The University of Sydney
School of Mathematics and Statistics
NSW 2006 Australia

SUMS Problem Competition 2007

1. Among the numbers $2^1, 2^2, \cdots, 2^{10}$, there are 3 whose first digit is 1 (namely, $2^4 = 16$, $2^7 = 128$, and $2^{10} = 1024$). It turns out that among the numbers $2^1, 2^2, \cdots, 2^{100}$, there are 30 whose first digit is 1; and among the numbers $2^1, 2^2, \cdots, 2^{1000}$, there are 301 whose first digit is 1. For any positive integer N, define a_N by the rule that among the numbers 2^n with $1 \le n \le 10^N$, there are a_N whose first digit is 1. Prove that a_{N+1} is always obtained from a_N by adding a single digit at the end.

Solution. A positive integer has first digit 1 exactly when it lies in the interval $[10^s, 2\times 10^s)$ for some $s \ge 0$. For any $s \ge 0$, there is exactly one positive integer k such that $10^s \le 2^k < 2 \times 10^s$: the reason is that on taking logarithms base 2, these inequalities become $\log_2(10^s) \le k < \log_2(10^s) + 1$, which means that $k = \lceil \log_2(10^s) \rceil$. Also note that for $N \ge 1$,

$$10^{\lfloor 10^N \log_{10} 2 \rfloor} \le 2^{10^N} < 10^{\lfloor 10^N \log_{10} 2 \rfloor + 1}.$$

Consequently, among the numbers 2^n with $1 \le n \le 10^N$, there is exactly one in the interval $[10^s, 2 \times 10^s)$ for every $s \in \{1, 2, \cdots, 10^N \log_{10} 2\}$. This proves that $a_N = \lfloor 10^N \log_{10} 2 \rfloor$, which is just the number formed by the first N digits after the decimal point in $\log_{10} 2 = 0.30102999566\cdots$. The result is now obvious.

2. The sisters Alice, Bess, and Cath are fighting over a triangular pizza, which may be imagined as a triangle PQR. Their father David proposes the following procedure for sharing it between the four of them. Alice will select a point A on the edge PQ, then Bess will select a point B on the edge PR, then Cath will select a point C on the edge QR. David will then cut the pizza along the lines AB, BC, and AC, and take the centre piece ABC for himself, leaving three corner pieces (some possibly empty, if endpoints of edges have been chosen). The sisters will then either all take the corner piece to the left of the point they selected, or all take the corner piece to the right of their point; Alice (as the eldest) will get to choose left or right. As everyone knows, each sister will make her choices purely to maximize the area of her own share, except that Alice and Bess, if their own shares are unaffected, will act to the advantage of the youngest sister Cath. If they all reason perfectly, what will they do?

Solution. Write $|XYZ|$ for the area of triangle XYZ, and normalize $|PQR|$ to be 1. Consider Cath's decision in selecting the point C. At this stage A and B have already been chosen. Let $\alpha = \frac{|AQ|}{|PQ|}$, $\beta = \frac{|BR|}{|PR|}$ (already determined), and $\gamma = \frac{|QC|}{|QR|}$ (to be chosen by Cath); then $0 \le \alpha, \beta, \gamma \le 1$, and

$$|ABP| = (1-\alpha)(1-\beta), \ |ACQ| = \alpha\gamma, \ |BCR| = \beta(1-\gamma).$$

Cath knows that if she selects γ such that $\alpha\gamma < (1-\alpha)(1-\beta)$, then Alice will choose ABP, leaving Cath with ACQ, of area $\alpha\gamma$. Cath's share in this case will be less than $(1-\alpha)(1-\beta)$. If she selects γ such that $\alpha\gamma > (1-\alpha)(1-\beta)$, then Alice will choose ACQ, leaving Cath with BCR, of area $\beta(1-\gamma)$. Her share in this case will be less than $\beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))$. If she selects $\gamma = \frac{1}{\alpha}(1-\alpha)(1-\beta)$, then Alice's two possible pieces will have the same area, so she will make her decision to favour Cath, and thus Cath will get a piece of area $\max\{(1-\alpha)(1-\beta), \beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))\}$. The last option is obviously preferable, if it is possible (i.e. if $\alpha \neq 0$ and the required value of γ is ≤ 1); if not, only the first option is possible and Cath will simply want to maximize $\alpha\gamma$ (there is a slight exception here if $\alpha = 0$ and $\beta = 1$). So Cath will choose γ according to the following rule:

a) if $\alpha = 0$ and $\beta = 1$, then $\gamma = 0$, which gives Alice 0, Cath 1, and Bess 0;

b) if $\alpha < (1-\alpha)(1-\beta)$, then $\gamma = 1$, which gives Alice $(1-\alpha)(1-\beta)$, Cath α, and Bess 0;

c) if $0 \neq \alpha \geq (1-\alpha)(1-\beta)$, then $\gamma = \frac{1}{\alpha}(1-\alpha)(1-\beta)$, which gives Alice $(1-\alpha)(1-\beta)$, Cath $\max\{(1-\alpha)(1-\beta), \beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))\}$, and Bess $\min\{(1-\alpha)(1-\beta), \beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))\}$.

