

**INFINITE REDUCED WORDS
AND
THE TITS BOUNDARY OF A COXETER GROUP**

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ABSTRACT. Let (W, S) be a finite rank Coxeter system with W infinite. We prove that the limit weak order on the blocks of infinite reduced words of W is encoded by the topology of the Tits boundary $\partial_T X$ of the Davis complex X of W . We consider many special cases, including W word hyperbolic, and X with isolated flats. We establish that when W is word hyperbolic, the limit weak order is the disjoint union of weak orders of finite Coxeter groups. We also establish, for each boundary point ξ , a natural order-preserving correspondence between infinite reduced words which “point towards” ξ , and elements of the reflection subgroup of W which fixes ξ .

INTRODUCTION

Let (W, S) be a finite rank Coxeter system with W infinite. In this paper we compare the *limit weak order* on the *infinite reduced words* of W with the topology of the Tits boundary $\partial_T X$ of the Davis complex X of W .

An infinite reduced word $\mathbf{i} = i_1 i_2 i_3 \cdots$ is an infinite word such that each initial finite subword $i_1 i_2 \cdots i_k$ is a reduced word for W . In Section 1 we review the notion of a braid limit $\mathbf{i} \rightarrow \mathbf{j}$ of two infinite reduced words, which roughly says that \mathbf{j} can be obtained from \mathbf{i} by an infinite sequence of braid moves. Braid limits give rise to an equivalence relation on infinite reduced words, and a partial order on the equivalence classes, called the limit weak order. This partial order generalises the usual weak order of a Coxeter group [1] to infinite reduced words. Lam and Pylyavskyy [14] studied the limit weak order of affine Weyl groups, where it was introduced in the context of the theory of total positivity of loop groups. When W is the Weyl group of a Kac–Moody group, our present work may have applications to the yet-to-be-developed theory of total positivity of Kac–Moody groups.

The Davis complex X is a proper, complete, CAT(0) metric space on which the Coxeter group W acts properly discontinuously and cocompactly by isometries [8]. The visual boundary ∂X , consisting of equivalence classes of geodesic rays in X , can be equipped with the Tits metric, giving a metric space $\partial_T X$ which we call the Tits boundary of X (see [2]).

There is a bijection between reflections $r \in W$ and *walls* $M(r) \subset X$. The (possibly empty) boundary ∂M of a wall is a subset of the boundary $\partial_T X$. These walls induce an arrangement (with some similarities to a hyperplane arrangement) in $\partial_T X$ and we define an equivalence relation on $\partial_T X$ essentially via the facial structure of this arrangement. For $\xi \in \partial_T X$, we let $C(\xi)$ denote the equivalence class of ξ .

Each infinite reduced word \mathbf{i} naturally gives rise to a path $\gamma : [0, \infty) \rightarrow X$, starting in the identity chamber of X . The walls crossed by γ are the inversions $\text{Inv}(\mathbf{i})$ of \mathbf{i} . To each infinite reduced word \mathbf{i} we associate a subset $\partial_T X(\mathbf{i}) \subset \partial_T X$ consisting of points $\xi \in \partial_T X$ such that there is a geodesic

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ray $[p, \xi) \subset X$ pointing in the direction of ξ with the property that $\text{Inv}(\mathbf{i})$ is the disjoint union of the walls crossed by $[p, \xi)$, and the walls separating the identity from p .

We say that two infinite reduced words \mathbf{i} and \mathbf{j} are *in the same block* if $\text{Inv}(\mathbf{i})$ and $\text{Inv}(\mathbf{j})$ differ by finitely many walls. The limit weak order descends to a partial order on blocks.

Our main theorem is the following:

Theorem 1. *The subsets $\partial_T X(\mathbf{i}) \subset \partial_T X$ have the following properties:*

- (1) *For each \mathbf{i} and \mathbf{j} we have $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$ or $\partial_T X(\mathbf{i}) \cap \partial_T X(\mathbf{j}) = \emptyset$. We have $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$ if and only if \mathbf{i} and \mathbf{j} are in the same block.*
- (2) *Each $\partial_T X(\mathbf{i})$ is an equivalence class $C(\xi)$, and each equivalence class $C(\xi)$ is of the form $\partial_T X(\mathbf{i})$ for some infinite reduced word \mathbf{i} . Thus the $\partial_T X(\mathbf{i})$ form a partition of $\partial_T X$.*
- (3) *Each $\partial_T X(\mathbf{i})$ is a path-connected, totally geodesic subset of $\partial_T X$.*
- (4) *The closure of $\partial_T X(\mathbf{i})$ in $\partial_T X$ is the following union:*

$$\overline{\partial_T X(\mathbf{i})} = \bigcup_{\mathbf{j} \leq \mathbf{i}} \partial_T X(\mathbf{j})$$

where \leq denotes the limit weak order.

Thus the topology of the Tits boundary encodes the limit weak order on the blocks of infinite reduced words. Theorem 1 is established in Propositions 18 and 21 and Theorem 23.

A natural direction to explore would be the question: *to what extent does the limit weak order of blocks (or the limit weak order itself) determine the homotopy or homeomorphism type of the Tits boundary?*

In Sections 1 and 2 we review background on infinite reduced words, the Davis complex and the Tits boundary. Section 3 studies the boundary ∂M of walls, and shows that the arrangement induced in $\partial_T X$ has properties similar to a hyperplane arrangement. An important technical tool here is the Parallel Wall Theorem of Brink and Howlett [3]. The partition of $\partial_T X$ by $C(\xi)$ is introduced in Section 4, where Theorem 1 is established.

In Section 5 we associate to each $\xi \in \partial_T X$ a reflection subgroup $W(\xi)$ whose reflections are walls M such that $\xi \in \partial M$. We then apply results of Deodhar [10] and Dyer [11] to establish in Proposition 40 that the set of equivalence classes of infinite reduced words \mathbf{i} such that $C(\xi) = \partial_T X(\mathbf{i})$ is in bijection with the set of elements of $W(\xi)$, and the partial order on this set of equivalence classes of infinite reduced words corresponds naturally to the weak order on $W(\xi)$. We also establish results connecting the geometry of X with the Coxeter group $W(\xi)$.

Section 6 discusses many special cases and examples. We briefly describe some of them here.

Suppose that (W, S) is an affine Coxeter group with corresponding finite Weyl group W_{fin} . Then $\partial_T X$ is isometric to a sphere with dimension equal to the rank of W , and the partition of $\partial_T X$ by the $C(\xi)$ or $\partial_T X(\mathbf{i})$ is essentially the (spherical) braid arrangement of W_{fin} . Furthermore, for each piece $C = C(\xi)$ the set $\{\mathbf{i} \mid \partial_T X(\mathbf{i}) = C\}$ of equivalence classes of infinite reduced words is either a singleton, or can be identified with a possibly reducible affine Coxeter group of lower rank. These statements follow from the works of Cellini–Papi [7], Ito [13] and Lam–Pylyavskyy [14].

Another interesting case is when W is word hyperbolic. Such Coxeter groups were characterised by Moussong (see [8]). In this case Theorem 1 is particularly elegant:

Theorem 2. *Let W be word hyperbolic. Then*

- (1) *For any infinite reduced word \mathbf{i} we have that $\partial_T X(\mathbf{i})$ consists of a single point $\xi = \xi(\mathbf{i})$.*
- (2) *Two infinite reduced words \mathbf{i} and \mathbf{j} are comparable only if \mathbf{i} and \mathbf{j} are in the same block.*
- (3) *The limit weak order restricted to a block $B(\mathbf{i})$ is isomorphic to the weak order of a (possibly trivial) finite Coxeter group $W(\xi(\mathbf{i}))$.*

Thus the limit weak order of a word hyperbolic W is the disjoint union of weak orders of finite Coxeter groups. We also show that W being word hyperbolic is equivalent to $W(\xi)$ being finite (possibly trivial) for every $\xi \in \partial_T X$. From the point of view of braid relations and infinite reduced words, these results seem far from obvious, as we illustrate in Example 1 below.

Example 1. Consider the Coxeter group

$$W = \langle s_1, s_2, s_3, s_4, s_5 \mid s_i^2 = (s_i s_{i+1})^2 = 1 \rangle.$$

The generators s_i may be viewed as reflections in the sides of a regular right-angled hyperbolic pentagon and the Davis complex X is the induced tessellation of the hyperbolic plane by such pentagons, depicted in Figure 1. In this case, the Tits boundary $\partial_T X$ is the boundary of the hyperbolic plane, that is, a circle, with the distance between any two points in $\partial_T X$ being infinite. Thus in particular, $\partial_T X$ has the discrete topology. Each wall is a hyperbolic geodesic connecting two distinct boundary points. The boundaries of walls are dense in $\partial_T X$; see [6, Example 2.7]. Furthermore, each point of $\partial_T X$ is in the boundary of at most one wall. For otherwise, we would have two disjoint walls that become arbitrarily close, which is impossible since the geometry of a chamber in X is fixed.

Suppose \mathbf{i} is an infinite reduced word with $\xi = \xi(\mathbf{i})$. If ξ is in the boundary of the wall $M = M(r)$ then the block $B(\mathbf{i})$ consists of two equivalence classes of infinite reduced words, corresponding to the two elements of the finite Coxeter group $W(\xi(\mathbf{i})) = \langle r \rangle$. One of these equivalence classes consists of infinite reduced words which do not cross M but which “follow M out to ξ ”, and this equivalence class corresponds to the trivial element of $W(\xi(\mathbf{i}))$. The other equivalence class in the block $B(\mathbf{i})$ consists of infinite reduced words which cross M and then immediately “follow M out to ξ ”. If ξ is not in the boundary of any wall, then $B(\mathbf{i})$ contains a single equivalence class of infinite reduced words and $W(\xi(\mathbf{i}))$ is trivial.

For example we have a braid limit

$$\mathbf{i} = 125252525 \cdots \rightarrow 25252525 \cdots = \mathbf{i}'$$

of two inequivalent infinite reduced words \mathbf{i} and \mathbf{i}' such that $\xi = \xi(\mathbf{i}) = \xi(\mathbf{i}')$ lies on the boundary of the wall $M = M(s_1)$ (see Section 1 for notation, and Figure 1). Thus $B(\mathbf{i}) = B(\mathbf{i}') = \{\mathbf{i}, \mathbf{i}'\}$ contains exactly two equivalence classes. On the other hand the infinite reduced word

$$\mathbf{j} = 1352413524 \cdots$$

is such that $\xi(\mathbf{j})$ is not in the boundary of any wall. To see this it is enough to check that there does not exist an infinite reduced word \mathbf{j}' , not braid equivalent to \mathbf{j} , satisfying either $\mathbf{j} \rightarrow \mathbf{j}'$ or $\mathbf{j}' \rightarrow \mathbf{j}$. The fact that $\mathbf{j} \rightarrow \mathbf{j}'$ is impossible is easy: each initial finite subword of \mathbf{j} is the *unique* reduced word of the corresponding element of W . So suppose $\mathbf{j}' = j'_1 j'_2 \cdots \rightarrow \mathbf{j}$. Then the letter 1 must occur in \mathbf{j}' in such a way that it can be moved to the front via the Coxeter relations of W . Thus \mathbf{j}' starts like one of $1 \cdots$, $21 \cdots$ or $51 \cdots$, or starts with a string of alternating 2s and 5s followed by a 1. The latter three cases are impossible, since s_5 and s_2 do not commute, and s_5 and s_3 do not commute. Thus \mathbf{j}' starts with $1 \cdots$. Continuing this argument we see that in fact $\mathbf{j}' = \mathbf{j} = 1352413524 \cdots$.

