 \\ d_{2,1}^2 & d_{2,2}^2 & d_{2,3}^2 & 1 \\ d_{3,1}^2 & d_{3,2}^2 & d_{3,3}^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

of the question. Hence the determinant of the matrix in the question is that of MN^T, and so is $\det(M)\det(N)$. Now $\det(M)=(x_1y_2-x_2y_1)-(x_1y_3-x_3y_1)+(x_2y_3-x_3y_2)$ and $\det(N)=-4((u_1v_2-u_2v_1)-(u_1v_3-u_3v_1)+(u_2v_3-u_3v_2))$. Provided that A_1, A_2 and A_3 are oriented in such a way that as we move from A_1 to A_2, then A_2 to A_3 and A_3 back to A_1 we are moving in an anti-clockwise direction, the quantity $(x_1y_2-x_2y_1)-(x_1y_3-x_3y_1)+(x_2y_3-x_3y_2)$ is twice the area of the triangle $\triangle A_1A_2A_3$. We can see this for example by considering the cross product $\mathbf{u}\times\mathbf{v}$ of the vectors $\mathbf{u}=(x_2-x_1)\,\mathbf{i}+(y_2-y_1)\,\mathbf{j}$ and $\mathbf{v}=(x_3-x_1)\,\mathbf{i}+(y_3-y_1)\,\mathbf{j}$, which equals $c\,\mathbf{k}$ for c equal to the area of the parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$. So $c/2$ equals the area of the triangle with vertices $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. So $\det(M)=2\text{Area}\,(\triangle A_1A_2A_3)$. For the same reasons, assuming that also $\triangle B_1B_2B_3$ is "positively oriented", we have $\det(N)=-8\text{Area}\,(\triangle B_1B_2B_3)$. So the formula in the question is proved, provided that the two triangles are positively oriented.

7. For $i=1,\dots,n$, let k_i denote the number of 1's in the i-th row of our matrix $A=(a_{i,j})$. Let S_i denote the set of ordered pairs (r,s) such that $1\le r<s\le n$ and $a_{i,r}=a_{i,s}=1$. The hypothesis on A means that the sets $S_1,\dots,S_n$ are pairwise disjoint. Clearly $|S_i|=k_i(k_i-1)/2$ for each i, and there are $n(n-1)/2$ ordered pairs (r,s) such that $1\le r<s\le n$. Hence

$$\sum_{i=1}^n \frac{k_i(k_i-1)}{2} \;=\; \sum_{i=1}^n |S_i| \;=\; \Big|\bigcup_{i=1}^n S_i\Big| \;\le\; \frac{n(n-1)}{2}.$$

The function $f(x)=x(x-1)/2$ satisfies $f''(x)=1>0$, and so is convex on $\mathbb{R}$. Hence $f(t_1x_1+\cdots+t_nx_n)\le t_1f(x_1)+\cdots+t_nf(x_n)$ if $t_1,\dots,t_n\ge 0$ and $t_1+\cdots+t_n=1$. In particular, taking $x_i=k_i$ and $t_i=1/n$ for each i, we get

$$\frac{\bar{k}(\bar{k}-1)}{2}\le\frac{1}{n}\sum_{i=1}^n\frac{k_i(k_i-1)}{2}$$

for $\bar{k}=(k_1+\cdots+k_n)/n$. Hence $\bar{k}(\bar{k}-1)\le n-1$. So $\bar{k}$ lies between the two roots of $x^2-x-(n-1)=0$, and therefore

$$\bar{k}\le\frac{1+\sqrt{1+4(n-1)}}{2}=\frac{1+\sqrt{4n-3}}{2}.$$

The total number N of zeroes in A is $k_1 + \cdots + k_n = n\bar{k}$, and so

$$N \leq \frac{n(1+\sqrt{4n-3}\,)}{2}.$$

This is a improvement of the estimate $N \leq n\sqrt{2n-1}$ in the question.