Knowing this, Bess reasons as follows. If $\alpha = 0$, then Bess is certain to get 0, so she should choose $\beta = 1$ to favour Cath; this gives Alice 0. If $\alpha = 1$, then Bess is certain to get 0, so again she should choose $\beta = 1$ to favour Cath; this too gives Alice 0. If $0 < \alpha < 1$, she should choose β so as to ensure that $\alpha \geq (1-\alpha)(1-\beta)$ and, subject to that constraint, maximize $\min\{(1-\alpha)(1-\beta), \beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))\}$. Now as β increases from 0 to 1, $(1-\alpha)(1-\beta)$ decreases from $1-\alpha$ to 0, and $\beta(1-\frac{1}{\alpha}(1-\alpha)(1-\beta))$ increases from 0 to 1. So the minimum is maximized when the two are equal, i.e. when

$$1 = (\frac{1}{\alpha} + \frac{1}{\beta})(1-\alpha)(1-\beta),$$

which must happen for a unique $\beta \in (0,1)$. A simple calculation shows that this unique β is nothing other than $1-\alpha$, and it does indeed satisfy the constraint $\alpha \geq (1-\alpha)(1-\beta)$. So in this case Bess should choose $\beta = 1-\alpha$, which will give Alice $\alpha(1-\alpha)$. Knowing that Bess and Cath will decide according to these rules, Alice's best option is clearly to maximize $\alpha(1-\alpha)$ by selecting $\alpha = \frac{1}{2}$, i.e. choosing A to be the midpoint of PQ. Bess and Cass will then also choose the midpoints, and Alice will have to flip a coin to decide on left or right, because all four pieces will be exactly a quarter of the total area.

3. The members of a tennis club are planning a doubles carnival consisting of several rounds. In the spirit of social tennis, results don't matter, but participation does; so in each round, every member is to play in exactly one game. Each round is to be either a mixed doubles round, in which every game involves two male and two female players, or an ordinary doubles round, in which every game involves four players of the same gender. There is a further requirement that over the whole carnival, any two members play in the same game exactly once; whether they are partners or opponents in this game is immaterial. If there are 2^k male and 2^k female members, for what (positive integer) values of k is this possible?

Solution. We will prove that this is possible if and only if k is odd (i.e. the total number of members is a power of 4). Firstly, suppose that it is possible, and let r be the number of rounds. In every round, Member A plays with three other members, and the total $3r$ is meant to equal $2^{k+1}-1$. So we must have $2^{k+1} \equiv 1 \bmod 3$, which forces k to be odd.

Now assume k is odd, and let $n = \frac{k+1}{2}$, so that the number of members is 4^n. Write $\mathbb{F}_2$ for the field with two elements $\{0,1\}$, and $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ for the degree-2 extension $\{0,1,\alpha,\alpha+1\}$, where $\alpha^2 = \alpha+1$. The $\mathbb{F}_4$-vector space $\mathbb{F}_4^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{F}_4\}$ has 4^n elements, which we can assign bijectively to the members of the tennis club in such a way that the males are those whose first coordinate x_1 is either 0 or 1, while the females are those such that x_1 is either α or $\alpha + 1$. To construct the schedule, we index the rounds by the one-dimensional $\mathbb{F}_4$-subspaces of $\mathbb{F}_4^n$. If L is such a one-dimensional subspace, we let the games in that round be the cosets $L + (a_1, a_2, \cdots, a_n)$. It is well known that every element of $\mathbb{F}_4^n$ is contained in exactly one coset of L, so every member will play in exactly one game in each round. If the elements of L all have zero first coordinate, then the elements of each coset $L + (a_1, a_2, \cdots, a_n)$ will have the same first coordinate, and hence the players in every game in that round will have the same gender. If some element of L has nonzero first coordinate, then the four elements of L must have the four different first coordinates, and the same is therefore true for each coset, so that gives a mixed doubles round. Finally, two distinct elements $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ of $\mathbb{F}_4^n$ belong to exactly one coset together, namely $\mathbb{F}_4(b_1 - a_1, b_2 - a_2, \cdots, b_n - a_n) + (a_1, a_2, \cdots, a_n)$. So any two members play in the same game exactly once.

