THE UNIVERSITY OF SYDNEY

Sydney University Mathematical Society Problem Competition 2011

1. Alice and Bess are playing a game with an ordinary six-sided die. Alice's target numbers are $1, 2, 3$, and Bess' target numbers are $4, 5, 6$. They take turns in rolling the die, with Alice going first. If the one whose turn it is rolls a target number which she has not previously rolled, she gets to roll again; if she rolls a target number which she has previously rolled, or a number which is not one of her target numbers, her turn ends. The winner is the first player to have rolled all three of her target numbers (not necessarily all in the one turn). What is the probability that Alice wins?

Solution. Although it is implicit in the question that the game ends when one of the players wins, we can imagine them continuing to go through their turns after that point, until the total number of rolls is some very large number N. It is clear that as N tends to infinity, the probability that any number remains unrolled tends to zero. So it makes no difference if we imagine the players continuing to play the game forever, and we can assume they both complete rolling their target numbers at some stage. We want the probability that Alice achieves this before Bess.

Let p_n be the probability that the turn in which Alice completes rolling her three target numbers is her nth turn. We can calculate p_n by thinking of the total sequence of numbers Alice rolls, say $a_1, a_2, a_3, \cdots$ where each $a_i \in \{1, 2, 3, 4, 5, 6\}$. If the last of her three target numbers occurs first as a_d, then $a_i \neq a_d$ for $i < d$, and the two elements of $\{1, 2, 3\} \setminus \{a_d\}$ definitely occur in the sequence $a_1, a_2, \cdots, a_{d-1}$. Moreover, apart from the two first occurrences of these elements of $\{1, 2, 3\} \setminus \{a_d\}$, every other roll in the sequence $a_1, a_2, \cdots, a_{d-1}$ results in a new turn being taken. So a_d is rolled in Alice's $(d-2)$th turn. Hence Alice rolls the last of her three target numbers in her nth turn if and only if $a_{n+2} \in \{1, 2, 3\}$, $a_i \neq a_{n+2}$ for $i < n+2$, and the two elements of $\{1, 2, 3\} \setminus \{a_{n+2}\}$ definitely occur in $a_1, a_2, \cdots, a_{n+1}$. The number of $(n+2)$-tuples $(a_1, \cdots, a_{n+2}) \in \{1, 2, 3, 4, 5, 6\}^{n+2}$ with these properties is $3(5^{n+1} - 2 \times 4^{n+1} + 3^{n+1})$, by a simple inclusion/exclusion count. So

$$p_n = \frac{3(5^{n+1} - 2 \times 4^{n+1} + 3^{n+1})}{6^{n+2}} = \frac{1}{2} \times (\frac{5}{6})^{n+1} - (\frac{2}{3})^{n+1} + (\frac{1}{2})^{n+2}.$$

Of course, this is also the probability that the turn in which Bess completes rolling her three target numbers is her nth turn. We clearly have $0 \leq p_n \leq 1$, $p_n \to 0$ as $n \to \infty$, and $\sum_{n \geq 1} p_n = 1$.

Now Alice wins the game if the turn in which she completes rolling her three target numbers is her nth turn, and the turn in which Bess completes rolling her three target numbers is her mth turn, for some pair (n, m) with $n \leq m$. The probability of this happening is $\sum_{n \leq m} p_n p_m$, where the sum is over all pairs (n, m) of positive numbers with $n \leq m$. Since

$$1 = (\sum_{n \geq 1} p_n)^2 = -\sum_{n \geq 1} p_n^2 + 2 \sum_{n \leq m} p_n p_m,$$

it suffices to calculate

$$\begin{aligned}\sum_{n\geq 1} p_n^2 &= \sum_{n\geq 1} \frac{1}{4}\times(\frac{25}{36})^{n+1}+(\frac{4}{9})^{n+1}+(\frac{1}{4})^{n+2}-(\frac{5}{9})^{n+1}-(\frac{1}{3})^{n+1}+\frac{1}{2}\times(\frac{5}{12})^{n+1}\\ &= \frac{625}{5184}\times\frac{36}{11}+\frac{16}{81}\times\frac{9}{5}+\frac{1}{64}\times\frac{4}{3}-\frac{25}{81}\times\frac{9}{4}-\frac{1}{9}\times\frac{3}{2}+\frac{25}{288}\times\frac{12}{7}\\ &= \frac{271}{4620}.\end{aligned}$$

Hence the probability of Alice winning is $\frac{1}{2}(1+\frac{271}{4620})=\frac{4891}{9240}$ (about 53%).