8. We show that the desired least common multiple is

$$Q(t) = \prod_{j=1}^{n} (t^j - 1).$$

To see this, let us first factorize $Q(t)$ over $\mathbb{C}$. Each factor $t^j - 1$ is the product of the j distinct factors $t - \alpha$, where $\alpha = e^{2\pi ri/j}$, $r = 0, 1, \ldots, j-1$. So each α is a root of unity, that is, $\alpha^k = 1$ for some integer $k \geq 0$. Furthermore, k can be taken less than or equal to n. Recall that α is called a primitive k-th root of unity if $\alpha^k = 1$ and if $\alpha^\ell = 1$ holds for no integer $\ell \geq 0$ such that $\ell < k$. In this case, we can write $\alpha = e^{2\pi\nu i/k}$ for some $k \geq 1$, and some $\nu \in \{0, \ldots, k-1\}$ with $\gcd(\nu, k) = 1$ (for example, if $\alpha = 1$, then $k = 1$ and $\nu = 0$). Suppose that $\alpha = e^{2\pi\nu i/k}$ is a primitive k-th root of unity. Then $\alpha^j = 1$ if and only if k divides j. So when $j \in \{1, \ldots, n\}$, $t - \alpha$ is a factor of $t^j - 1$ if and only if k divides j; that is, for $j = k, 2k, \ldots, mk$, where $m = \lfloor n/k \rfloor$. Hence the multiplicity of $t - \alpha$ in $Q(t)$ is $\lfloor n/k \rfloor$. Thus

$$Q(t) = \prod_{k=1}^{n} \Big(\prod_{\substack{\alpha \text{ primitive} \\ k\text{-th root of } 1}} (t - \alpha) \Big)^{\lfloor n/k \rfloor}. \tag{1}$$

Now $Q(t)$ is divisible by all the polynomials $P_{n_1,\ldots,n_r}(t) = (t^{n_1} - 1) \cdots (t^{n_r} - 1)$, where $n_1, \ldots, n_r$ are positive integers and $n_1 + \cdots + n_r = n$. For if $\alpha \in \mathbb{C}$ is a root of $P_{n_1,\ldots,n_r}(t)$, then α is clearly a root of unity. If α is a primitive k-th root of unity, then $t^{n_j} - 1$ contains a single factor $t - \alpha$ for each j such that k divides n_j. Note that $k \leq n_j$ for each such j. If k divides n_j for m j's, then mk is at most the sum of these n_j's and so $mk \leq n$. So the multiplicity of $t - \alpha$ in $P_{n_1,\ldots,n_r}(t)$ is at most $\lfloor n/k \rfloor$. It follows that $P_{n_1,\ldots,n_r}(t)$ divides $Q(t)$.

Now let $F(t)$ be any polynomial which is divisible by all the $P_{n_1,\ldots,n_r}(t)$. For $k \in \{1, \ldots, n\}$, let $m = \lfloor n/k \rfloor$, and let $(n_1, \ldots, n_r)$ consist of m k's and (if $mk < n$) one $n - mk$. Then $P_{n_1,\ldots,n_r}(t) = (t^k - 1)^m (t^{n-mk} - 1)$ must divide $F(t)$. But this $P_{n_1,\ldots,n_r}(t)$ is divisible by $(t - \alpha)^m$, and so $F(t)$ must be too. In view of (1), we see that $Q(t)$ divides $F(t)$. So $Q(t)$ is the least common multiple of the polynomials $P_{n_1,\ldots,n_r}(t)$.

9. Recall that a function $f(x, y)$ of two variables is called *differentiable* at (x_0, y_0) if (a) there is a $\delta_0 > 0$ such that $f(x, y)$ is defined at least on the square of (x, y)'s satisfying $|x - x_0| < \delta_0$ and $|y - y_0| < \delta_0$, and (b) we can write

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + \epsilon_1(x, y)(x - x_0) + \epsilon_2(x, y)(y - y_0),$$

where A, B are constants and where $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ are two functions which tend to zero as $(x, y) \to (x_0, y_0)$. Recall that $\epsilon_1(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$ means that if $\epsilon > 0$

is given, there is a $\delta > 0$ such that $|\epsilon_1(x, y)| < \epsilon$ whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$. If f is differentiable at (x_0, y_0), then the A and B above are the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, respectively. The following fact is well-known:

Lemma 1. *Suppose that the partial derivative $f_x(x, y)$ exists at each point of a square centred on (x_0, y_0), and that $f_x(x, y)$ is continuous at (x_0, y_0). Suppose also that the partial derivative $f_y(x, y)$ exists at (x_0, y_0). Then f is differentiable at (x_0, y_0). The same is true if we interchange the roles of $f_x(x, y)$ and $f_y(x, y)$.*

Proof. By the hypotheses, there is a $\delta > 0$ such that $f(x, y)$ is defined and $f_x(x, y)$ exists if $|x - x_0| < \delta$ and $|y - y_0| < \delta$. Write

$$f(x, y) = f(x_0, y_0) + \big(f(x, y) - f(x_0, y)\big) + \big(f(x_0, y) - f(x_0, y_0)\big).$$

By the Mean Value Theorem, for each x, y such that $|x - x_0| < \delta$ and $|y - y_0| < \delta$, there is a point ξ (depending on y) between x and x_0 such that $f(x, y) - f(x_0, y) = f_x(\xi, y)(x - x_0)$. By continuity of $f_x(x, y)$ at (x_0, y_0), if $\epsilon > 0$ is given, there is a $\delta_1 > 0$ such that $|f_x(x, y) - f_x(x_0, y_0)| < \epsilon$ if $|x - x_0| < \delta_1$ and $|y - y_0| < \delta_1$. So we can write $f(x, y) - f(x_0, y) = f_x(x_0, y_0)(x - x_0) + \epsilon_1(x, y)(x - x_0)$, where $\epsilon_1(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$.

By definition of $f_y(x_0, y_0)$, if $\epsilon > 0$ is given, there is a $\delta_2 > 0$ such that

$$\Big|\frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} - f_y(x_0, y_0)\Big| < \epsilon$$

if $0 < |y - y_0| < \delta_2$. Hence $f(x_0, y) - f(x_0, y_0) = f_y(x_0, y_0)(y - y_0) + \epsilon_2(y)(y - y_0)$ for a function $\epsilon_2(y)$ which tends to 0 as $y \to y_0$.

Combining the above steps, we have

$$\begin{aligned} f(x, y) &= f(x_0, y_0) + \big(f(x, y) - f(x_0, y)\big) + \big(f(x_0, y) - f(x_0, y_0)\big) \\ &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + \epsilon_1(x, y)(x - x_0) \\ &\quad + f_y(x_0, y_0)(y - y_0) + \epsilon_2(y)(y - y_0), \end{aligned}$$

and so f is differentiable at (x_0, y_0). $\square$

Lemma 2. *Suppose that f is defined on the square $|x-x_0| < \delta_0$ and $|y-y_0| < \delta_0$. Suppose that the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist on this square and are differentiable functions at (x_0, y_0). Then $f_{x,y}(x_0, y_0) = f_{y,x}(x_0, y_0)$.*

Proof. By the hypotheses, we can write the equations

$$f_x(x, y) = f_x(x_0, y_0) + A(x - x_0) + B(y - y_0) + \epsilon_1(x, y)(x - x_0) + \epsilon_2(x, y)(y - y_0),$$

$$f_y(x, y) = f_y(x_0, y_0) + C(x - x_0) + D(y - y_0) + \epsilon_3(x, y)(x - x_0) + \epsilon_4(x, y)(y - y_0),$$

for constants A, B, C, D and functions $\epsilon_j(x, y)$ which tend to 0 as $(x, y) \to (x_0, y_0)$. We have $A = f_{x,x}(x_0, y_0)$, $B = f_{x,y}(x_0, y_0)$, $C = f_{y,x}(x_0, y_0)$ and $D = f_{y,y}(x_0, y_0)$, and so in particular we know that all these second order partial derivatives exist. By replacing $f(x,y)$ by $f(x,y) - \frac{1}{2}\big(A(x-x_0)^2 + D(y-y_0)^2\big)$, we can assume that $A = D = 0$. By the Mean Value Theorem, we can write $f(x,y) - f(x_0,y) = f_x(\xi,y)(x-x_0)$ for some ξ between x and x_0. So

$$\begin{aligned} &f(x,y) - f(x_0,y) \\ &\quad = \big[f_x(x_0,y_0) + A(\xi - x_0) + B(y-y_0) + \epsilon_1(\xi,y)(\xi-x_0) + \epsilon_2(\xi,y)(y-y_0)\big](x-x_0). \end{aligned} \tag{1}$$