4. Let T be a tree with n vertices. (A tree is a connected graph with no cycles.) Fix $1 \le k \le n$, and let $\mathcal{S}_k$ be the set of all k-element subsets of the set of vertices of T. For any $S \in \mathcal{S}_k$, let $c(S)$ be the number of connected components of the subgraph obtained by restricting to the vertices in S (i.e. deleting all of the tree except the vertices in S and the edges between them). Prove that $\sum_{S\in\mathcal{S}_k} c(S) = (n-k+1)\binom{n-1}{k-1}$.

Solution. The subgraph of T obtained by restricting to the vertices in S is a forest (a disconnected union of trees). It is easy to prove by induction that the number of connected components of a forest equals the number of vertices minus the number of edges. Hence $c(S) = k - e(S)$, where $e(S)$ is the number of edges between elements of S. Moreover, there are $n-1$ edges in T, each of which belongs to $\binom{n-2}{k-2}$ subsets in $\mathcal{S}_k$. Hence

$$\sum_{S\in\mathcal{S}_k} c(S) = k\binom{n}{k} - (n-1)\binom{n-2}{k-2} = n\binom{n-1}{k-1} - (k-1)\binom{n-1}{k-1} = (n-k+1)\binom{n-1}{k-1},$$

as required.

5. Say that a rational number r is splittable if the cubic polynomial $x^3 - 3x - r$ factorizes as $(x - r_1)(x - r_2)(x - r_3)$ where r_1, r_2, r_3 are rational. Find polynomials f and g with integer coefficients such that r is splittable if and only if $r = \frac{f(t)}{g(t)}$ for some rational number t.

Solution. Clearly r is splittable if and only if there exist rational r_1, r_2, r_3 such that

$$\begin{aligned} r_1 + r_2 + r_3 &= 0, \\ r_1r_2 + r_1r_3 + r_2r_3 &= -3, \\ r_1r_2r_3 &= r. \end{aligned}$$

For a fixed r_1, the first two equations can be rewritten as

$$\begin{aligned} r_2 + r_3 &= -r_1, \\ r_2r_3 &= r_1^2 - 3. \end{aligned}$$

Hence r_2 and r_3 are the roots of the quadratic $x^2 + r_1x + (r_1^2 - 3)$, and they are rational if and only if the discriminant $r_1^2 - 4(r_1^2 - 3) = 12 - 3r_1^2$ is the square of a rational number. Now

$$\begin{aligned} & 12 - 3r_1^2 \text{ is the square of a rational number} \\ & \iff r_1 = 2 \text{ or } \frac{12 - 3r_1^2}{(2 - r_1)^2} \text{ is the square of a rational number} \\ & \iff r_1 = 2 \text{ or } \frac{6 + 3r_1}{2 - r_1} = t^2 \text{ for some } t \in \mathbb{Q} \\ & \iff r_1 = 2 \text{ or } r_1 = \frac{2t^2 - 6}{t^2 + 3} \text{ for some } t \in \mathbb{Q}. \end{aligned}$$

Suppose that r is splittable. Since it is impossible for all of r_1, r_2, r_3 to equal 2, we can renumber if necessary to ensure that $r_1 \neq 2$, so

$$\begin{aligned} r &= r_1 r_2 r_3 = r_1(r_1^2 - 3) \\ &= \frac{(2t^2 - 6)((2t^2 - 6)^2 - 3(t^2 + 3)^2)}{(t^2 + 3)^3} \\ &= \frac{(2t^2 - 6)(t^4 - 42t^2 + 9)}{(t^2 + 3)^3}, \text{ for some } t \in \mathbb{Q}. \end{aligned}$$

Conversely, if r has this form for some $t \in \mathbb{Q}$, we can let $r_1 = \frac{2t^2-6}{t^2+3}$ and find r_2, r_3 by solving the quadratic. (The solutions are $\frac{-t^2 \pm 6t + 3}{t^2+3}$.) So one solution to the problem is

$$f(t) = (2t^2 - 6)(t^4 - 42t^2 + 9),\ g(t) = (t^2 + 3)^3.$$

6. Let n be a positive integer. This question concerns sets of n integers $\{a_1, a_2, \cdots, a_n\}$ which have the property that the $\binom{n}{2}$ differences $|a_j - a_i|$, $1 \leq i < j \leq n$, are all distinct. For example, the set of the first n powers of 2, namely $\{1, 2, 4, 8, \cdots, 2^{n-1}\}$, has this property. Construct a set with this property for which the differences $|a_j - a_i|$ are all less than $e^{\frac{n+2}{2}}$.