2. Determine all pairs of positive integers a, b such that $4^a + 4^b + 1$ is an integer square.

Solution. Let $4^a + 4^b + 1 = n^2$ for a positive integer n, which is clearly odd. We can assume without loss of generality that $a \geq b$. Since $n^2 > 4^a = (2^a)^2$, we have $n^2 \geq (2^a+1)^2 = 4^a + 2^{a+1} + 1$, so $2b \geq a+1$. On the other hand, $(n-1)(n+1) = 4^b(4^{a-b}+1)$ is divisible by 2^{2b}. One of $n-1$ and $n+1$ must be congruent to 2 modulo 4, so the other one is divisible by 2^{2b-1}. Hence $4^b(4^{a-b}+1) = (n-1)(n+1) \geq 2^{2b-1}(2^{2b-1}-2) = 4^b(4^{b-1}-1)$, which forces $a-b \geq b-1$ or $2b \leq a+1$. We conclude that $2b = a+1$. Conversely, $a = 2b-1$, $n = 2^{2b-1}+1$ is a solution for any b. So the solutions are precisely $(a,b) = (2m-1, m)$ and $(a,b) = (m, 2m-1)$ for positive integers m.

3. Let m and n be positive integers with $m \geq n$. Let A be the $n \times n$ matrix with (i,j)-entry equal to the binomial coefficient $\binom{mj}{i}$. Find the determinant of A.

Solution. If we use the standard convention that $\binom{x}{i}$ means $x(x-1)\cdots(x-i+1)/i!$, then $\binom{mj}{i}$ makes sense for all complex numbers m. In this way we can define the matrix A for any m, whether or not m is a positive integer $\geq n$, and the following argument applies to this generality.

First suppose that $m = 1$. Then A is an upper-triangular matrix, since $\binom{j}{i} = 0$ if $j < i$ where i and j are positive integers. Moreover, the diagonal entries $\binom{i}{i}$ in this case are all 1, which implies that $\det(A) = 1$.

Now revert to the case of general m. The (i,j)-entry of A is $\sum_{k=1}^n b_{ik}(mj)^k$, where b_{ik} denotes the coefficient of x^k in the polynomial $\binom{x}{i}$ (which clearly has zero constant term when i is a positive integer). So by definition of matrix multiplication, $A = BC$ where B is the $n \times n$ matrix whose (i,k)-entry is b_{ik}, and C is the $n \times n$ matrix whose (k,j)-entry is $(mj)^k$.

But it is clear that $b_{ik} = 0$ unless $k \leq i$, and $b_{ii} = 1/i!$ for all i. So the matrix B is lower triangular with diagonal entries $1/1!, 1/2!, \cdots, 1/n!$. This implies that $\det(B) = (1!2!\cdots n!)^{-1}$, so $\det(C) = 1!2!\cdots n! \det(A)$.

We now know that in the $m = 1$ case, $\det(C) = 1!2!\cdots n!$. But the matrix C for general m is obtained from the matrix C in the $m = 1$ case simply by multiplying the kth row by m^k. So $\det(C)$ for general m equals $1!2!\cdots n! m^1 m^2 \cdots m^n$. From this we deduce that $\det(A) = m^1 m^2 \cdots m^n = m^{\frac{n^2+n}{2}}$.

4. The power series $\displaystyle\sum_{n=0}^{\infty} \cos(\frac{\pi n}{6})\,\frac{z^n}{n!}$ converges for all z, to $f(z)$ say. Let $\displaystyle\sum_{n=0}^{\infty} a_n\, z^n$ be the power series expansion of $\displaystyle\frac{3}{1+2f(z)}$ about 0. Prove that $a_{3n} = 0$ for all positive integers n.