More generally, one can deduce the following result: an infinite reduced word $\mathbf{i} = i_1 i_2 \cdots$ satisfies $\xi(\mathbf{i}) \in \partial M$ for some wall M if and only if there exists $N > 0$ such that $i_N i_{N+1} i_{N+2} i_{N+3} \cdots = a(a+2)a(a+2) \cdots$ for some $a \in \mathbb{Z}/5\mathbb{Z}$.

Another case where the topology of the Tits boundary $\partial_T X$ is known reasonably explicitly is the case where X has *isolated flats* (this definition is reviewed in Section 6). Hruska and Kleiner [12] have shown that the Tits boundary $\partial_T X$ of such spaces is a disjoint union of isolated points and standard Euclidean spheres. Caprace [4] has classified Coxeter groups W where X has isolated flats. Following Caprace’s work, we show that the partition of each Euclidean sphere in $\partial_T X$ into equivalence classes $C(\xi)$ is the arrangement induced by an affine Coxeter group.

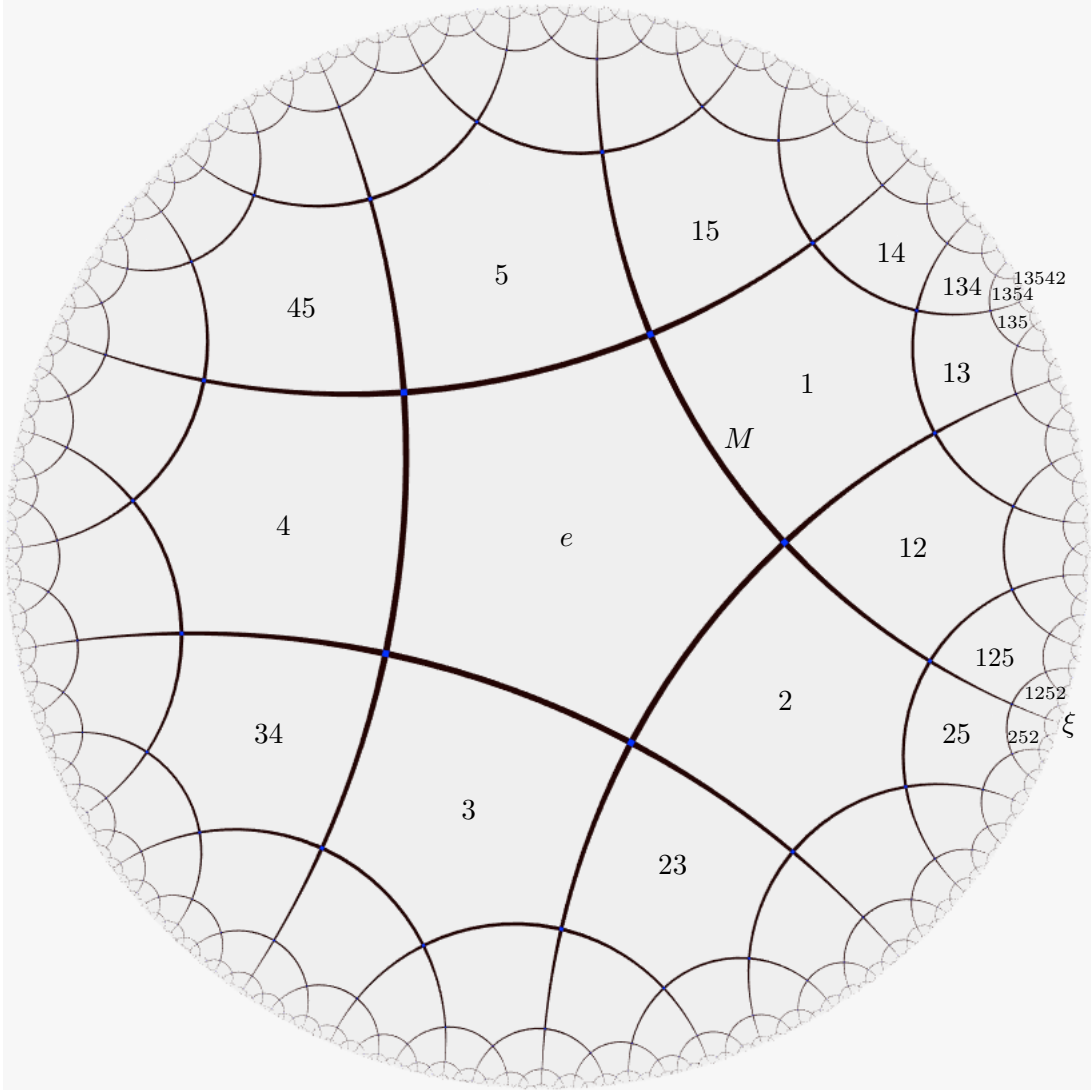


FIGURE 1. The Davis complex for Example 1

These examples suggest the following questions: *does the Tits boundary of a Coxeter group always have the homotopy type of a disjoint union of a wedge of spheres? When is the limit weak order (or limit weak order of blocks) a disjoint union of posets P that are graded, or Eulerian, or Cohen–Macaulay, or shellable?* See for example [1] for a discussion of these poset properties in the Coxeter setting.

Throughout this paper, we make use of results about CAT(0) spaces which may be found in [2].

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1. INFINITE REDUCED WORDS AND THE LIMIT WEAK ORDER

Let (W, S) be a Coxeter system of finite rank, where $S = \{s_i \mid i \in I\}$. Thus S is finite and W has presentation

$$W = \langle \{s_i\}_{i \in I} \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ for all $i \in I$, and for all distinct $i, j \in I$, $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$. We shall assume that W is an infinite group.

An *infinite reduced word* in W is a sequence

$$\mathbf{i} = i_1 i_2 i_3 \cdots$$

where each $i_k \in I$, and for each $n \geq 1$, the finite initial subword $i_1 i_2 \cdots i_n$ is a reduced word, or equivalently, the product $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression.

Let Φ^+ denote the set of positive roots of W . For each infinite reduced word $\mathbf{i} = i_1 i_2 i_3 \cdots$, we define an inversion set $\text{Inv}(\mathbf{i}) \subset \Phi^+$ by

$$\text{Inv}(\mathbf{i}) = \{s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k} \mid k = 1, 2, \dots\}$$

where α_{i_k} denotes the simple root indexed by i_k . The fact that all the roots in $\text{Inv}(\mathbf{i})$ are positive is a consequence of the reducedness condition.

There are bijections between the set Φ^+ of positive roots, the set R of reflections r of W and the set \mathcal{M} of walls $M = M(r)$ in the Davis complex of Section 2. We shall often identify $\text{Inv}(\mathbf{i})$ with a set of reflections, or a set of walls, without further comment. Thus $\text{Inv}(\mathbf{i})$ also denotes the set of walls crossed by \mathbf{i} in the Davis complex.

Let \mathbf{j} and \mathbf{i} be infinite reduced words for W . We shall say that \mathbf{j} is a *braid limit* of \mathbf{i} if it can be obtained from \mathbf{i} by a possibly infinite sequence of braid moves. More precisely, we require that one has $\mathbf{i} = \mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \dots$ such that $\lim_{k \rightarrow \infty} \mathbf{j}_k = \mathbf{j}$ and for each k , there exists ℓ such that the finite initial subword $(\mathbf{j}_k)_1 (\mathbf{j}_k)_2 (\mathbf{j}_k)_3 \cdots (\mathbf{j}_k)_\ell$ of \mathbf{j}_k is a reduced word for the same element of W as the finite initial subword $(\mathbf{j}_{k+1})_1 (\mathbf{j}_{k+1})_2 (\mathbf{j}_{k+1})_3 \cdots (\mathbf{j}_{k+1})_\ell$ of \mathbf{j}_{k+1} , and we have $(\mathbf{j}_k)_r = (\mathbf{j}_{k+1})_r$ for $r > \ell$. Here, the limit $\lim_{k \rightarrow \infty} \mathbf{j}_k = \mathbf{j}$ of words is taken coordinate-wise: $j_r = \lim_{k \rightarrow \infty} (\mathbf{j}_k)_r$. We write $\mathbf{i} \rightarrow \mathbf{j}$ to mean that there is a braid limit from \mathbf{i} to \mathbf{j} . We say that \mathbf{i} and \mathbf{j} are (limit) *braid equivalent* if $\mathbf{i} \rightarrow \mathbf{j}$ and $\mathbf{j} \rightarrow \mathbf{i}$.

The following result is established in [14, Section 4]. See also [13].

Lemma 3. *We have $\text{Inv}(\mathbf{j}) \subset \text{Inv}(\mathbf{i})$ if and only if there is a braid limit from \mathbf{i} to \mathbf{j} . Two infinite reduced words \mathbf{i} and \mathbf{j} are braid equivalent if and only if $\text{Inv}(\mathbf{i}) = \text{Inv}(\mathbf{j})$.*

Define a partial preorder on infinite reduced words by $\mathbf{i} \leq \mathbf{j}$ if and only if $\mathbf{j} \rightarrow \mathbf{i}$. This preorder descends to a partial order on braid equivalence classes of infinite reduced words, called the *limit weak order* in [14]. We let (\mathcal{W}, \leq) denote the set of equivalence classes of infinite reduced words in W , equipped with the limit weak order.

We say that two infinite reduced words \mathbf{i} and \mathbf{j} are *in the same block* if $\text{Inv}(\mathbf{i})$ and $\text{Inv}(\mathbf{j})$ differ by finitely many roots. We write $B(\mathbf{i}) \subset \mathcal{W}$ to denote the set of equivalence classes of infinite reduced words in the same block as \mathbf{i} . The partial order \leq on \mathcal{W} induces a partial order on blocks: $B(\mathbf{i}) \leq B(\mathbf{j})$ if there exists $\mathbf{i}' \in B(\mathbf{i})$ and $\mathbf{j}' \in B(\mathbf{j})$ so that $\mathbf{i}' \leq \mathbf{j}'$.

2. THE TITS BOUNDARY OF THE DAVIS COMPLEX

This section contains brief background on notions from geometric group theory, from the references [2] and [8].

2.1. Geodesics and the CAT(0) condition. We briefly review the CAT(0) condition, referring the reader to [2] for details.

Let (X, d) be a metric space. A geodesic path from $x \in X$ to $y \in X$ is a map $\gamma : [0, l] \rightarrow X$ such that $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Similarly we define geodesic rays $\gamma : [0, \infty) \rightarrow X$ and geodesic lines $\gamma : (-\infty, \infty) \rightarrow X$. The metric space X is called *geodesic* if every pair of points $x, y \in X$ is connected by a geodesic path.