Solution. It suffices to construct an increasing sequence $a_1 < a_2 < a_3 < \cdots$ of positive integers such that $a_n < e^{\frac{n+2}{2}}$ and all the differences $a_j - a_i$, $i < j$, are distinct. Given this, it is obvious that $\{a_1, \cdots, a_n\}$ is a set with the required properties, for any $n \geq 1$.

If "construct" is interpreted no more strongly than "define", one can use the Mian-Chowla sequence, which is defined by setting $a_1 = 1$, $a_2 = 2$, and then successively choosing $a_3, a_4, \cdots$ to be the smallest positive integers such that all the differences of the terms chosen so far are distinct. It is easy to see that in choosing a_n, at most $(n-1)^3$ values are ruled out, so $a_n \leq (n-1)^3 + 1$ for all n. Since $(n-1)^3 + 1 < e^{\frac{n+2}{2}}$ for $n \geq 13$, one need only check that $a_n < e^{\frac{n+2}{2}}$ holds for n up to 12. However, there is no known formula for the nth term of the Mian-Chowla sequence, so this is arguably unsatisfactory as a "construction".

One of many possible alternatives is the following sequence defined by a second-order recurrence relation and initial conditions:

$$a_1 = 1,\ a_2 = 4,\ a_n = a_{n-1} + a_{n-2} + (n-2) \text{ for } n \geq 3.$$

This sequence begins $1, 4, 6, 12, 21, 37, 63, \cdots$ and is clearly an increasing sequence of positive integers. An explicit formula, which can be found by standard methods (and is in any case easy to prove by induction) is:

$$a_n = \frac{2\sqrt{5}+1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{2\sqrt{5}-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n - n - 1, \text{ for all } n \geq 1.$$

From this it is easy to see that

$$a_n < \frac{2\sqrt{5}+1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n < e(\sqrt{e})^n = e^{\frac{n+2}{2}}.$$

We now prove by contradiction that the differences $a_j - a_i$, $i < j$, are distinct. Suppose that $a_j - a_i = a_l - a_k$, where $i < j$, $k < l$, and $j < l$. (It is clear that $j = l$ would force $i = k$, so there is no need to allow this possibility.) There are two cases.
Case 1: $k \leq l - 2$. In this case, we have

$$a_l - a_k \geq a_l - a_{l-2} = a_{l-1} + (l-2) > a_{l-1} \geq a_j > a_j - a_i,$$

giving the required contradiction.
Case 2: $k = l - 1$. In this case, we have

$$a_j > a_j - a_i = a_l - a_{l-1} = a_{l-2} + (l-2) \geq a_{l-2},$$

which forces $j = l - 1$. If $l = 3$, this means that $4 - a_i = 6 - 4$, clearly impossible; and if $l \geq 4$,

$$a_i = a_{l-1} - (a_j - a_i) = a_{l-1} - (a_{l-2} + (l-2)) = a_{l-3} - 1,$$

which is also clearly impossible.

7. Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of real numbers which is skew-symmetric, meaning that $a_{ij} = -a_{ji}$ for all i, j. Define a new skew-symmetric matrix $B = (b_{ij})_{i,j=1}^n$ as follows:

$$b_{ij} = \begin{cases} -a_{ij} + \sum_{p\geq 1} (-1)^{p+1} \sum_{\substack{i_1,i_2,\cdots,i_p \in \mathbb{Z} \\ i<i_1<i_2<\cdots<i_p<j}} a_{ii_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_pj}, & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -b_{ji}, & \text{if } i > j. \end{cases}$$

Prove that $\det(B) = \det(A)$.

Solution. For any skew-symmetric matrix A, we have

$$\det(A) = \det(A^{\mathrm{t}}) = \det(-A) = (-1)^n \det(A).$$

So if n is odd, we must have $\det(A) = 0$. Thus the question only has content when n is even, which we assume henceforth.

Let $\widetilde{A} = (\widetilde{a}_{ij})_{i,j=1}^n$ and $\widetilde{B} = (\widetilde{b}_{ij})_{i,j=1}^n$ be the upper-triangular matrices obtained from A and B respectively by setting all below-diagonal entries to 0 and all diagonal entries to 1. Then

$$A = \widetilde{A} - \widetilde{A}^{\mathrm{t}} \text{ and } B = \widetilde{B} - \widetilde{B}^{\mathrm{t}}.$$