Also, by the Mean Value Theorem, for some η between y and y_0, we have

$$\begin{aligned} f(x_0,y) &= f(x_0,y_0) + f_y(x_0,\eta)(y-y_0) \\ &= f(x_0,y_0) + \big[f_y(x_0,y_0) + D(\eta-y_0) + \epsilon_4(x_0,\eta)(\eta-y_0)\big](y-y_0). \end{aligned} \tag{2}$$

When we substitute (2) into (1) and remember that $A = D = 0$, we find that

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + B(x-x_0)(y-y_0) + E(x,y),$$

where

$$E(x,y) = \big[\epsilon_1(\xi,y)(\xi-x_0) + \epsilon_2(\xi,y)(y-y_0)\big](x-x_0) + \epsilon_4(x_0,\eta)(\eta-y_0)(y-y_0).$$

If $\epsilon > 0$, there is a $\delta > 0$ such that $|\epsilon_j(x,y)| < \epsilon$ for each j if $|x-x_0| < \delta$ and $|y-y_0| < \delta$. Since $|\xi - x_0| \le |x-x_0|$ and $|\eta - y_0| \le |y-y_0|$, we see that

$$|E(x,y)| \le \epsilon\big(|x-x_0|^2 + |x-x_0|\,|y-y_0| + |y-y_0|^2\big)$$

if $|x-x_0| < \delta$ and $|y-y_0| < \delta$.

Similarly, starting from $f(x,y) - f(x,y_0) = f_y(x,\eta)(y-y_0)$ and $f_y(x,\eta) = f_y(x_0,y_0) + C(x-x_0) + D(\eta-y_0) + \epsilon_3(x,\eta)(x-x_0) + \epsilon_4(x,\eta)(\eta-y_0)$, and writing $f(x,y_0) = f(x_0,y_0) + f_x(\xi,y_0)(x-x_0) = f(x_0,y_0) + \big(f_x(x_0,y_0) + A(\xi-x_0) + \epsilon_1(\xi,y_0)(\xi-x_0)\big)(x-x_0)$, we find that

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + C(x-x_0)(y-y_0) + F(x,y),$$

where

$$|F(x,y)| \le \epsilon\big(|x-x_0|^2 + |x-x_0|\,|y-y_0| + |y-y_0|^2\big)$$

if $|x-x_0| < \delta$ and $|y-y_0| < \delta$. Subtracting these two expressions for $f(x,y)$, we get

$$(B-C)(x-x_0)(y-y_0) = F(x,y) - E(x,y).$$

If we now take $|x-x_0| = |y-y_0| < \delta$ and divide both sides of the last formula by $(x-x_0)(y-y_0)$, we see that $|B-C| \le 6\epsilon$. But $\epsilon > 0$ was arbitrary, and so $B = C$. $\square$

Lemma 3. *Suppose that the conditions of Problem 9 are satisfied. That is, suppose that (a) for some* $\delta_0 > 0$ *the partial derivatives* $f_x(x,y)$, $f_y(x,y)$, $f_{x,x}(x,y)$ *and* $f_{y,y}(x,y)$ *all* *exist if $|x - x_0| < \delta_0$ and $|y - y_0| < \delta_0$, and (b) $f_{x,x}(x,y)$ and $f_{y,y}(x,y)$ are continuous at (x_0, y_0) and (c) $f_{x,y}(x_0, y_0)$ and $f_{y,x}(x_0, y_0)$ exist. Then the conditions of Lemma 2 are satisfied, and so $f_{x,y}(x_0, y_0) = f_{y,x}(x_0, y_0)$.*