Let X be a metric space. Let Δ be a geodesic triangle in X , and $\bar{\Delta}$ be a Euclidean comparison triangle: a triangle in the Euclidean plane with the same side lengths as Δ . There is then a bijection $x \mapsto \bar{x}$ between points of Δ and points of $\bar{\Delta}$. We say that Δ is CAT(0) if for any $x, y \in \Delta$ we have $d(x, y) \leq d(\bar{x}, \bar{y})$. We say that X is CAT(0) if X is a geodesic metric space all of whose geodesic triangles are CAT(0). Similarly one defines CAT(1) spaces using comparison triangles on the unit sphere in \mathbb{R}^3 .

Let $\gamma : [0, a] \rightarrow X$ and $\gamma' : [0, a'] \rightarrow X$ be two geodesic paths with the same start point $\gamma(0) = p = \gamma'(0)$. The (*Alexandrov*) angle between γ and γ' is defined as

$$\angle_p(\gamma, \gamma') := \limsup_{t, t' \rightarrow 0} \bar{Z}_p(\gamma(t), \gamma(t'))$$

where $\bar{Z}_p(\gamma(t), \gamma(t'))$ denotes the angle at p of a Euclidean triangle which has side lengths equal to the pairwise distances between $\{p, \gamma(t), \gamma(t')\}$.

2.2. The Davis complex. Let (W, S) be a Coxeter system. In this section we briefly recall the construction and relevant properties of the Davis complex X for (W, S) , following [8]. This includes a discussion of the ends of W and the geodesic extension property for X .

For each $T \subset S$, let W_T be the *special subgroup* of W generated by the elements of T . By convention, W_\emptyset is the trivial group. We say $T \subset S$ is *spherical* if W_T is finite. Denote by \mathcal{S} the set of all spherical subsets of S , partially ordered by inclusion. The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex, denoted L and called the *nerve* of (W, S) . Thus the vertex set of L is S , and a nonempty set T of vertices spans a simplex σ_T in L if and only if T is spherical.

We denote by K the geometric realisation of the poset \mathcal{S} . Equivalently, K is the cone on the barycentric subdivision of the nerve L . Note that K is compact, since it is the cone on a finite simplicial complex. We call the cone point of K its *centre*. For each $s \in S$ let K_s be the union of the (closed) simplices in K which contain the vertex s but do not contain the centre. In other words, K_s is the closed star of the vertex s in the barycentric subdivision of L . Note that K_s has nonempty intersection with K_t if and only if s and t generate a finite subgroup of W . For each $x \in K$, let

$$S(x) := \{s \in S \mid x \in K_s\}.$$

Now define an equivalence relation \sim on the set $W \times K$ by $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$. The *Davis complex* $X = X(W, S)$ for (W, S) is then the quotient space

$$X := (W \times K) / \sim.$$

The natural W -action on $W \times K$ descends to an action on X .

We identify K with the subcomplex (e, K) of X , where e is the identity element of W . Then K , as well as any of its translates by an element of W , is called a *chamber* of X . It is immediate that each chamber is compact, the set of chambers is in bijection with the set of elements of W and W acts transitively on the set of chambers. We denote by 0 the centre of the chamber $K = (e, K)$ in X .

If (W, S) is irreducible affine then the Davis complex $X = X(W, S)$ is just the barycentric subdivision of the Coxeter complex for (W, S) .

The Davis complex X may be equipped with a piecewise Euclidean metric so that it is a proper, complete CAT(0) metric space. Then in particular, given any two points $x, y \in X$, there is a

unique geodesic segment from x to y , denoted $[x, y]$. By construction, the Coxeter group W acts properly discontinuously and cocompactly by isometries on X . Hence W with its word metric is quasi-isometric to X .

In the special case that W is generated by the set of reflections in codimension one faces of a compact convex hyperbolic polytope P , the Davis complex X may instead be equipped with a piecewise hyperbolic metric, so that X is isometric to the induced tessellation of hyperbolic space by copies of P . This natural metrisation of X is used in Example 1 above. Although we work with a piecewise Euclidean metric on X below, our results hold for this piecewise hyperbolic metric as well.

Recall that for a finitely generated group G , the *ends* of G counts the number of path components of the Cayley graph of G as larger and larger finite subgraphs are removed. By a result of Hopf, a finitely generated group G has 0, 1, 2 or infinitely many ends. We have the following result [8, Section 8].

Theorem 4. *Let W be a Coxeter group.*

- (1) W has 0 ends if and only if W is spherical (that is, finite).
- (2) W is one-ended if and only if, for each $T \in \mathcal{S}$, the punctured nerve $L - \sigma_T$ is connected.
- (3) W is two-ended if and only if (W, \mathcal{S}) decomposes as the direct product $(W, \mathcal{S}) = (W_0 \times W_1, \mathcal{S}_0 \cup \mathcal{S}_1)$ where W_1 is finite and W_0 is the infinite dihedral group.
- (4) W has infinitely many ends if and only if it is infinite, not as in (3) and there is at least one $T \in \mathcal{S}$ so that the punctured nerve $L - \sigma_T$ is disconnected.

Since X is a complete CAT(0) metric space, by Lemma II.5.8(2) of [2] we may define X to have the *geodesic extension property* if every geodesic path $\gamma : [a, b] \rightarrow X$ (with $a < b$) can be extended to a geodesic line $(-\infty, \infty) \rightarrow X$.

Proposition 5. *If W is one-ended then X has the geodesic extension property.*

Proof. By [2, Proposition II.5.10], it suffices to show that X has no free faces. Let K be the standard chamber of X and L the nerve of W . Then by construction of the Davis complex, X has no free faces if and only if every free face of K is contained in a mirror of K . Since K is the cone on the barycentric subdivision of L , there is a free face of K not contained in a mirror if and only if there is an $s \in \mathcal{S}$ such that the vertex $\{s\}$ is contained in a unique maximal simplex of L of dimension ≥ 1 . (Such a free face will contain the edge between $\{s\}$ and the cone point of K .)

Now suppose $s \in \mathcal{S}$ is contained in a unique maximal simplex. Then the link of s in L is this maximal simplex with s removed, which is itself a simplex $\sigma_T \in L$. Thus L is either a simplex (in which case W has 0 ends), or $L - \sigma_T$ is disconnected. Then by Theorem 4, W cannot be one-ended. \square

2.3. Visual boundary and Tits boundary of the Davis complex. We briefly recall definitions and key results concerning these boundaries. See [2] for further details. We will use some other boundaries of X in Section 4 below.

We denote by ∂X the visual boundary of the Davis complex X . That is, ∂X is the set of equivalence classes of geodesic rays in X , where two rays $c, c' : [0, \infty) \rightarrow X$ are defined to be *equivalent* if there is a $0 < K < \infty$ so that for all $t \geq 0$, $d(c(t), c'(t)) < K$. We say that a point $\xi \in \partial X$ is *represented* by a geodesic ray c if c is in the equivalence class ξ , and we may then write $c(\infty) = \xi$. Since X is complete and CAT(0), for any $\xi \in \partial X$ and $p \in X$, there exists a unique geodesic ray from p to ξ , denoted $[p, \xi]$.

We denote by $\partial_T X$ the Tits boundary of the Davis complex X , that is, the visual boundary ∂X equipped with the length metric d_T induced by the angular metric \angle . By definition, the distance between points $\xi, \xi' \in \partial X$ in the angular metric is $\angle(\xi, \xi') = \sup_{p \in X} \angle_p(\xi, \xi')$, where $\angle_p(\xi, \xi')$ is the (Alexandrov) angle at p between the geodesic rays $[p, \xi]$ and $[p, \xi']$.

Lemma 6. *Let ξ and ξ' be distinct points in $\partial_T X$. Then at least one of the following holds:*

- (1) ξ and ξ' are connected by a geodesic arc in $\partial_T X$;
- (2) ξ and ξ' are connected by a geodesic in X .

Moreover, if ξ and ξ' are not connected by a geodesic in X and $d_T(\xi, \xi') < \pi$, then there is a unique geodesic arc in $\partial_T X$ which connects ξ and ξ' .

We note that it is possible for both (1) and (2) in Lemma 6 to occur. For example if W is irreducible affine of rank n , then $\partial_T X$ is the sphere S^{n-1} with its usual metric, and every pair of antipodal points in $\partial_T X$ will be connected by infinitely many geodesic arcs of length π in $\partial_T X$ as well as by infinitely many (parallel) geodesics in X .

Proof of Lemma 6. Since X is a proper CAT(0) space, Proposition II.9.21(1) and (2) of [2] say that if ξ and ξ' are not connected by a geodesic in X , they are connected by a geodesic arc in the boundary $\partial_T X$, of length $d_T(\xi, \xi') = \angle(\xi, \xi') \leq \pi$. By Theorem 9.20 of [2], since X is a complete CAT(0) space the boundary $\partial_T X$ is a complete CAT(1) space, so $d_T(\xi, \xi') < \pi$ implies that this geodesic arc is unique. \square

3. WALLS

In this section we investigate walls M and their boundaries ∂M . We show that the arrangement induced in $\partial_T X$ has properties similar to a hyperplane arrangement and define the equivalence class $C(\xi) \subset \partial_T X$ for $\xi \in \partial_T X$.

We first discuss walls in X . A *wall* in X is by definition the fixed set of a reflection $r \in W$. Each wall $M = \text{Fix}(r)$ is a closed, convex subcomplex of X , which determines two closed half-spaces M^+ and M^- of X such that $M = M^+ \cap M^-$ and the reflection r interchanges M^+ and M^- . The half-space M^+ is by definition that containing the point 0. The collection of walls in X is locally finite, and the maximum number of walls intersecting at any point of X is bounded by the maximum size of a subset $T \subset S$ for which W_T is finite.

We will say that two points $x, y \in X$ are on *opposite sides* of a wall M if $x \in M^+ \setminus M$ and $y \in M^- \setminus M$, or vice versa. We then say that a wall M *separates* two points $x, y \in X$ if x and y are on opposite sides of M , so in particular neither x nor y is in M . By the local structure of walls, if M separates x and y then the geodesic segment $[x, y]$ intersects M in a single point. Given $x, y \in X$, we denote by $\mathcal{M}(x, y)$ the set of walls which separate x and y .

Lemma 7. *For each $L > 0$ there is a constant $f(L)$ so that a geodesic segment of length L intersects at most $f(L)$ walls.*

Proof. A sufficiently small ball in X intersects at most as many walls as the maximum number incident at a point. A geodesic segment of length L is covered by a fixed number (depending on L) of these sufficiently small balls, which completes the proof. \square

The *angle* between a geodesic c and a wall M that it meets is by definition the infimum of the angles between c and geodesics contained in M . The following result will be used in Section 4.