We claim that $\widetilde{B} = \widetilde{A}^{-1}$; since these are square matrices, it suffices to prove that $\widetilde{A}\widetilde{B}$ is the identity matrix. It is clear that $\widetilde{A}\widetilde{B}$ is upper-triangular with all diagonal entries equal to 1. For any $i < k$, its (i, k)-entry is

$$\begin{aligned}\sum_j \widetilde{a}_{ij}\widetilde{b}_{jk} &= a_{ik} + b_{ik} + \sum_{i<j<k} a_{ij}b_{jk} \\ &= \sum_{p\geq 1}(-1)^{p+1} \sum_{\substack{i_1,i_2,\cdots,i_p\in\mathbb{Z} \\ i<i_1<i_2<\cdots<i_p<k}} a_{ii_1}a_{i_1i_2}\cdots a_{i_pk} \\ &\quad + \sum_{i<j<k} a_{ij} \sum_{q\geq 0}(-1)^{q+1} \sum_{\substack{j_1,j_2,\cdots,j_q\in\mathbb{Z} \\ j<j_1<j_2<\cdots<j_q<k}} a_{jj_1}a_{j_1j_2}\cdots a_{j_qk} \\ &= 0,\end{aligned}$$

because the term indexed by $p, i_1, i_2, \cdots, i_p$ cancels that indexed by $j = i_1, q = p-1, j_1 = i_2, \cdots, j_{p-1} = i_p$. So the claim is proved, and it implies that

$$B = \widetilde{B} - \widetilde{B}^{\mathrm{t}} = (\widetilde{B}^{\mathrm{t}}\widetilde{A}^{\mathrm{t}})\widetilde{B} - \widetilde{B}^{\mathrm{t}}(\widetilde{A}\widetilde{B}) = \widetilde{B}^{\mathrm{t}}(\widetilde{A}^{\mathrm{t}} - \widetilde{A})\widetilde{B} = -\widetilde{B}^{\mathrm{t}}A\widetilde{B}.$$

Hence

$$\det(B) = \det(-\widetilde{B}^{\mathrm{t}}A\widetilde{B}) = (-1)^n \det(\widetilde{B}^{\mathrm{t}})\det(A)\det(\widetilde{B}) = \det(A),$$

since $\det(\widetilde{B})$ is clearly 1.

8. Take a cube with edges of length 1. Fix a length $0 < \ell < \frac{1}{\sqrt{2}}$, and attach a square to each face whose centre is the centre of the face, whose sides have length ℓ, and whose edges are (initially) parallel to the edges of the face. Now rotate each of these six squares anti-clockwise about the centre of its face, through some angle $0 < \theta < \frac{\pi}{4}$ (the same angle for all six). Let A be a vertex of the square on face F, let F' be the face which is closest to A of those adjacent to F, and let B and C be the vertices of the square on face F' which are closest to A. Prove that there are unique values of ℓ and θ (subject to the above bounds) for which ABC is an equilateral triangle, and that this value of ℓ is irrational.

Solution. Set up coordinates so that the vertices of the cube are $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ (independent signs). Let $\alpha = \frac{\ell}{\sqrt{2}}\cos(\theta + \frac{\pi}{4})$, $\beta = \frac{\ell}{\sqrt{2}}\sin(\theta + \frac{\pi}{4})$. We can assume that face F is the one with equation $x = \frac{1}{2}$, and then the vertices of the square on face F are $(\frac{1}{2}, \pm\alpha, \pm\beta)$ and $(\frac{1}{2}, \pm\beta, \mp\alpha)$. By symmetry it does not matter which one we call A, so let $A = (\frac{1}{2}, \beta, -\alpha)$. The face F' is then the one with equation $y = \frac{1}{2}$, and $B = (\beta, \frac{1}{2}, \alpha)$, $C = (\alpha, \frac{1}{2}, -\beta)$ are the two closest vertices. We have

$$|AB|^2 = 2(\beta - \tfrac{1}{2})^2 + 4\alpha^2,\ |AC|^2 = (\alpha - \tfrac{1}{2})^2 + (\beta - \tfrac{1}{2})^2 + (\alpha - \beta)^2,\ |BC|^2 = \ell^2 = 2\alpha^2 + 2\beta^2.$$

So $|AB| = |BC|$ if and only if

$$\begin{aligned}2(\beta - \tfrac{1}{2})^2 + 4\alpha^2 &= 2\alpha^2 + 2\beta^2, \text{ i.e.} \\ -2\beta + \tfrac{1}{2} + 2\alpha^2 &= 0, \text{ i.e. } \beta = \alpha^2 + \tfrac{1}{4},\end{aligned}$$

and $|AC| = |BC|$ if and only if

$$\begin{aligned}(\alpha - \tfrac{1}{2})^2 + (\beta - \tfrac{1}{2})^2 + (\alpha - \beta)^2 &= 2\alpha^2 + 2\beta^2, \text{ i.e.} \\ -\alpha - \beta - 2\alpha\beta + \tfrac{1}{2} &= 0.\end{aligned}$$