Solution. Let $\omega = \frac{-1+\sqrt{3}\mathbf{i}}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}\mathbf{i}}{2}$ be the two complex cube roots of 1. Then

$$\exp(\mathbf{i}\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}(\omega^2 - 1),\ \exp(-\mathbf{i}\frac{\pi}{6}) = -\frac{1}{\sqrt{3}}(\omega - 1).$$

Using de Moivre's theorem, we obtain

$$\cos(\frac{\pi n}{6}) = \frac{\exp(\mathbf{i}\frac{\pi n}{6}) + \exp(-\mathbf{i}\frac{\pi n}{6})}{2} = \frac{1}{2}(-\frac{1}{\sqrt{3}})^n[(\omega-1)^n + (\omega^2-1)^n].$$

Substituting this in the definition of $f(z)$, and setting $y = -\frac{z}{\sqrt{3}}$, gives

$$f(z) = \frac{1}{2}[\exp(-(\omega-1)\frac{z}{\sqrt{3}}) + \exp(-(\omega^2-1)\frac{z}{\sqrt{3}})] = \frac{1}{2}[\exp((\omega-1)y) + \exp((\omega^2-1)y)].$$

Hence the function whose power series we are interested in is

$$\begin{aligned}\frac{3}{1+2f(z)} &= \frac{3}{1+\exp((\omega-1)y)+\exp((\omega^2-1)y)} \\ &= \frac{3\exp(y)}{\exp(y)+\exp(\omega y)+\exp(\omega^2 y)} \\ &= 1 + \frac{\exp(y)+\omega\exp(\omega y)+\omega^2\exp(\omega^2 y)}{\exp(y)+\exp(\omega y)+\exp(\omega^2 y)} + \frac{\exp(y)+\omega^2\exp(\omega y)+\omega\exp(\omega^2 y)}{\exp(y)+\exp(\omega y)+\exp(\omega^2 y)},\end{aligned}$$

where the last step uses the fact that $1+\omega+\omega^2 = 0$. Let $g_1(y)$ and $g_2(y)$ be the fractions appearing in the last line. Then

$$g_1(\omega y) = \frac{\exp(\omega y)+\omega\exp(\omega^2 y)+\omega^2\exp(y)}{\exp(\omega y)+\exp(\omega^2 y)+\exp(y)} = \omega^2 g_1(y),$$

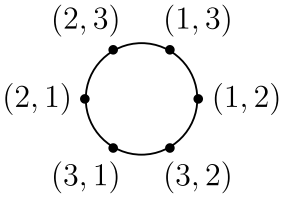
which means that in the power series expansion of g_1 about 0, the only powers y^m which occur with nonzero coefficient are those where $m \equiv 2 \pmod 3$. Similarly, in the power series expansion of g_2 about 0, the only powers y^m which occur with nonzero coefficient are those where $m \equiv 1 \pmod 3$. So in $g_1(y) + g_2(y)$ there are no y^{3n} terms for $n \geq 1$, and correspondingly there are no z^{3n} terms in $\dfrac{3}{1+2f(z)}$.

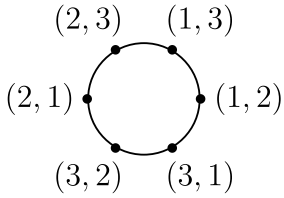
5. 2011 is a prime number. Let $N = 2^{2011} - 1$, a 606-digit number which can be shown to be composite by computer calculations. Using elementary number theory (and maybe a pocket calculator), prove that N has no prime factors less than $80,000$.

Solution. Let p be a prime factor of N. Then $2^{2011} \equiv 1 \pmod p$, and since 2011 is prime, 2011 must be the multiplicative order of 2 modulo p. By Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$, so we conclude that $2011 \mid p-1$, i.e. $p = 1 + 2011k$ for some positive integer k. We must show that $k \geq 40$.

Since $(2^{1006})^2 = 2^{2012} \equiv 2 \pmod p$, 2 is a quadratic residue mod p. It is well known that this forces $p \equiv \pm 1 \pmod 8$. Since $2011 \equiv 3 \pmod 8$, this tells us that $k \equiv 0 \pmod 8$ or $k \equiv 2 \pmod 8$. Also, from the facts that $p \not\equiv 0 \pmod 3$, $p \not\equiv 0 \pmod 5$, $p \not\equiv 0 \pmod 7$, we deduce that $k \not\equiv 2 \pmod 3$, $k \not\equiv 4 \pmod 5$, $k \not\equiv 3 \pmod 7$. The only values of k less than 40 which satisfy all these congruences are $k = 16$ and $k = 18$. But $1 + 2011 \times 16 = 32177$ is a multiple of 23, and $1 + 2011 \times 18 = 36199$ is a multiple of 53. So we must have $k \geq 40$.