Lemma 8. *There exist constants $L > 0$ and $\epsilon > 0$ such that any geodesic segment of length L must intersect some wall at angle greater than ϵ .*

Proof. Suppose otherwise, that is, suppose that for all $L > 0$ and $\epsilon > 0$ there is a geodesic segment of length L such that every wall it intersects is met at angle $\leq \epsilon/2$. By Lemma 7, this geodesic segment intersects at most $f(L)$ walls. By reflecting this geodesic segment, we may consider the images of its subsegments inside the fundamental chamber. Let this “broken geodesic” have initial point a_1 and final point a_n and let a_2, \dots, a_{n-1} be the points at which the path turns. In other words the maximal geodesic subsegments inside the fundamental chamber are $[a_k, a_{k+1}]$ for $1 \leq k < n$. Note

that $n \leq f(L)$, $\sum_{k=1}^{n-1} d(a_k, a_{k+1}) = L$, and for $1 \leq k < n$ the angles between the geodesic segments $[a_k, a_{k+1}]$ and the walls containing the points a_k and a_{k+1} are both $\leq \epsilon/2$. By [2, I.1.13(2)] it follows that for $1 \leq k \leq n-2$ the angle between $[a_k, a_{k+1}]$ and $[a_{k+1}, a_{k+2}]$ is greater than $\pi - \epsilon$. To simplify notation we will denote by $\angle(a_i a_j, a_j a_k)$ the angle between the geodesic segments $[a_i, a_j]$ and $[a_j, a_k]$, for $1 \leq i, j, k \leq n$.

We claim that for every $L > 0$, there is an $\epsilon = \epsilon(L)$ such that $d(a_0, a_n) > L/\sqrt{2}$. This suffices to complete the proof, since for L large enough there are no points inside the fundamental chamber at distance $L/\sqrt{2}$ apart.

We first show that for $1 \leq k < n$, the angle between $[a_0, a_k]$ and $[a_k, a_{k+1}]$ is greater than $\pi - k\epsilon$. Suppose by induction that $\angle(a_0 a_k, a_k a_{k+1}) > \pi - k\epsilon$. In a CAT(0) space the angle sum of a triangle is at most π , so it follows that $\angle(a_0 a_{k+1}, a_{k+1} a_k) \leq k\epsilon$. Let a'_0 be a point such that the geodesic segment $[a_0, a'_0]$ contains a_{k+1} in its interior and let a'_k be a point such that the geodesic segment $[a_k, a'_k]$ contains a_{k+1} in its interior. Since $\angle(a_k a_{k+1}, a_{k+1} a_{k+2}) > \pi - \epsilon$, we have $\angle(a'_k a_{k+1}, a_{k+1} a_{k+2}) \leq \epsilon$, and since $\angle(a_0 a_{k+1}, a_{k+1} a_k) \leq k\epsilon$ we also have $\angle(a'_k a_{k+1}, a_{k+1} a'_0) \leq k\epsilon$. By [2, Proposition I.1.14] it follows that $\angle(a'_0 a_{k+1}, a_{k+1} a_{k+2})$ is less than or equal to $(k+1)\epsilon$, so $\angle(a_0 a_{k+1}, a_{k+1} a_{k+2}) > \pi - (k+1)\epsilon$ as required.

In particular, $\angle(a_0 a_k, a_k a_{k+1}) > \pi - n\epsilon$ for each k . Now using [2, Proposition II.1.7(5)], we get that for every k we have

$$(1) \quad d(a_0, a_{k+1}) > C(n\epsilon) (d(a_0, a_k) + d(a_k, a_{k+1}))$$

for some constant C depending only on $n\epsilon$. The function $C = C(\delta)$ has the property that $C \rightarrow 1$ as $\delta \rightarrow 0$. In fact, one can pick $C(\delta) = \sqrt{1 - (1 - \cos(\pi - \delta))/4}$. Repeatedly using (1), we obtain

$$d(a_0, a_n) > C(n\epsilon)^n \sum_{k=0}^{n-1} d(a_k, a_{k+1}) = C(n\epsilon)^n L.$$

Now pick $\epsilon = \epsilon(L) > 0$ so that $1 > C(f(L)\epsilon) > (1/\sqrt{2})^{1/f(L)}$. Then with this ϵ , every geodesic segment of length L must intersect some wall at angle greater than ϵ . \square

We now consider walls and the boundary. We define \overline{M} to be the closure of M in $\overline{X} = X \cup \partial X$, and define $\partial M = \overline{M} \cap \partial X = \overline{M} \setminus M$. Similarly we define \overline{M}^+ , ∂M^+ and \overline{M}^- , ∂M^- .

Lemma 9. *Let M be a wall in X . Then ∂M and ∂M^\pm are closed in $\partial_T X$ and $\partial M^\pm \setminus \partial M$ is open in $\partial_T X$.*

Proof. By [2, Proposition 9.7(1)], it is enough to establish the claims for ∂X . But then the claims follow from the definitions. \square

We say that two points $\xi, \xi' \in \partial X$ are on *opposite sides* of ∂M if $\xi \in \partial M^+ \setminus \partial M$ and $\xi' \in \partial M^- \setminus \partial M$, or vice versa.

Lemma 10. *Let M be a wall.*

- (1) *The intersection of ∂M^+ and ∂M^- is ∂M .*
- (2) *Suppose $\xi, \xi' \in \partial X \setminus \partial M$ are on opposite sides of ∂M . If we are in the situation of Lemma 6(1), then the geodesic arc in $\partial_T X$ crosses ∂M . If we are in the situation of Lemma 6(2), then the geodesic in X crosses M .*

Proof. For (1), the inclusion $\partial M \subset \partial M^+ \cap \partial M^-$ is clear. Now suppose $\xi \in \partial M^+ \cap \partial M^-$. Since M^+ and M^- are totally geodesic sets, ξ can be represented by geodesic rays c, c' where c (respectively c') stays completely inside M^+ (respectively M^-). By definition we have $d(c(t), c'(t)) < K < \infty$ so in particular both c and c' are a bounded distance from M . It follows that $\xi \in \partial M$.

For (2), in the case of Lemma 6(1), let $\gamma \subset \partial_T X$ be the geodesic arc. Then $\gamma \cap \partial M^+$ and $\gamma \cap \partial M^-$ are both closed subsets of γ , and must thus intersect. Hence $\gamma \cap \partial M^- \cap \partial M^+ \neq \emptyset$. In the case

of Lemma 6(2), if the geodesic $\gamma \subset X$ does not intersect M , then it must stay completely inside either M^+ or M^- . But then ξ, ξ' will both lie in ∂M^+ or ∂M^- , contradicting the assumption. \square

Thus for each wall M , its boundary ∂M determines two closed half-spaces ∂M^+ and ∂M^- of ∂X , such that $\partial M = \partial M^+ \cap \partial M^-$. If $\xi, \xi' \in \partial X \setminus \partial M$ are on opposite sides of ∂M , then using (2) of Lemma 10, we will abuse terminology and say that ξ and ξ' are on *opposite sides of M* and are *separated by M* . Two points $\xi, \xi' \in \partial X \setminus \partial M$ are defined to be on the *same side of M* if they are not on opposite sides.

We next consider geodesic rays from various points in the Davis complex to various points in its boundary.

Lemma 11. *Let M be a wall.*

- (1) *The geodesic ray from any $p \in M$ to $\xi \in \partial M$ must be contained completely in M .*
- (2) *Let c be a geodesic ray which intersects M but is not contained in M . Then $c(\infty)$ is not in ∂M .*
- (3) *Let $\xi \in \partial M$ and $p \in X \setminus M$. Then the geodesic ray $[p, \xi)$ never intersects M .*

Proof. For (1), if $[p, \xi)$ is not contained in M then reflecting this ray in M gives another geodesic ray from p to ξ , contradicting the uniqueness of the geodesic ray. For (2), by local considerations there is a unique point of intersection of c with M , at say $c(t_0) = p_0$. The unique geodesic ray from p_0 to $\xi = c(\infty)$ is then given by $c(t)$ for $t \geq t_0$. If $\xi \in \partial M$ this contradicts (1). For (3), if $[p, \xi)$ intersects M this contradicts (2). \square

We thus make the following definition: a wall M *separates* $p \in X$ from $\xi \in \partial X$ if p is not in M and the geodesic ray $[p, \xi)$ intersects M , hence by Lemma 11(3) the boundary point ξ is not in ∂M .

Lemma 12. *Suppose $\xi, \xi' \in \partial X$ and M is a wall. If $\xi, \xi' \in \partial X \setminus \partial M$ are separated by M then ξ and ξ' are separated by infinitely many walls. If $\xi \notin \partial M$ and $\xi' \in \partial M$ then ξ and ξ' are also separated by infinitely many walls.*

Proof. Let c be a geodesic ray such that $c(0) \in M$ and $c(\infty) = \xi$, and assume $\xi \notin \partial M$. By the Parallel Wall Theorem [3, Theorem 2.8], there is a wall M_1 which separates $c(t_1)$ from M , for some sufficiently large t_1 . Now, c must intersect M_1 at some time $t'_1 \in (0, t_1)$, so in particular ξ does not lie in ∂M_1 , by Lemma 11(2). It follows that c does not stay bounded distance from M_1 , so we can repeat the argument to find M_2, M_3 , and so on. All of the walls M_i separate ξ from ∂M , and so in particular separate ξ from ξ' . \square

We next consider pairs of boundary points which are connected by a geodesic in X .

Lemma 13. *Let ξ and ξ' be distinct points in $\partial_T X$, with $\xi = c(\infty)$ and $\xi' = c'(\infty)$ where c and c' are geodesic rays based at the same point $c(0) = c'(0) = p \in X$. If ξ and ξ' are connected by a geodesic γ in X then the set of walls crossed by both c and c' is finite.*

Proof. Suppose M is a wall such that c and c' both intersect M , and M does not contain $c(0) = c'(0)$. Thus M separates $c(\infty)$ and $c'(\infty)$ from $c(0) = c'(0)$. Since γ is a geodesic it can intersect M at most once, but $c(\infty)$ and $c'(\infty)$ are on the same side of M , so γ does not intersect M at all. Thus, M separates γ from $c(0) = c'(0)$.

Now let α be any geodesic segment connecting $c(0) = c'(0)$ to some point on γ . Every wall M which intersects both c and c' must intersect α . But only finitely walls go through each point, and the intersection points on α are a discrete set. Since α has finite length the statement of the lemma follows. \square

For $\xi \in \partial X$, define $\text{Inv}(\xi)$ to be the set of walls separating ξ from 0. By our definition of separation, if ξ lies in ∂M , then $M \notin \text{Inv}(\xi)$ regardless of the relative position of 0 and M .

Similarly, define $\text{Inv}(p)$ for $p \in X$ to be the set of walls separating p from 0. We also define $\text{Inv}(p, \xi)$ to be the set of walls separating p from ξ , so that $\text{Inv}(\xi) = \text{Inv}(0, \xi)$.

Lemma 14. *Let $\xi \in \partial X$ and $p \in X$. Then*

$$(\text{Inv}(0, \xi) \setminus \text{Inv}(p, \xi)) \cup (\text{Inv}(p, \xi) \setminus \text{Inv}(0, \xi))$$

is finite.