Eliminating β from these two equations, we find that

$$\begin{aligned}-\alpha - \alpha^2 - \tfrac{1}{4} - 2\alpha^3 - \tfrac{1}{2}\alpha + \tfrac{1}{2} &= 0, \text{ i.e.} \\ 8\alpha^3 + 4\alpha^2 + 6\alpha - 1 &= 0.\end{aligned}$$

Elementary calculus shows that $8x^3 + 4x^2 + 6x - 1$ has exactly one real root, which lies in the open interval $(0, \frac{1}{6})$. So there are unique values of α and β for which the triangle ABC is equilateral; but we need to check that ℓ and θ are also uniquely determined and satisfy the bounds in the question. We have

$$\begin{aligned}\ell^2 &= 2\alpha^2 + 2\beta^2 = 2\alpha^4 + 3\alpha^2 + \tfrac{1}{8} \\ &= \alpha(-\alpha^2 - \tfrac{3}{2}\alpha + \tfrac{1}{4}) + 3\alpha^2 + \tfrac{1}{8} \\ &= -\alpha^3 + \tfrac{3}{2}\alpha^2 + \tfrac{1}{4}\alpha + \tfrac{1}{8} \\ &= 2\alpha^2 + \alpha < 2(\tfrac{1}{6})^2 + \tfrac{1}{6} < \tfrac{1}{2},\end{aligned}$$

so ℓ is uniquely determined and satisfies $0 < \ell < \frac{1}{\sqrt{2}}$. The fact that $0 < \alpha < \beta$ means that $\theta = \tan^{-1}(\frac{\beta}{\alpha}) - \frac{\pi}{4}$ is in the interval $(0, \frac{\pi}{4})$ as required. Finally we need to prove that ℓ is irrational, for which it suffices to show that ℓ^2 is irrational. If $\ell^2 = 2\alpha^2 + \alpha$ were rational, then α would be either a rational number or a quadratic irrational (the latter meaning that the minimal polynomial of α over $\mathbb{Q}$ has degree 2). But a rational root of $8x^3 + 4x^2 + 6x - 1$ would have to have denominator dividing 8, and it is clear that $\frac{1}{8}$ is not a root, so the unique real root α is irrational. This also shows that $8x^3 + 4x^2 + 6x - 1$ is irreducible in $\mathbb{Q}[x]$, so $x^3 + \frac{1}{2}x^2 + \frac{3}{4}x - \frac{1}{8}$ is the minimal polynomial of α, and we are finished.

The polyhedron formed by the vertices of the six squares, with these particular values of ℓ and θ, is an Archimedean solid called the *snub cube*; apart from the six squares, its other faces are 32 equilateral triangles, all congruent to ABC. The fact that α has degree 3 over $\mathbb{Q}$ shows that the snub cube cannot be constructed by straightedge and compass.

9. Let $h(n)$ denote the number of permutations of the set $\{1, \cdots, n\}$ which do not preserve any two-element subset $\{i, j\}$. (In other words, there are no two elements i, j which the permutation leaves fixed, nor are there two elements i, j which the permutation swaps.) Show that

$$\lim_{n\to\infty} \frac{h(n)}{n!} = 2e^{-3/2}.$$

Solution. Let $f(n)$ denote the number of permutations of $\{1, \cdots, n\}$ which have no 1-cycles (i.e. fix no elements) and no 2-cycles (i.e. there are no two elements which the permutation swaps). Then $h(n)$ counts the same permutations as $f(n)$, but in addition counts the permutations with a single 1-cycle and no 2-cycles. The number of permutations of the latter kind is clearly $nf(n-1)$, so

$$h(n) = f(n) + nf(n-1), \text{ and } \frac{h(n)}{n!} = \frac{f(n)}{n!} + \frac{f(n-1)}{(n-1)!}.$$

Thus the desired limit will follow if we can show that $\lim_{n\to\infty} \frac{f(n)}{n!} = e^{-3/2}$. This in turn will follow from the following formula:

$$\begin{aligned}\frac{f(n)}{n!} &= \text{sum of the coefficients of } x^0, x^1, \cdots, x^n \text{ in the Taylor series of } \exp(-x)\exp(-\frac{x^2}{2}) \\ &= \sum_{\substack{m_1,m_2\geq 0\\ m_1+2m_2\leq n}} \frac{(-1)^{m_1+m_2}}{m_1! 2^{m_2} m_2!}.\end{aligned}$$

One can obtain the second expression directly by inclusion/exclusion counting. A slicker approach uses the following identity of formal power series in the variables $p_1, p_2, p_3, \cdots$:

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{w\in S_n} p_w = \exp(\sum_{i\geq 1} \frac{p_i}{i}),$$

where S_n denotes the group of permutations of $\{1, \cdots, n\}$, and p_w is defined to be $p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots$ if w has a_1 1-cycles, a_2 2-cycles, a_3 3-cycles, and so on. Specializing this identity at the values $p_1 = 0$, $p_2 = 0$, $p_i = x$ for $i \geq 3$, we get an equality of power series in x:

$$\begin{aligned}\sum_{n\geq 0} \frac{f(n)}{n!} x^n &= \exp(\sum_{i\geq 3} \frac{x^i}{i}) \\ &= \exp(-\log(1-x) - x - \frac{x^2}{2}) \\ &= \frac{\exp(-x)\exp(-\frac{x^2}{2})}{1-x},\end{aligned}$$

from which the formula follows. There is an obvious generalization to counting permutations with other length cycles excluded.