6. Let $n \geq 3$ be an integer. Consider the $n(n-1)$ ordered pairs (i, j), where $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$. Show that there is a way to arrange these pairs around a circle, equally spaced, so that for any distinct $i, j, k \in \{1, 2, \cdots, n\}$, the arc from (i, j) to (j, k) which passes through (i, k) is less than half the circumference of the circle. For example, the first of the following pictures for $n = 3$ has this property; the second does not, because (to name one of its failings) the arc from $(1, 3)$ to $(3, 2)$ which passes through $(1, 2)$ is equal to half the circumference.





Solution. We claim that the following arrangement works: going clockwise from an arbitrary starting point, put

$$(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), \cdots, (1, n), (2, n), \cdots, (n-1, n),$$

and then the reversals of all these pairs in the same order, namely

$$(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), \cdots, (n, 1), (n, 2), \cdots, (n, n-1).$$

Notice that the first semicircle we filled contains all the pairs (i, j) with $i < j$, and the second semicircle contains all the pairs (i, j) with $i > j$. Moreover, (i, j) and (j, i) are diametrically opposed for all $i \neq j$.

We must check the condition stated in the question, which we will call $C(i, j, k)$, for any disjoint i, j, k. It is convenient to divide into cases based on the relative order of i, j, k. Because rotating by $180°$ reverses every pair, $C(i, j, k)$ holds if and only if $C(k, j, i)$ holds, so we only need to consider three of the six possible orders.

Case 1: $i < j < k$. Then $(i, j), (i, k), (j, k)$ are all in the first semicircle, and our construction placed them in that clockwise order, with (i, k) between (i, j) and (j, k). So $C(i, j, k)$ holds.

Case 2: $i < k < j$. Then by Case 1, (i, j) lies between (i, k) and (k, j) in the first semicircle. Hence if we follow the semicircular arc from (k, j) to its opposite point (j, k) which passes through (i, j), we will reach (i, k) after reaching (i, j). So the sub-arc from (i, j) to (j, k) contains (i, k) and is less than half the circumference, as required for $C(i, j, k)$ to hold.

Case 3: $j < i < k$. Then by Case 1, (j, k) lies between (j, i) and (i, k) in the first semicircle. Hence if we follow the semicircular arc from (j, i) to its opposite point (i, j) which passes through (j, k), we will reach (i, k) after reaching (j, k). So the sub-arc from (j, k) to (i, j) contains (i, k) and is less than half the circumference, as required for $C(i, j, k)$ to hold.

Notice that in our first semicircle we could have used any order of the pairs which satisfied Case 1 (that is, such that (i, k) is between (i, j) and (j, k) whenever $i < j < k$), as long as we used the same order for the reversed pairs in the second semicircle. There are also arrangements satisfying the conditions $C(i, j, k)$ which do not have the property that (i, j) and (j, i) are diametrically opposed for all $i \neq j$: for example, take the above arrangement for $n = 4$ and swap the adjacent pairs $(1, 2)$ and $(4, 3)$.

7. For a positive integer n, let b_n denote the number of binary strings consisting of n zeroes and n ones which have no three consecutive zeroes and no three consecutive ones. Show that

$$b_n = \sum_{k=0}^{n} \left(\binom{k}{n-k} + \binom{k+1}{n-k-1} \right)^2,$$

where the binomial coefficient $\binom{k}{j}$ is defined to be zero if $j < 0$ or $j > k$.

Solution. For any nonnegative integers m, n, let $a_{m,n}$ be the number of binary strings consisting of m zeroes and n ones which start with a zero and have the property that there are no three consecutive zeroes and no three consecutive ones. If $m = n = 0$, we set $a_{0,0} = 1$, thus declaring that the empty string does "start with a zero". We have $a_{0,n} = 0$ for $n \geq 1$, and $a_{m,0} = 0$ for $m \geq 3$, with $a_{1,0} = a_{2,0} = 1$. For convenience, set $a_{m,n} = 0$ if $m = -1$ or $n = -1$.