Proof. Assume that there are infinitely many walls $M_i \in \text{Inv}(0, \xi) \setminus \text{Inv}(p, \xi)$. Note that by definition $\xi \notin \partial M_i$, so $\xi \in \partial M_i^- \setminus \partial M_i$ for each i . Since only finitely many walls cross each point of the finite length geodesic segment $[0, p]$, and the intersection points are a discrete set, we may assume that $[0, p]$ does not cross any of the walls M_i . Thus in particular $p \in M_i^+ \setminus M_i$ for each i . But $\xi \notin \partial M_i$ and $[p, \xi]$ does not cross M_i , so this implies $\xi \in \partial M_i^+ \setminus \partial M_i$ for each i , a contradiction. The argument if there are infinitely many walls in $\text{Inv}(p, \xi) \setminus \text{Inv}(0, \xi)$ is similar. \square

Definition 1 (The set $C(\xi)$). Define an equivalence relation on $\partial_T X$ by $\xi \sim \xi'$ if and only if for each wall M , $\xi \in \partial M^\pm \Leftrightarrow \xi' \in \partial M^\pm$. Let $C(\xi) \subset \partial_T X$ denote the equivalence class of ξ .

Lemma 15. *Suppose ξ and ξ' are not equivalent. Then*

$$(\text{Inv}(\xi) \setminus \text{Inv}(\xi')) \cup (\text{Inv}(\xi') \setminus \text{Inv}(\xi))$$

is infinite.

Proof. Without loss of generality, we may consider the following two cases: $\xi \in \partial M^- \setminus \partial M$ and $\xi' \in \partial M^+$, and $\xi \in \partial M^+ \setminus \partial M$ and $\xi' \in \partial M$, for some wall M . In the first case, by the same argument as in Lemma 12, the geodesic ray $[0, \xi)$ crosses infinitely many walls which separate ξ from M , and so there are infinitely many walls which separate ξ from ξ' . In the second case, let p be a point in M . By Lemma 14 and the argument in Lemma 12, there are infinitely many walls M_i which are in both $\text{Inv}(p, \xi)$ and $\text{Inv}(\xi) = \text{Inv}(0, \xi)$, and which separate M from ξ . None of these walls M_i are in $\text{Inv}(p, \xi')$, since $[p, \xi')$ is contained in M , so again by Lemma 14 only finitely many of the M_i can be in $\text{Inv}(\xi') = \text{Inv}(0, \xi')$. Hence there are infinitely many M_i in $\text{Inv}(\xi) \setminus \text{Inv}(\xi')$. This completes the proof. \square

4. PARTITION OF TITS BOUNDARY

In this section we prove our main result, Theorem 1, which is stated in the introduction.

Definition 2 (The set $\partial_T X(\mathbf{i})$). Let \mathbf{i} be an infinite reduced word. We define $\partial_T X(\mathbf{i})$ to be the set of $\xi \in \partial_T X$ such that there exists a geodesic ray c with $c(\infty) = \xi$ such that when $c(0) = p \in X$, the walls crossed by c , together with the walls separating p from 0, are exactly the inversions of \mathbf{i} (and no wall is crossed twice).

By Lemma 3, if \mathbf{i} and \mathbf{j} are equivalent infinite reduced words, then $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$.

Lemma 16. *For each \mathbf{i} , the set $\partial_T X(\mathbf{i})$ is nonempty.*

Proof. Let $0 = x_0, x_1, x_2, \dots$ be the centres of the chambers visited by the infinite reduced word \mathbf{i} . Then for each x_i there is a distance function $d_{x_i} : X \rightarrow [0, \infty)$ given by $d_{x_i}(y) = d(x_i, y)$ where d is the metric on X . Let $\mathcal{C}(X)$ be the space of continuous functions on X with the topology of uniform convergence on compact subsets, and let \bar{d}_{x_i} be the equivalence class of d_{x_i} in $\mathcal{C}_*(X) := \mathcal{C}(X)/\{\text{constants}\}$. Denote by \hat{X} the closure in $\mathcal{C}_*(X)$ of the set of equivalence classes $\iota(X) := \{\bar{d}_x : x \in X\}$.

We will first show that a subsequence of the sequence $\{\bar{d}_{x_i}\}$ converges to a point $h \in \hat{X} \setminus X$, hence by [2, Proposition II.8.19], a subsequence of the points $\{x_i\}$ converges to some ξ in the visual boundary ∂X . We then finish the proof by showing that $\xi \in \partial_T X(\mathbf{i})$.

Since the Davis complex X is a proper metric space, the space $\overline{X} = X \cup \partial X$ obtained by adjoining to X its visual boundary is compact [2, p. 264]. By [2, Theorem II.8.13], \overline{X} and \hat{X} are homeomorphic. Hence \hat{X} is compact (and first-countable, by definition of the topology on \overline{X}), and so $\{\overline{d_{x_i}}\}$ has a convergent subsequence say $\{\overline{d_{x_{i_n}}}\}$. Let h be the limit of this subsequence.

We now identify X with $\iota(X)$ and show that $h \in \hat{X} \setminus X$. Assume by contradiction that $h \in X$. Then there is a point $x \in X$ and a constant $a \in \mathbb{R}$ such that for all $z \in X$, $h(z) = d_x(z) + a = d(x, z) + a$. By definition of the topology on \hat{X} , there are also constants $a_n \in \mathbb{R}$ such that the sequence $\{d_{x_{i_n}} + a_n\}$ converges uniformly to h on bounded subsets of X . We obtain a contradiction from the observation that $d(x_{i_n}, x) \rightarrow \infty$ as $n \rightarrow \infty$. Thus $h \in \hat{X} \setminus X$.

By [2, Proposition II.8.19], the geodesic segments $[x_0, x_{i_n}]$ converge to a geodesic ray $c = [x_0, \xi)$. We now complete the proof by showing that this ray crosses the same set of walls as \mathbf{i} , hence $\xi \in \partial_T X(\mathbf{i})$ and so $\partial_T X(\mathbf{i})$ is nonempty. Suppose that a wall M is in $\text{Inv}(\mathbf{i})$. Then there is an $n > 0$ such that M separates 0 from all of $x_{i_n}, x_{i_{n+1}}, \dots$. Thus for all large enough n the geodesic segment $[x_0, x_{i_n}]$ crosses M , and so c crosses M . Now suppose c crosses a wall M . Then for all n large enough, the geodesic segment $[x_0, x_{i_n}]$ crosses M , and thus the wall M separates x_0 from x_{i_n} , hence $M \in \text{Inv}(\mathbf{i})$. \square

Lemma 17. *Every $\xi \in \partial_T X$ lies in some $\partial_T X(\mathbf{i})$.*

Proof. Let $c : [0, \infty) \rightarrow X$ be the geodesic ray in the direction of ξ such that $c(0)$ lies in the identity chamber. The sequence of chambers encountered by c will give rise to sequence $w^{(1)}, w^{(2)}, \dots \in W$ of elements of the Coxeter group such that for each i , $\ell(w^{(i)}) = \ell(w^{(i-1)}) + \ell((w^{(i-1)})^{-1}w^{(i)})$. In particular such a sequence arises from taking finite subsequences of a single infinite reduced word. (Note that $\ell(w^{(i)}) - \ell(w^{(i-1)}) > 1$ can occur if multiple walls are crossed at one time.) Finally, we remark that the sequence $w^{(1)}, w^{(2)}, \dots \in W$ must be infinite by Lemma 11(2). \square

Proposition 18. *The partition of $\partial_T X$ into the $C(\xi)$ is exactly the partition of $\partial_T X$ into the $\partial_T X(\mathbf{i})$. We have $\partial_T X(\mathbf{i}) = \partial_T X(\mathbf{j})$ if and only if $\text{Inv}(\mathbf{i}) \setminus \text{Inv}(\mathbf{j})$ and $\text{Inv}(\mathbf{j}) \setminus \text{Inv}(\mathbf{i})$ are both finite sets.*

Proof. It is obvious that each $\partial_T X(\mathbf{i})$ is a union of the $C(\xi)$. Suppose that ξ and ξ' are not in the same equivalence class. Then by Lemma 15, there are infinitely many walls which separate one but not both of the two points ξ and ξ' from 0. For any $p \in X$, there are only finitely many walls separating p from 0. It follows from the definition of $\partial_T X(\mathbf{i})$ that not both of ξ and ξ' can lie in $\partial_T X(\mathbf{i})$. The last statement also follows. \square

Lemma 19. *For each \mathbf{i} , if $\xi, \xi' \in \partial_T X(\mathbf{i})$ and $p \in X$ then the geodesic rays $[p, \xi)$ and $[p, \xi')$ cross the same set of walls.*

Proof. By Proposition 18, for each wall M , $\xi \in \partial M^\pm$ if and only if $\xi' \in \partial M^\pm$. Suppose first that $M \in \text{Inv}(p)$, that is, that $p \in M^- \setminus M$. Then $\xi \in \partial M^+ \setminus \partial M$ (respectively, $\xi' \in \partial M^+ \setminus \partial M$) if and only if $[p, \xi)$ (respectively, $[p, \xi')$) crosses M , and $\xi \in \partial M$ (respectively, $\xi' \in \partial M$) if and only if $[p, \xi)$ (respectively, $[p, \xi')$) does not cross M . The case that $p \in M^+ \setminus M$ is similar. If $p \in M$ then neither $[p, \xi)$ nor $[p, \xi')$ crosses M . \square

Lemma 20. *Suppose $\xi, \xi' \in \partial_T X(\mathbf{i})$. Then $d_T(\xi, \xi') < \pi$.*

Proof. It follows from Lemma 13 that if $\xi' \in C(\xi)$ then ξ and ξ' are not connected by a geodesic in X . Hence by Proposition 9.21(1) and (2) of [2], $d_T(\xi, \xi') = \angle(\xi, \xi') \leq \pi$. Suppose that $d_T(\xi, \xi') = \pi$. Fix $\varepsilon > 0$. Then there is a point $p \in X$ such that $\angle_p(\xi, \xi') > \pi - \varepsilon$. Let c be the geodesic ray from p to ξ and c' the geodesic ray from p to ξ' . Then by Lemma 19, c and c' cross the same set of walls. By Lemma 8, there is a wall M that intersects c at angle greater than ε . This wall M also

intersects c' , and the triangle with vertices the two intersection points and p will have total angle greater than π . This is a contradiction. \square

Proposition 21. *Each $C(\xi)$ is a path-connected, totally geodesic subset of $\partial_T X$.*

Proof. It follows from Lemma 13 that if $\xi' \in C(\xi)$ then ξ and ξ' are not connected by a geodesic in X . Hence by Lemma 6 and Lemma 20, there is a unique geodesic arc in $\partial_T X$ connecting ξ and ξ' , of length strictly less than π . We will show that each point η on the geodesic arc in $\partial_T X$ connecting ξ' and ξ in fact lies in $\partial_T X(\mathbf{i})$.