Proof. The function f_x is differentiable at (x_0, y_0) because (i) $f_{x,x}(x,y) = \big(f_x\big)_x(x,y)$ exists in a square about (x_0, y_0) and $f_{x,x}(x,y)$ is continuous at (x_0, y_0), and (ii) $f_{x,y}(x_0, y_0) = \big(f_x\big)_y(x_0, y_0)$ exists. Similarly, applying the last sentence of Lemma 1, the function f_y is differentiable at (x_0, y_0) because (i) $f_{y,y}(x,y) = \big(f_y\big)_y(x,y)$ exists in a square about (x_0, y_0) and $f_{y,y}(x,y)$ is continuous at (x_0, y_0), and (ii) $f_{y,x}(x_0, y_0) = \big(f_y\big)_x(x_0, y_0)$ exists. □

10. Let A_n denote the number of "alternating" permutations of $1, 2, \ldots, n$, i.e., those for which 1 comes before 2, 2 after 3, 3 before 4, etc. The first few of these numbers are $A_1 = 1$, $A_2 = 1$, $A_3 = 2$ and $A_4 = 5$. It is convenient to also define $A_0 = 1$. The condition on alternating permutations may be stated in this way: we require that $2k+1$ comes before $2k$ for $k = 1, \ldots, \lfloor (n-1)/2 \rfloor$ and that $2k+1$ comes before $2k+2$ for $k = 0, 1, \ldots, \lfloor (n-2)/2 \rfloor$. We consider the cases n odd and n even separately.

If $n = 2m+1$ is odd, then the last letter in any alternating permutation must be even, $2k$ say. Amongst the remaining letters, the letters $1, \ldots, 2k-1$ must form an alternating sequence, and so must the letters $2k+1, \ldots, 2m+1$. Taking any of the $\binom{2m}{2k-1}$ choices of $2k-1$ positions from the first $2m$, any of the A_{2k-1} alternating permutations of $1, \ldots, 2k-1$, and any of the $A_{2m-2k+1}$ alternating permutations of $2k+1, \ldots, 2m+1$, we obtain an alternating permutation of $1, \ldots, 2m+1$ with last letter $2k$. It follows that

$$A_{2m+1} = \sum_{k=1}^{m} \binom{2m}{2k-1} A_{2k-1} A_{2(m-k)+1}. \tag{1}$$

Form the generating function $A(t) = \sum_{m=0}^{\infty} A_{2m+1} t^{2m+1}/(2m+1)!$. In terms of $A(t)$, (1) says that

$$A(t)^2 = \sum_{m=1}^{\infty} A_{2m+1} \frac{t^{2m}}{(2m)!} = A'(t) - 1.$$

Since $A(0) = 0$, it is elementary that $A(t) = \tan(t)$.

Similarly, if $n = 2m$ is even, then the last letter in any alternating permutation must still be even, $2k$ say. In the same way, we find that

$$A_{2m} = \sum_{k=1}^{m} \binom{2m-1}{2k-1} A_{2k-1} A_{2(m-k)}. \tag{2}$$

Form the generating function $B(t) = \sum_{m=0}^{\infty} A_{2m} t^{2m}/(2m)!$. In terms of $A(t)$ and $B(t)$, (2) says that

$$A(t)B(t) = \sum_{m=1}^{\infty} 2m\, A_{2m} \frac{t^{2m-1}}{(2m)!} = B'(t).$$

Since $B(0) = A_0 = 1$, it is elementary that $B(t) = \sec(t)$.

Hence the generating function of all the A_n's is given by

$$\sum_{n=0}^{\infty} A_n \frac{t^n}{n!} = \tan(t) + \sec(t).$$

The numbers A_n are therefore closely related to various well-known (but complicated) numbers. For example,

$$A_{2m-1} = \frac{(-1)^{m+1} 2^{2m} (2^{2m} - 1) B_{2m}}{2m},$$

where B_n is the n-th Bernoulli number. This formula can be derived from the generating function $t/(e^t - 1) = \sum_{n=0}^{\infty} B_n t^n / n!$ for the Bernoulli numbers.