From any string of the type counted by $a_{m,n}$ where $m, n \geq 1$, we obtain a smaller one by removing the initial 0 or 00 and the subsequent 1 or 11. Hence we have a recurrence relation

$$a_{m,n} = a_{m-1,n-1} + a_{m-1,n-2} + a_{m-2,n-1} + a_{m-2,n-2}, \text{ for all } m, n \geq 1.$$

We can use this to determine the generating function $A(x, y) = \sum_{m,n\geq 0} a_{m,n}x^m y^n$, a formal power series in the indeterminates x and y. We have

$$\begin{aligned} A(x, y) &= 1 + x + x^2 + \sum_{m,n\geq 1} a_{m,n}x^m y^n \\ &= 1 + x + x^2 + \sum_{m,n\geq 1} (a_{m-1,n-1} + a_{m-1,n-2} + a_{m-2,n-1} + a_{m-2,n-2})x^m y^n \\ &= 1 + x + x^2 + (xy + xy^2 + x^2y + x^2y^2)A(x, y), \end{aligned}$$

which implies that

$$\begin{aligned} A(x, y) &= (1 + x + x^2)(1 - xy(1 + x)(1 + y))^{-1} \\ &= (1 + x + x^2) \sum_{k\geq 0} x^k y^k (1 + x)^k (1 + y)^k. \end{aligned}$$

Extracting the coefficient of $x^m y^n$, we deduce that

$$\begin{aligned} a_{m,n} &= \sum_{k\geq 0} \left(\binom{k}{m-k} + \binom{k}{m-k-1} + \binom{k}{m-k-2} \right) \binom{k}{n-k} \\ &= \sum_{k\geq 0} \left(\binom{k}{m-k} + \binom{k+1}{m-k-1} \right) \binom{k}{n-k}. \end{aligned}$$

Now for a positive integer n, the binary strings counted by b_n divide evenly into those that begin with a zero and those that being with a one, so

$$\begin{aligned} b_n = 2a_{n,n} &= 2 \sum_{k\geq 0} \left(\binom{k}{n-k} + \binom{k+1}{n-k-1} \right) \binom{k}{n-k} \\ &= 2 \sum_{k\geq 0} \binom{k}{n-k}^2 + \sum_{k\geq 0} 2 \binom{k}{n-k} \binom{k+1}{n-k-1} \\ &= \sum_{k\geq 0} \binom{k}{n-k}^2 + \sum_{k\geq 0} \binom{k+1}{n-k-1}^2 + \sum_{k\geq 0} 2 \binom{k}{n-k} \binom{k+1}{n-k-1} \\ &= \sum_{k\geq 0} \left(\binom{k}{n-k} + \binom{k+1}{n-k-1} \right)^2. \end{aligned}$$

It is clear that every term with $k > n$ is zero, so we get the formula in the question.

8. Let $(x_1, y_1), \cdots, (x_n, y_n)$ be n distinct points in the plane with $0 < x_i < 1$ and $0 < y_i < 1$ for all i. Let V be the set of all integer translates of these points, i.e. points of the form (x_i+a, y_i+b) for $1 \leq i \leq n$, $a, b \in \mathbb{Z}$. A *periodic hexagon tessellation* with vertex set V consist of a set of continuous curves in the plane called *edges* such that:

- the endpoints of each edge are distinct elements of V, and edges do not intersect except at their endpoints,
- every element of V is the endpoint of exactly three edges,
- every one of the regions between the edges has exactly six edges on its boundary, and
- whenever C is an edge, every integer translate $C + (a, b)$ for $a, b \in \mathbb{Z}$ is also an edge.

Show that a periodic hexagon tessellation with vertex set V exists if and only if n is even.