By Proposition 18, it suffices to show that $\eta \in C(\xi)$. Suppose that there is a wall M such that $\xi, \xi' \in \partial M^+$ but $\eta \in \partial M^- \setminus \partial M$. Then the arc $[\xi, \xi']$ has a (possibly equal) subarc $\gamma = [\zeta, \zeta']$ such that $\zeta, \zeta' \in \partial M$ and every point in the interior of γ is in $\partial M^- \setminus \partial M$. Let r be the reflection in the wall M . Then r induces an isometry of the Tits boundary, which fixes ζ and ζ' but does not fix γ . But then γ and $r(\gamma)$ are two distinct geodesic arcs in $\partial_T X$ connecting ζ and ζ' of length strictly less than π . This is impossible, so there is no wall M such that $\xi, \xi' \in \partial M^+$ but $\eta \in \partial M^- \setminus \partial M$. The argument is similar if $\xi, \xi' \in \partial M^-$ but $\eta \in \partial M^+ \setminus \partial M$. Therefore $\eta \in C(\xi)$, and so $C(\xi)$ is a path-connected totally geodesic subset of $\partial_T X$. \square

For each M , let

$$\epsilon_M(\xi) = \begin{cases} 0 & \text{if } \xi \in \partial M \\ + & \text{if } \xi \in \partial M^+ \setminus \partial M \\ - & \text{if } \xi \in \partial M^- \setminus \partial M. \end{cases}$$

Define $\partial M^\circ^\pm := \partial M^\pm \setminus \partial M$ and $\partial M^\circ^0 := \partial M$. Then

$$C(\xi) = \cap_M \partial M^{\circ \epsilon_M(\xi)}.$$

Define

$$C'(\xi) = \cap_M \partial M^{\epsilon_M(\xi)}$$

where $\partial M^0 := \partial M$.

Proposition 22. *The closure $\overline{C(\xi)}$ is equal to $C'(\xi)$.*

Proof. Let $\eta' \in C'(\xi)$ and $\eta \in C(\xi)$. Suppose η' and η are in the situation of Lemma 6(1). Then as in the proof of Lemma 15, there are infinitely many walls M that separate η' from 0 which do not separate η from 0. This is impossible from the definition of $C'(\xi)$. Thus there is a geodesic arc $c : [0, 1] \rightarrow \partial_T X$ with $c(0) = \eta'$ and $c(1) = \eta$. The interior of this geodesic cannot intersect any walls, again by the definition of $C'(\xi)$. Thus $c((0, 1]) \subset C(\xi)$. Thus $c(0) \in \overline{c((0, 1])} \subset \overline{C(\xi)}$. Finally, we note that $C'(\xi)$ is closed, since it is an intersection of closed subspaces (Lemma 9). \square

The following result completes the proof of Theorem 1.

Theorem 23. *The $\partial_T X(\mathbf{i})$ form a partition of $\partial_T X$ satisfying*

$$\overline{\partial_T X(\mathbf{i})} = \bigsqcup \{ \partial_T X(\mathbf{j}) \mid \mathbf{j} \text{ a set of representatives for the blocks } \leq B(\mathbf{i}) \} = \bigcup_{\mathbf{j} \leq \mathbf{i}} \partial_T X(\mathbf{j}).$$

Remark 1. Note that it is not true that if $\partial_T X(\mathbf{j}') \subset \overline{\partial_T X(\mathbf{i})}$ then $\mathbf{j}' \leq \mathbf{i}$. One may have to first increase $\text{Inv}(\mathbf{i})$ by a finite set of walls before this inequality holds.

Proof. Let $\partial_T X(\mathbf{i}) = C(\xi)$. Suppose $C(\xi') \subset \overline{C(\xi)}$. It follows from Proposition 22 that $\text{Inv}(\xi') \subset \text{Inv}(\xi)$. Suppose we have chosen $p \in X$ so that the set of walls crossed from 0 to p , and then from p to ξ , is equal to $\text{Inv}(\mathbf{i})$ (with no wall crossed twice). Then using Lemma 11(3), one sees that the set of walls crossed from 0 to p , and then from p to ξ' , includes no wall twice, and is equal to $\text{Inv}(p) \cup \text{Inv}(\xi')$. Thus $\overline{\partial_T X(\mathbf{i})} \subset \bigcup_{\mathbf{j} \leq \mathbf{i}} \partial_T X(\mathbf{j})$.

Conversely, suppose \mathbf{j} is such that $C(\xi') = \partial_T X(\mathbf{j}) \notin \overline{C(\xi)}$. If ξ' is not in the same path-component of $\partial_T X$ as $C(\xi)$, then by Lemma 13, we know that \mathbf{j} and \mathbf{i} are incomparable. Now assume that ξ and ξ' are in the same path component of $\partial_T X$. Let $c : [0, l] \rightarrow \partial_T X$ be the geodesic arc such that $c(0) = \xi'$ and $c(l) = \xi$. If the interior of c intersects some wall ∂M , then by Lemma 12 and its proof there is a line of pairwise non-intersecting walls M_i separating ξ and ξ' with $M = M_0$. Note that the origin (or any other point) cannot lie on the same side of all these walls. Otherwise any (finite length) geodesic joining the origin to M will intersect infinitely many of these walls. It follows that $\text{Inv}(\xi) - \text{Inv}(\xi')$ and $\text{Inv}(\xi') - \text{Inv}(\xi)$ are both infinite, and so \mathbf{j} and \mathbf{i} are incomparable.

Finally, we consider the case that $c((0, l))$ intersects no walls, but that ξ is contained in a wall ∂M , and ξ' is not contained in this wall. In this case, the same argument shows that $\text{Inv}(\xi') - \text{Inv}(\xi)$ is infinite and so we cannot have $\mathbf{j} < \mathbf{i}$. \square

5. INTERSECTIONS OF WALLS, AND REFLECTION SUBGROUPS

We first consider the intersections of walls and their boundaries in X and in $\partial_T X$. We then study the group $W(\xi)$ generated by reflections in the walls which have $\xi \in \partial_T X$ in their boundary. The main result of this section is Proposition 40.

5.1. Intersections of walls in X . Our aim in this section is to prove Corollary 25 below. We could not find Corollary 25 in the literature, although it may well be known to experts.

Let V be the geometric realisation of the Coxeter system (W, S) . Then V is a vector space with basis $\{\alpha_s \mid s \in S\}$ and symmetric (possibly degenerate) bilinear form $B : V \times V \rightarrow \mathbb{R}$. The root system $\Phi \subset V$ consists of vectors $\alpha \in V$, all satisfying $B(\alpha, \alpha) = 1$.

Lemma 24. *Let A be a finite set of real numbers, not including 1. Suppose $\Psi \subset \Phi$ is a subset of roots such that $B(\alpha, \beta) \in A$ for every $\alpha \neq \beta \in \Psi$. Then Ψ is finite.*

Proof. Suppose Ψ is infinite. Let $\alpha_1 \in \Psi$. Then there must be an infinite set $\Psi_1 \subset \Psi$ such that $B(\alpha_1, \beta) = a_1$ for some $a_1 \in A$ and all $\beta \in \Psi_1$. Note that the linear function $B(\alpha_1, \cdot)$ is not constant since it takes value 1 on α_1 and the value $a_1 \neq 1$ on some other vector. Thus $\Psi_1 \subset V_1 \subsetneq V$, where V_1 is an affine subspace with codimension 1 in V . We repeat this argument to define $\Psi_k \subset \Psi_{k-1} \subset \dots \subset \Psi_2 \subset \Psi_1 \subset \Psi$ lying inside $V_k \subset V_{k-1} \subset \dots \subset V_2 \subset V_1 \subset V$, where V_k has codimension k in V .

But the dimension of V is finite, so we obtain a contradiction. \square

Recall that two walls intersect in X if and only if the corresponding reflections generate a spherical subgroup.

Corollary 25. *Any collection of pairwise intersecting walls in X is finite.*

Proof. Apply Lemma 24 with A the set of possible inner products between pairs of roots generating a spherical subgroup of W , and Ψ the subset of roots corresponding to the walls in this collection. The set A will be finite since each pair of intersecting walls will intersect at an angle belonging to a finite set determined by the local geometry of a chamber in X . Also A does not contain 1 for otherwise the corresponding pair of roots will generate an infinite group. \square

5.2. Intersections of walls in the boundary. Given $\xi \in \partial_T X$ we define

$$\mathcal{M}(\xi) := \{M \in \mathcal{M} \mid \xi \in \partial M\}.$$

In this section we investigate the set $\mathcal{M}(\xi)$ and its implications for the geometry of X .

Proposition 26. *For each $\xi \in \partial_T X$, the set $\mathcal{M}(\xi)$ is infinite if and only if there are disjoint walls $M, M' \in \mathcal{M}(\xi)$.*

Proof. Since the collection of walls is locally finite in X , if $\mathcal{M}(\xi)$ is infinite the conclusion follows from Corollary 25. Conversely, suppose $M, M' \in \mathcal{M}(\xi)$ are disjoint and let r, r' be the reflections fixing M, M' respectively. Then r and r' generate an infinite dihedral group which fixes ξ . Apply this group to the wall M to obtain infinitely many walls in $\mathcal{M}(\xi)$. \square

Proposition 27. *Suppose that M, M' are disjoint walls such that $\partial M \cap \partial M'$ contains a point ξ . Then X contains an isometrically embedded Euclidean plane.*

Proof. Since M and M' are closed subsets of the complete metric space X , M and M' are complete in the induced metric. Hence by [2, Proposition II.2.4(1)], we may choose points $p \in M$ and $p' \in M'$ such that $d(p, p') = d(M, M')$. Let $c = [p, \xi)$ and $c' = [p', \xi)$ and consider the function $t \mapsto d(c(t), c'(t))$. Since the geodesic rays c and c' are equivalent and $d(p, p')$ realises the distance $d(M, M')$, this function is bounded and non-decreasing. As X is CAT(0), this function is convex. Hence the function $t \mapsto d(c(t), c'(t))$ is constant.

Now for any $t > 0$, consider the four points $p, p', q = c(t)$ and $q' = c'(t)$. Since the wall M' is a convex and complete subset of X , and $d(p, p')$ realises the distance $d(p, M')$, by [2, Proposition II.2.4] the Alexandrov angle between the geodesic segments $[p, p']$ and $[p', q']$ is $\geq \pi/2$. By considering $d(p', M)$ instead, we obtain that the Alexandrov angle between the geodesic segments $[p', p]$ and $[p, q]$ is also $\geq \pi/2$. But since $d(q, q') = d(p, p') = d(M, M')$, the same argument shows that the Alexandrov angles between $[q, q']$ and the segments $[q, p]$ and $[q', p']$ are also $\geq \pi/2$. Thus by the Flat Quadrilateral Theorem [2, II.2.11], each of these four angles is equal to $\pi/2$, and moreover the convex hull of the four points p, p', q, q' is isometric to the convex hull of a rectangle in the Euclidean plane. Therefore the convex hull of c and c' is a flat half-strip which is orthogonal to both M and M' .

Let r and r' be the reflections fixing M and M' respectively. Since M and M' are disjoint these reflections generate an infinite dihedral group. The union of the images of the flat half-strip bounded by c and c' under the action of $\langle r, r' \rangle$ is a flat half-plane in X (which is orthogonal to all of the images of M and M' under this action). Hence for all $n \geq 1$ there is an isometric embedding into X of the ball of radius n centred at the origin in the Euclidean plane. Thus by [2, Lemma 9.34], there is an isometric embedding of the Euclidean plane into X . \square

Corollary 28. *If ∂X contains a point ξ such that $\mathcal{M}(\xi)$ is infinite, then X contains an isometrically embedded Euclidean plane.*

We also note the following basic properties of walls and their intersections.