10. For any positive integers $m \leq n$, let $a_{n,m}$ denote the number of surjective functions from $\{1, 2, \cdots, n\}$ to $\{1, 2, \cdots, m\}$. Define a polynomial $p_n(x)$ by the formula

$$p_n(x) = a_{n,1}(x-1)^{n-1} + a_{n,2}(x-1)^{n-2} + \cdots + a_{n,n-1}(x-1) + a_{n,n}.$$

Let $b_{n,d}$ denote the coefficient of x^d in $p_n(x)$. Prove that $b_{n,d} \geq 0$ for all $0 \leq d \leq n-1$.

Solution. **(First Method)** We seek a recurrence relation for $p_n(x)$. The first step is to note that any surjective function $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, m\}$ is of one of two types: either $f(n) = f(j)$ for some $j < n$ or not. In the first case, the restriction of f to $\{1, \cdots, n-1\}$ is still surjective, so it can be chosen in $a_{n-1,m}$ ways, and then there are m choices for $f(n)$. In the second case, there are still m choices for $f(n)$, and the restriction of f to $\{1, \cdots, n-1\}$ is surjective onto $\{1, 2, \cdots, m\} \setminus \{f(n)\}$, so it can be chosen in $a_{n-1,m-1}$ ways. Thus we have the recurrence relation

$$a_{n,m} = m(a_{n-1,m} + a_{n-1,m-1}) \text{ for all } n \geq 2,$$

where $a_{n-1,i}$ is interpreted as 0 if $i = 0$ or $i = n$. If we define a polynomial $q_n(x) = \sum_{m=1}^{n} a_{n,m} x^m$, this recurrence relation can be reformulated

$$\begin{aligned}q_n(x) &= \sum_{m=1}^{n-1} m a_{n-1,m} x^m + \sum_{m=2}^{n} m a_{n-1,m-1} x^m \\ &= x \sum_{m=1}^{n-1} m a_{n-1,m} x^{m-1} + \sum_{m=1}^{n-1} (m+1) a_{n-1,m} x^{m+1} \\ &= x q_{n-1}'(x) + x^2 q_{n-1}'(x) + x q_{n-1}(x),\end{aligned}$$

for all $n \geq 2$. Now if we define $r_n(x) = \sum_{m=1}^{n} a_{n,m} x^{n-m}$, we have

$$q_n(x) = x^n r_n(x^{-1}), \text{ so } q_n'(x) = n x^{n-1} r_n(x^{-1}) - x^{n-2} r_n'(x^{-1}).$$

Substituting these expressions, we find that

$$\begin{aligned}x^n r_n(x^{-1}) &= (x + x^2)[(n-1)x^{n-2} r_{n-1}(x^{-1}) - x^{n-3} r_{n-1}'(x^{-1})] + x^n r_{n-1}(x^{-1}) \\ &= (n x^n + (n-1)x^{n-1}) r_{n-1}(x^{-1}) - (x^{n-1} + x^{n-2}) r_{n-1}'(x^{-1}).\end{aligned}$$

Dividing through by x^n and replacing x by x^{-1}, this says

$$r_n(x) = ((n-1)x + n)r_{n-1}(x) - x(x+1)r'_{n-1}(x).$$

Since $p_n(x)$ is by definition $r_n(x-1)$, this implies that

$$p_n(x) = ((n-1)x + 1)p_{n-1}(x) - x(x-1)p'_{n-1}(x),$$

for all $n \geq 2$. This is our desired recurrence relation.

Now define $b_{n,d}$ to be the coefficient of x^d in $p_n(x)$ as in the question. Our recurrence relation implies that

$$\begin{aligned} b_{n,d} &= (n-1)b_{n-1,d-1} + b_{n-1,d} - (d-1)b_{n-1,d-1} + db_{n-1,d} \\ &= (n-d)b_{n-1,d-1} + (d+1)b_{n-1,d}, \end{aligned}$$

for all $n \geq 2$. From this recurrence and the fact that $p_1(x) = 1$, it easily follows by induction that all $b_{n,d}$ are nonnegative.