Solution. We first assume that we have a periodic hexagon tessellation with vertex set V, and show that n is even. Let E be the set of edges of the tessellation which have both endpoints in the square $S = [0, 1] \times [0, 1]$, and let E' be the set of edges which have exactly one endpoint in S. Since every endpoint in S is one of the n points (x_i, y_i), and each such point is the endpoint of exactly three edges, we have $3n = 2|E| + |E'|$. We have a map τ from E' to itself defined as follows. For $C \in E'$, let (x_i, y_i) be the endpoint of C which lies in S. The other endpoint of C has the form $(x_j + a, y_j + b)$ where $a, b \in \mathbb{Z}$ are not both zero. By periodicity, $C - (a, b)$ is also an edge, with endpoints $(x_i - a, y_i - b)$ and (x_j, y_j), and hence belonging to E'. We set $\tau(C) = C - (a, b)$. Clearly $\tau(C) \neq C$, and $\tau(\tau(C)) = (C - (a, b)) - (-a, -b) = C$. So E' is partitioned into 2-element subsets $\{C, \tau(C)\}$, and $|E'|$ is even. Hence $3n = 2|E| + |E'|$ is even, forcing n to be even.

Conversely, let n be even; we will show there exists a periodic hexagon tessellation with vertex set V. Set $(x'_i, y'_i) = (\frac{2i-1}{2n}, \frac{1}{2})$, and let V' be the set of integer translates of these points. There is a homeomorphism from the square $[0, 1] \times [0, 1]$ to itself which fixes the boundary and sends (x_i, y_i) to (x'_i, y'_i) for all i. Applying such a homeomorphism to every integer translate of the square would transform a periodic hexagon tessellation with vertex set V into one with vertex set V', and applying the inverse homeomorphism would do the reverse. So we may assume that $(x_i, y_i) = (x'_i, y'_i)$.

Assume for a moment that $n = 2$. It is easy to see that in this case we have a periodic hexagon tessellation with vertex set V whose edges are all the integer translates of the following three line segments: that joining $(\frac{1}{4}, \frac{1}{2})$ to $(\frac{3}{4}, \frac{1}{2})$, that joining $(\frac{1}{4}, \frac{1}{2})$ to $(\frac{3}{4}, \frac{3}{2})$, and that joining $(\frac{1}{4}, \frac{1}{2})$ to $(-\frac{1}{4}, -\frac{1}{2})$.

For general even n, we obtain a periodic hexagon tessellation with vertex set V by taking the tessellation constructed for the $n = 2$ case and dividing every x-coordinate by $n/2$. The result is indeed preserved by addition of (a, b) for any $a, b \in \mathbb{Z}$, because the $n = 2$ tessellation is preserved by addition of $(\frac{n}{2}a, b)$.

9. By a *word* in this problem we mean a (possibly empty) string of lowercase letters in the usual alphabet `a`–`z`. If W_1 and W_2 are words then we write W_1W_2 for the concatenation of W_1 and W_2. We say that there is an *elementary transition* between two words W and W' if W has the form $W_1W_2W_3$ and W' equals $W_1W_2W_2W_3$ (in other words, W' is obtained from W by repeating some sub-word), or if W has the form $W_1W_2W_2W_3$ and W' equals $W_1W_2W_3$ (in other words, W' is obtained from W by deleting one copy of a repeated sub-word). We say that two words W and W' are *equivalent* if they are connected by a finite sequence of such elementary transitions. For example, `barbaric` is equivalent to `baariric` because of the following sequence of elementary transitions:

$$\texttt{barbaric} \longleftrightarrow \texttt{baric} \longleftrightarrow \texttt{baaric} \longleftrightarrow \texttt{baariric}$$

Find the number of equivalence classes of words in which every letter of the alphabet appears.

Solution. This problem was first solved in the paper 'On semi-groups in which $x^r = x$' by J. A. Green and D. Rees, *Proc. Camb. Philos. Soc.* **48** (1952).

We need to identify various features of a word which are unchanged by an elementary transition. The first, implicit in the question, is the set of letters which appear in the word, which we will call the *content* of the word. The content is not altered by repeating a sub-word or deleting one copy of a repeated sub-word, so two equivalent words must have the same content. For example, `barbaric` and `baariric` both have content $\{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{i}, \texttt{r}\}$. It is clear that the number of equivalence classes of words with a fixed content C depends only on the cardinality $|C|$; let $f(n)$ be the number of equivalence classes of words with content C where $|C| = n$. The question asks for $f(26)$, but there is nothing special about 26, so we will find a formula for $f(n)$. Our formula will initially be recursive, so we aim to relate $f(n)$ to $f(n-1)$. As the base case we note that $f(0) = 1$ (there is a unique empty word).