Lemma 29. *Assume X has the geodesic extension property. If M and M' are distinct walls such that $M \cap M' \neq \emptyset$, then $\partial M \neq \partial M'$.*

Proof. Let $p \in M \cap M'$. Since M and M' are distinct walls, we may choose $q \in M \setminus M'$. Then as X has the geodesic extension property, the geodesic segment $[p, q]$ extends to a geodesic ray $[p, \xi)$. By the local properties of walls, $\xi \in \partial M$. If $\xi \in \partial M'$ as well, then by Lemma 11 the entire geodesic ray $[p, \xi)$ is contained in M' , a contradiction. \square

Proposition 30. *Assume X has the geodesic extension property. Let M and M' be disjoint walls such that $\partial M = \partial M'$ is nonempty. Then M and M' are constant distance apart and X contains an isometrically embedded Euclidean plane.*

Proof. Let $\xi \in \partial M = \partial M'$. As in the proof of Proposition 27, we may choose points $p \in M$ and $p' \in M'$ such that $d(p, p') = d(M, M')$, and the geodesic rays $[p, \xi)$ and $[p', \xi)$ then bound a flat half-strip in X .

Now since X has the geodesic extension property, there exists a geodesic line $c : \mathbb{R} \rightarrow X$ such that $c([0, \infty)) = [p, \xi)$. By the local structure of walls, the entire image of c must be contained in

M . Put $\eta = c(-\infty)$. Then $\eta \in \partial M = \partial M'$. Since $d(p, p') = d(M, M')$, the geodesic rays $[p, \eta)$ and $[p', \eta)$ also bound a flat half-strip.

Let r' be the reflection which fixes the wall M' . Then $r'c$ is a geodesic line which is contained in the wall $r'M$ and passes through the point $r'p$. The geodesic ray $[r'p, \xi)$ is constant distance from $[p', \xi)$, and the geodesic ray $[r'p, \eta)$ is constant distance from $[p', \eta)$. Thus the geodesic line $r'c$ is at uniformly bounded distance from the geodesic line c . Therefore by the Flat Strip Theorem [2, II.2.13], the convex hull of c and $r'c$ is isometric to a flat strip. Note that the width of this flat strip is $d(p, r'p) = 2d(p, p')$.

We have so far shown that there is a geodesic line c in M , and passing through p , which is at constant distance from the geodesic line $r'c$ in $r'M$.

Now let x be any point in M which is distinct from p . Then by the geodesic extension property again, the geodesic segment $[p, x]$ may be extended to a geodesic line c_x passing through p , and moreover by the local structure of walls the line c_x is contained in M . The above argument may be repeated with the endpoints of c_x to show that c_x is at constant distance $2d(p, p')$ from the geodesic line $r'c_x$ in $r'M$. In particular, x is at distance $2d(p, p')$ from $r'M$. Therefore we have that the walls M and $r'M$ are at constant distance $2d(p, p')$.

Now for any point $x' \in M'$, we have $d(x', M) = d(x', r'M)$. Thus the wall M' is exactly halfway in between M and $r'M$, and we conclude that M and M' are constant distance $d(p, p')$.

To obtain an isometrically embedded Euclidean plane which is orthogonal to both M and M' , keep on reflecting the flat strip between lines c and $r'c$. \square

5.3. Intersections of boundaries of walls in the boundary. Given $\xi \in \partial_T X$ we define

$$\partial\mathcal{M}(\xi) := \{\partial M \mid M \in \mathcal{M}, \xi \in \partial M\} = \{\partial M \mid M \in \mathcal{M}(\xi)\}.$$

That is, $\partial\mathcal{M}(\xi)$ is the set of boundaries of walls which contain the point ξ . It is immediate that for all $\xi \in \partial M$, $|\partial\mathcal{M}(\xi)| \leq |\mathcal{M}(\xi)|$.

5.4. The group $W(\xi)$. For each $\xi \in \partial_T X$, define $R(\xi)$ to be the set of reflections in walls in $\mathcal{M}(\xi)$. That is, $R(\xi)$ is the set of reflections in walls which have ξ in their boundary. Let $W(\xi)$ be the subgroup of W generated by the elements of $R(\xi)$. If $R(\xi)$ is empty, then define $W(\xi)$ to be the trivial group.

By a theorem of Deodhar [10], proved independently by Dyer [11], the group $W(\xi)$ is itself a Coxeter group, with respect to a canonical system of generators. Dyer [11, Corollary (3.4)(i)] also established the following, where ℓ denotes word length with respect to the Coxeter generators.

Lemma 31. *Let $w \in W(\xi)$ be arbitrary and $r \in W(\xi)$ be a reflection. Then $\ell(rw) < \ell(w)$ in $W(\xi)$ if and only if $\ell(rw) < \ell(w)$ in W .*

In order to say more about $W(\xi)$, we first show that $R(\xi)$ is closed under conjugation.

Lemma 32. *If r, t lie in $R(\xi)$, then rtr^{-1} lies in $R(\xi)$.*

Proof. We have to show that if we reflect M_t in M_r then the resulting wall $M' = rM_t$, which is fixed by the reflection rtr^{-1} , lies in $\mathcal{M}(\xi)$. Let c be a geodesic contained in M_t such that $c(\infty) = \xi$. Let $c' = r.c$ be its reflection, which is contained in M' . Since ξ is fixed by the reflection r , we have $c'(\infty) = \xi = c(\infty)$, and so $M' \in \mathcal{M}(\xi)$. \square

Lemma 33. *Every reflection in $W(\xi)$ is conjugate via an element of $W(\xi)$ to a reflection in $R(\xi)$.*

Proof. This is Corollary (3.11)(ii) in Dyer [11]. \square

Corollary 34. *The set $R(\xi)$ is exactly the set of reflections of $W(\xi)$. Hence $W(\xi)$ is finite if and only if $\mathcal{M}(\xi)$ is finite.*

Proof. This follows from the previous two lemmas, and the observation that $W(\xi)$ is finite if and only if it contains finitely many reflections. \square

Lemma 35. *If $\mathcal{M}(\xi)$ is finite and nonempty then $\bigcap_{M \in \mathcal{M}(\xi)} M$ is nonempty. If in addition X has the geodesic extension property, then this intersection contains a geodesic line.*

Proof. Since $W(\xi)$ is finite and X is CAT(0), there is a point $p \in X$ which is fixed by all $w \in W(\xi)$. In particular, p is fixed by all reflections in $R(\xi)$, hence p is contained in every wall $M \in \mathcal{M}(\xi)$. That is, $\bigcap_{M \in \mathcal{M}(\xi)} M$ is nonempty. Now consider the geodesic ray $[p, \xi)$. By Lemma 11, this geodesic ray is contained in every $M \in \mathcal{M}(\xi)$. Let η be the other endpoint of a geodesic extension of the ray $[p, \xi)$. Then by the local properties of walls, the ray $[p, \eta)$ is also contained in every $M \in \mathcal{M}(\xi)$. Thus $\bigcap_{M \in \mathcal{M}(\xi)} M$ contains a geodesic line. \square

Lemma 36. *Suppose $W(\xi)$ is nontrivial and take any $w \in W(\xi)$. Let p_0 be in the interior of the chamber of X corresponding to w . Then there is a point p on the geodesic ray joining p_0 to ξ , such that every wall intersecting the geodesic segment $[0, p]$ either has ξ in its boundary, or separates 0 from ξ .*

Proof. Let M be a wall intersecting the geodesic segment $[0, p_0]$. If M does not intersect the geodesic ray $[p_0, \xi)$ and M does not separate 0 from ξ , then since neither of the geodesic rays $[p_0, \xi)$ and $[0, \xi)$ cross M , and M separates 0 from p_0 , we must have $\xi \in \partial M$, that is, $M \in \mathcal{M}(\xi)$.

Now suppose the geodesic segment $[0, p_0]$ intersects a wall M_1 such that $M_1 \notin \mathcal{M}(\xi)$ and M_1 does not separate 0 from ξ . Then by the above argument, M_1 intersects $[p_0, \xi)$. So we may define p_1 to be a point on the geodesic ray $[p_0, \xi)$ such that p_1 is on the same side of M_1 as ξ . Note that the geodesic segment $[0, p_1]$ intersects fewer walls that do not have ξ in their boundary and do not separate 0 from ξ than does the geodesic segment $[0, p_0]$. So repeating the construction, we build p_2, p_3, \dots , until we obtain $p = p_k$ with the desired property. (Note that the initial geodesic segment from 0 to p_0 intersects only finitely many walls, so this process stops.) \square

For $w \in W$, denote by $\text{Inv}_{\mathcal{M}(\xi)}(w)$ the set $\text{Inv}(w) \cap \mathcal{M}(\xi)$. Suppose w lies in the reflection subgroup $W(\xi) \subset W$. Then it follows from Lemma 31 that $\text{Inv}_{\mathcal{M}(\xi)}(w)$ can naturally be identified with the inversion set $\text{Inv}_{W(\xi)}(w)$ of w , considered as an element of $W(\xi)$.

Lemma 37. *Suppose $w \in W$. Then there exists $w' \in W(\xi)$ such that $\text{Inv}_{\mathcal{M}(\xi)}(w)$ can naturally be identified with $\text{Inv}_{W(\xi)}(w')$.*

Proof. We proceed by induction on $k = |\text{Inv}_{\mathcal{M}(\xi)}(w)|$. The base case $k = 0$ is clear. For $k > 0$, pick a wall $M \in \text{Inv}_{\mathcal{M}(\xi)}(w)$ so that no other walls in $\mathcal{M}(\xi)$ separate w from M . Let the reflection corresponding to M be denoted $r \in W(\xi)$. Let $v \in W$ be such that $\text{Inv}_{\mathcal{M}(\xi)}(v) = \text{Inv}_{\mathcal{M}(\xi)}(w) - \{M\}$, and by induction we suppose that we have found $v' \in W(\xi) \subset W$ such that $\text{Inv}_{\mathcal{M}(\xi)}(v) = \text{Inv}_{W(\xi)}(v')$. We claim that $w' = rv'$ works, where geometrically w' is obtained from v' by reflecting in M . This follows easily from the observation that no wall in $\mathcal{M}(\xi)$ separates v' from M , which in turn follows from the same property of v . \square

Corollary 38. *Let w and p be as in Lemma 36 and let M be a wall crossed by $[0, p]$ such that $\xi \in \partial M$. Then $M \in \text{Inv}_{\mathcal{M}(\xi)}(w)$. Thus $\text{Inv}(p)$ is the disjoint union of $\text{Inv}_{\mathcal{M}(\xi)}(w)$ with a subset of $\text{Inv}(\xi)$.*

Proof. Since $M \in \mathcal{M}(\xi)$ it suffices to show that $M \in \text{Inv}(w)$, that is, that M is crossed by $[0, p_0]$. But the geodesic ray $[p_0, \xi)$ does not cross M , so p_0 and p are on the same side of M . Since M separates 0 from p the result follows. \square

Corollary 39. *The set of walls crossed by $[0, p]$, together with the set of walls crossed by $[p, \xi)$, is the disjoint union of $\text{Inv}_{\mathcal{M}(\xi)}(w)$ with $\text{Inv}(\xi)$ (and no wall is crossed twice).*

Proof. By construction, no wall crosses both $[0, p]$ and $[p, \xi]$. If $M \in \text{Inv}(\xi)$ and M is not crossed by $[0, p]$, then M separates both 0 and p from ξ , and so M is crossed by $[p, \xi]$. \square

Proposition 40. *Suppose $W(\xi)$ is nontrivial. Then the set of equivalence classes of infinite reduced words \mathbf{i} such that $C(\xi) = \partial_T X(\mathbf{i})$ is in bijection with the set of elements of $W(\xi)$, and the partial order on this set of equivalence classes of infinite reduced words corresponds naturally to the weak partial order on $W(\xi)$.*

Proof. Let $w \in W(\xi)$ and construct p as in Lemma 36. Then by considering the set of chambers visited by $[0, p] \cup [p, \xi]$, we may construct an infinite reduced word \mathbf{i} such that $\xi \in \partial_T X(\mathbf{i})$, hence $C(\xi) = \partial_T X(\mathbf{i})$, and $\text{Inv}(\mathbf{i}) = \text{Inv}_{\mathcal{M}(\xi)}(w) \sqcup \text{Inv}(\xi)$.