(Second Method) This method starts from the observation that

$$a_{n,m} = \sum_{\substack{n_1,n_2,\cdots,n_m \geq 1 \\ n_1+n_2+\cdots+n_m=n}} \frac{n!}{n_1!n_2!\cdots n_m!}.$$

This is because any surjective function $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, m\}$ determines such a sequence $n_1, \cdots, n_m$ by $n_i = |f^{-1}(i)|$, and for fixed n_i's the number of corresponding functions is the multinomial coefficient $\frac{n!}{n_1!n_2!\cdots n_m!}$.

Now let S_n be the group of all permutations of $\{1, \cdots, n\}$, and set $I = \{1, 2, \cdots, n-1\}$. For $\sigma \in S_n$, we say that $i \in I$ is a descent of σ if $\sigma(i+1) < \sigma(i)$, and we write $D(\sigma)$ for the set of descents of σ. For $n_1, \cdots, n_m$ as above, define

$$J(n_1, \cdots, n_m) = \{1, \cdots, n\} \setminus \{n_1, n_1 + n_2, n_1 + n_2 + n_3, \cdots, n_1 + n_2 + \cdots n_m\}.$$

Clearly $J(n_1, \cdots, n_m) \subseteq I$. We then have another interpretation of the above multinomial coefficient:

$$\frac{n!}{n_1!n_2!\cdots n_m!} = |\{\sigma \in S_n \mid J(n_1, \cdots, n_m) \subseteq D(\sigma)\}|.$$

This is clear, once you rewrite the condition on the right in the following equivalent form:

$$\begin{aligned} &\sigma(1) > \sigma(2) > \cdots > \sigma(n_1), \\ &\sigma(n_1+1) > \sigma(n_1+2) > \cdots > \sigma(n_1+n_2), \\ &\cdots \\ &\sigma(n_1+n_2+\cdots n_{m-1}+1) > \sigma(n_1+n_2+\cdots n_{m-1}+2) > \cdots > \sigma(n). \end{aligned}$$

Hence we can write

$$\begin{aligned} a_{n,m} &= \sum_{\substack{n_1,n_2,\cdots,n_m \geq 1 \\ n_1+n_2+\cdots+n_m=n}} |\{\sigma \in S_n \mid J(n_1, \cdots, n_m) \subseteq D(\sigma)\}| \\ &= \sum_{\substack{J \subseteq I \\ |J|=n-m}} |\{\sigma \in S_n \mid J \subseteq D(\sigma)\}|. \end{aligned}$$

Substituting this in the definition of $p_n(x)$, we obtain

$$\begin{aligned}
p_n(x) &= \sum_{m=1}^{n} \sum_{\substack{J\subseteq I\\ |J|=n-m}} |\{\sigma \in S_n \mid J \subseteq D(\sigma)\}| \, (x-1)^{n-m} \\
&= \sum_{J\subseteq I} |\{\sigma \in S_n \mid J \subseteq D(\sigma)\}| \, (x-1)^{|J|} \\
&= \sum_{J\subseteq I} \sum_{k=0}^{|J|} |\{\sigma \in S_n \mid J \subseteq D(\sigma)\}| \binom{|J|}{k} x^k (-1)^{|J|-k} \\
&= \sum_{J\subseteq I} \sum_{K\subseteq J} |\{\sigma \in S_n \mid J \subseteq D(\sigma)\}| \, x^{|K|} (-1)^{|J|-|K|}.
\end{aligned}$$

Reversing the order of summation and setting $J' = I \setminus J$, this becomes

$$p_n(x) = \sum_{K\subseteq I} x^{|K|} \sum_{J'\subseteq (I\setminus K)} (-1)^{|J'|-|I\setminus K|} |\{\sigma \in S_n \mid (I \setminus D(\sigma)) \subseteq J'\}|.$$

Now the sum over J' is exactly the inclusion/exclusion formula for

$$|\{\sigma \in S_n \mid (I \setminus D(\sigma)) = (I \setminus K)\}|.$$

So we have

$$\begin{aligned}
p_n(x) &= \sum_{K\subseteq I} x^{|K|} |\{\sigma \in S_n \mid D(\sigma) = K\}| \\
&= \sum_{d=0}^{n-1} |\{\sigma \in S_n \mid |D(\sigma)| = d\}| \, x^d.
\end{aligned}$$

So $b_{n,d}$ is not only nonnegative, it has a simple combinatorial interpretation: it is the number of permutations of $\{1, \cdots, n\}$ which have d descents. This is usually called the *Eulerian number* $A(n, d+1)$ (see Stanley, *Enumerative Combinatorics* volume 1, page 22).