In any word W with nonempty content C, there is one of the letters in C which is the last to appear as you read the word W from left to right; we call this letter the *left-last* letter of W, written $\ell(W)$. For example, $\ell(\texttt{falafel}) = \texttt{e}$. A moment's thought reveals that the order in which the letters first appear as you read W from left to right is unchanged by an elementary transition. In particular, two equivalent words must have the same left-last letter.

Define the *left prefix* of W, written $L(W)$, to be the sub-word obtained by starting at the left-hand end of W and reading up to, but not including, the left-last letter. For example, $L(\texttt{falafel}) = \texttt{falaf}$. Note that $L(W)$ is a word with content $C \setminus \{\ell(W)\}$. We claim that the equivalence class of $L(W)$ is unchanged under an elementary transition. We can assume that $W = W_1W_2W_3$ and the elementary transition is to $W' = W_1W_2W_2W_3$. If the left-last letter of W occurs in W_1 or W_2, then $L(W) = L(W')$. If the left-last letter of W does not occur in W_1 or W_2, then $L(W)$ has the form $W_1W_2W_4$ where W_4 is a sub-word of W_3, and $L(W') = W_1W_2W_2W_4$, so $L(W)$ and $L(W')$ are equivalent. It follows that if W and W' are equivalent, then $L(W)$ and $L(W')$ are equivalent.

We can also define the *right-last letter* $r(W)$ of W to be the letter which is last to appear as you read W from right to left, and the *right prefix* $R(W)$ to be the sub-word to the right of this letter. For example, $r(\texttt{falafel}) = \texttt{a}$ and $R(\texttt{falafel}) = \texttt{fel}$. By the same argument as for the left versions, two equivalent words must have the same right-last letter, and equivalent right prefixes.

We can now prove a recursive criterion for when two words are equivalent. Our claim is that if W and W' are words with the same nonempty content C, then W and W' are equivalent if and only if $\ell(W) = \ell(W')$, $r(W) = r(W')$, $L(W)$ and $L(W')$ are equivalent, and $R(W)$ and $R(W')$ are equivalent. We have already shown the "only if" direction, so what remains is the "if" direction: assuming that the four conditions hold, we must show that W is equivalent to W'. The assumption certainly implies that $L(W)\ell(W)r(W)R(W)$ is equivalent to $L(W')\ell(W')r(W')R(W')$, since we can perform the elementary transitions needed to transform $L(W)$ into $L(W')$ while leaving the remainder $\ell(W)r(W)R(W)$ unchanged, and then the elementary transitions needed to transform $R(W)$ into $R(W')$.

So it suffices to show that W is equivalent to $L(W)\ell(W)r(W)R(W)$. Note that W has the form $L(W)\ell(W)W_1$ for some word W_1. Let $a_1, a_2, \cdots, a_s$ be any sequence of letters from C. Since a_1 occurs somewhere in the word $L(W)\ell(W)$, we can repeat a sub-word of W beginning with a_1 and finishing with the first occurrence of $\ell(w)$, to give an elementary transition from W to a word of the form $L(W)\ell(W)a_1W_2$, where W_2 is some word. Since a_2 occurs somewhere in the word $L(W)\ell(W)$, we can repeat a sub-word of $L(W)\ell(W)a_1W_2$ beginning with a_2 and finishing with the letter a_1 succeeding the first occurrence of $\ell(W)$, to give an elementary transition from $L(W)\ell(W)a_1W_2$ to $L(W)\ell(W)a_1a_2W_3$, where W_3 is some word. Then there is an elementary transition from $L(W)\ell(W)a_1a_2W_3$ to $L(W)\ell(W)a_1a_2a_3W_4$, and so on. Hence W is equivalent to a word of the form $L(W)\ell(W)a_1a_2\cdots a_sW_{s+1}$. Since our sequence $a_1, a_2, \cdots, a_s$ was an arbitrary sequence of letters from C, this shows in particular that W is equivalent to $L(W)\ell(W)r(W)R(W)W'$ for some word W'. The same argument, but starting with the word $L(W)\ell(W)r(W)R(W)$ and using the expression of W as $b_s\cdots b_2b_1r(W)R(W)$, shows that $L(W)\ell(W)r(W)R(W)$ is equivalent to $W''W$ for some word W''.

We are now reduced to showing that if two words W and Y are such that W is equivalent to YW' and Y is equivalent to $W''W$ for some words W' and W'', then W and Y are equivalent. This holds by the following chain of equivalences:

$$W \sim YW' \sim YYW' \sim YW \sim W''WW \sim W''W \sim Y.$$

As a consequence of our recursive criterion for equivalence, specifying an equivalence class of words with nonempty content C amounts to choosing two letters $\ell, r \in C$ (possibly equal) to be the left-last and right-last letters, an equivalence class of words with content $C \setminus \{\ell\}$ to be the left prefixes, and an equivalence class of words with content $C \setminus \{r\}$ to be the right prefixes. These choices can be made arbitrarily, since for any $\ell, r \in C$ and any words W_1 with content $C \setminus \{\ell\}$ and W_2 with content $C \setminus \{r\}$, the word $W = W_1\ell rW_2$ has $\ell(W) = \ell$, $r(W) = r$, $L(W) = W_1$, $R(W) = W_2$.

Consequently, we have a recurrence relation $f(n) = n^2 f(n-1)^2$, valid for all $n \geq 1$. Iterating this recursion gives:

$$\begin{aligned} f(n) &= f(n-1)^2 n^2 \\ &= f(n-2)^4(n-1)^4 n^2 \\ &= f(n-3)^8(n-2)^8(n-1)^4 n^2 \\ &\vdots \\ &= 1^{2^n} 2^{2^{n-1}} 3^{2^{n-2}} \cdots (n-2)^{2^3}(n-1)^{2^2} n^{2^1}. \end{aligned}$$

This number grows quite fast: $f(3) = 1^8.2^4.3^2 = 144$, $f(4) = 1^{16}.2^8.3^4.4^2 = 341056$.

10. Let A be a finite set. Suppose we have a real-valued function f on the set of subsets of A with the property that $\displaystyle\sum_{i\in I} f(I \setminus \{i\}) = 0$ for every $I \subseteq A$. Prove that $f(I) = 0$ whenever $|I| < \dfrac{|A|}{2}$.

Solution. (Based on the solution submitted by Oliver Chambers, University of Melbourne.) Let I_1 be a subset of A such that $|I_1| = k$, $|A| = n > 2k$. For any $j \in \{0, 1, \cdots, k\}$, define

$$f_j = \sum_{\substack{I\subset A \\ |I|=k \\ |I\cap I_1|=j}} f(I).$$

We will prove that $f_j = 0$ for all j. Since $f_k = f(I_1)$, this includes the desired result $f(I_1) = 0$.

For any $j \in \{0, 1, \cdots, k\}$, we know that

$$\sum_{\substack{J \subset A \\ |J|=k+1 \\ |J \cap I_1|=j}} \sum_{i \in J} f(J \setminus \{i\}) = 0.$$

Now every term in this left-hand side is $f(I)$ for some k-element subset $I \subset A$ such that either $|I \cap I_1| = j$ or $|I \cap I_1| = j - 1$ (the latter being impossible if $j = 0$). If $|I \cap I_1| = j$, then the term $f(I)$ occurs $n - 2k + j$ times, because J must be obtained by adding to I an element of $A \setminus (I \cup I_1)$ which has cardinality $n - (2k - j)$. If $|I \cap I_1| = j - 1$, then the term $f(I)$ occurs $k - j + 1$ times, because J must be obtained by adding to I an element of $(A \setminus I) \cap I_1$ which has cardinality $k - (j - 1)$. So the equation becomes

$$(n - 2k + j)f_j + (k - j + 1)f_{j-1} = 0,$$

where f_{-1} is defined to be 0. Since $n > 2k$, the coefficient of f_j in this equation is nonzero. So the $j = 0, 1, \cdots, k$ cases of this equation imply successively that $f_0 = 0, f_1 = 0, \cdots, f_k = 0$.