Conversely, if \mathbf{i} is an infinite reduced word such that $\xi \in \partial_T X(\mathbf{i})$, let p be a point such that the set of walls crossed by $[0, p] \cup [p, \xi]$ is equal to $\text{Inv}(\mathbf{i})$ (and no wall is crossed twice). Without loss of generality we may choose p to be in the interior of a chamber of X . Let p belong to the interior of the chamber for $w' \in W$, so that $\text{Inv}(p) = \text{Inv}(w')$. Using Lemma 37, we define $w \in W(\xi)$ to be the unique element such that $\text{Inv}_{\mathcal{M}(\xi)}(w) = \text{Inv}(w') \cap \mathcal{M}(\xi)$. We claim that $\text{Inv}(\mathbf{i})$ is the disjoint union of $\text{Inv}_{\mathcal{M}(\xi)}(w)$ with $\text{Inv}(\xi)$. To see this, let $M \in \text{Inv}(\mathbf{i})$. If M intersects $[p, \xi]$ then M separates 0 from ξ so $M \in \text{Inv}(\xi)$. If M does not intersect $[p, \xi]$ then M intersects $[0, p]$, so either $\xi \in \partial M$ or $M \in \text{Inv}(\xi)$. If M intersects $[0, p]$ and $\xi \in \partial M$ then $M \in \text{Inv}_{\mathcal{M}(\xi)}(w)$ by definition. Therefore $\text{Inv}(\mathbf{i}) \subset \text{Inv}_{\mathcal{M}(\xi)}(w) \sqcup \text{Inv}(\xi)$. The opposite inclusion is clear. Thus to each \mathbf{i} such that $\partial_T X(\mathbf{i}) = C(\xi)$, we have associated an element $w \in W(\xi)$ such that $\text{Inv}(\mathbf{i}) = \text{Inv}_{\mathcal{M}(\xi)}(w) \sqcup \text{Inv}(\xi)$.

These constructions are inverses, and the natural correspondence between the partial order on equivalence classes of infinite reduced words in $\partial_T X(\mathbf{i})$ and the weak partial order on $W(\xi)$ follows from these constructions. \square

6. SPECIAL CASES

We conclude by discussing numerous special cases and examples. In particular we prove Theorem 2, concerning word hyperbolic Coxeter groups, and describe the case in which X has isolated flats.

6.1. Irreducible affine Coxeter groups. Infinite reduced words for affine Weyl groups were studied by Cellini-Papi [7] and Ito [13] from a root system point of view. The limit weak partial order on equivalence classes of infinite reduced words was studied in [14]. We summarise the results here in the notation of the present paper.

Let (W, S) be an irreducible affine Coxeter group with corresponding finite Weyl group W_{fin} . Then the boundaries of walls $\partial M \subset \partial_T X$ are in bijection with the reflections of W_{fin} . The pieces of the partition $\partial_T X = \sqcup C(\xi)$ are in bijection with the faces of the Coxeter arrangement of W_{fin} (with the origin of the Coxeter arrangement excluded). This arrangement could also be described as the (thin) spherical building at infinity.

The pieces $C(\xi)$ which are open (and thus not contained in the closure of any other $C(\xi')$) are in bijection with $w \in W_{\text{fin}}$. For any other piece $C(\xi)$, we have that $\mathcal{M}(\xi)$ is infinite but is partitioned into finitely many parallelism classes. These parallelism classes correspond to a collection of walls in the Coxeter arrangement of W_{fin} . Furthermore, $W(\xi)$ is a (possibly reducible) affine Coxeter group.

Thus the maximal elements of (\mathcal{W}, \leq) are in bijection with W_{fin} . The blocks of \mathcal{W} are in bijection with the faces of the Coxeter arrangement of W_{fin} with the origin excluded. The limit weak order on each block $B(\mathbf{i})$ is isomorphic to the weak order on some affine Coxeter group.

6.2. Reducible Coxeter groups, and virtually abelian Coxeter groups. Suppose that (W, S) is a reducible Coxeter system, with $S = S_1 \sqcup S_2$ so that $W = W_1 \times W_2$ where $W_k = \langle S_k \rangle$

for $k = 1, 2$. Let $I_k = \{i \mid s_i \in S_k\}$. Assume first that W_1 and W_2 are both infinite and let \mathbf{i} be an infinite reduced word in W . Then exactly one of the following occurs:

- (1) \mathbf{i} is equivalent to an infinite reduced word consisting of a finite reduced word in W_2 followed by an infinite reduced word in W_1 ;
- (2) \mathbf{i} is equivalent to an infinite reduced word consisting of a finite reduced word in W_1 followed by an infinite reduced word in W_2 ; or
- (3) \mathbf{i} contains infinitely many elements of I_1 and infinitely many elements of I_2 .

If case (1) holds for two infinite reduced words \mathbf{i} and \mathbf{i}' , denote by w_2 and w'_2 the elements of W_2 and by \mathbf{j} and \mathbf{j}' the infinite reduced words in W_1 such that $\text{Inv}(\mathbf{i}) = \text{Inv}(w_2) \sqcup \text{Inv}(\mathbf{j})$ and $\text{Inv}(w'_2) \sqcup \text{Inv}(\mathbf{j}')$. Then $\mathbf{i} \leq \mathbf{i}'$ if and only if both $w_2 \leq w'_2$ in the weak partial order on W_2 and $\mathbf{j} \leq \mathbf{j}'$ in the partial order on infinite reduced words in W_1 , with strict inequality $\mathbf{i} < \mathbf{i}'$ if and only if, in addition, at least one of the containments $\text{Inv}(w_2) \subseteq \text{Inv}(w'_2)$ or $\text{Inv}(\mathbf{j}) \subseteq \text{Inv}(\mathbf{j}')$ is strict. Similarly for the partial order on pairs of infinite reduced words for which (2) holds. If (1) holds for \mathbf{i} and (2) holds for \mathbf{i}' , then \mathbf{i} and \mathbf{i}' are clearly incomparable.

If case (3) holds then for $k = 1, 2$, the infinite reduced word \mathbf{i} is strictly greater in the partial order than any infinite reduced word \mathbf{j}_k obtained by concatenating a finite initial subword of \mathbf{i} with the infinite reduced word consisting of all subsequent elements of I_k in \mathbf{i} . Note that in this case the set $\text{Inv}(\mathbf{i})$ differs from $\text{Inv}(\mathbf{j}_k)$ by infinitely many walls.

Now let us discuss the Tits boundary $\partial_T X$ of the Davis complex X of W . The Davis complex X is a product $X = X_1 \times X_2$, and thus the Tits boundary is the spherical join $\partial_T X = \partial_T X_1 * \partial_T X_2$ (see [2, Corollary II.9.11]). We shall write $\xi = \cos \theta \xi_1 + \sin \theta \xi_2$ with $\theta \in [0, \pi/2]$ for a point of $\partial_T X$, where $\xi_1 \in \partial_T X_1$ and $\xi_2 \in \partial_T X_2$. If $\xi = \xi_1 + 0$ (respectively $\xi = 0 + \xi_2$), then $C(\xi)$ lies in all the walls of W_2 (respectively all the walls of W_1), and $C(\xi) = \partial_T X(\mathbf{i})$ for an infinite reduced word \mathbf{i} of the form (1) (respectively (2)). A point of the form $\xi = \cos \theta \xi_1 + \sin \theta \xi_2$ with $\theta \in (0, \pi/2)$ has equivalence class given by

$$C(\xi) = \{\cos \theta \xi'_1 + \sin \theta \xi'_2 \mid \xi'_1 \in C(\xi_1) \text{ and } \xi'_2 \in C(\xi_2)\}$$

and we then have $C(\xi) = \partial_T X(\mathbf{i})$ for an infinite reduced word \mathbf{i} of the form (3).

If W_2 is finite (and W_1 is infinite), then only case (1) can occur, and similarly if W_1 is finite only case (2) can occur. In particular, it follows from [8, Theorem 12.3.5] that a Coxeter group is virtually abelian if and only if it is a direct product of finite and affine Coxeter groups. It is thus straightforward to extend the results for irreducible affine Coxeter groups to all affine and all virtually abelian Coxeter groups.

6.3. Coxeter groups with two ends. Recall from Theorem 4(3) above that it is equivalent for W to have two ends and for W to be reducible of the form $W_0 \times W_1$ where $W_0 = \langle s, t \rangle$ is the infinite dihedral group and W_1 is spherical. Any infinite reduced word in W is then obtained by inserting into either $ststst \dots$ or $tsstst \dots$ the letters of a reduced word for some $w_1 \in W_1$. The boundary of X thus consists of two points, say ξ_+ and ξ_- . The only walls of X with nonempty boundary are the finitely many walls corresponding to the reflections in W_1 , and for each such wall M we have $\partial M = \{\xi_+, \xi_-\}$. In other words, $R(\xi_{\pm})$ is equal to the set of reflections in W_1 , and $W(\xi_{\pm}) = W_1$. Clearly $C(\xi_+) = \{\xi_+\}$ and $C(\xi_-) = \{\xi_-\}$. There are $2|W_1|$ equivalence classes of infinite reduced words, with $\mathbf{i} \sim \mathbf{j}$ if and only if \mathbf{i} and \mathbf{j} project to the same infinite reduced word in W_0 and \mathbf{i} and \mathbf{j} are eventually on the same side of every wall in W_1 (equivalently, the projections of \mathbf{i} and \mathbf{j} to W_1 define the same group element). The partial order on the set of infinite reduced words \mathbf{i} with $\partial_T X(\mathbf{i}) = \{\xi_+\}$ is then just the weak partial order on W_1 , and similarly for ξ_- .

6.4. Coxeter groups with infinitely many ends. Assume now that W has infinitely many ends. In particular, the case that W is virtually free and the case that W is a free product of

special subgroups are contained in the case that W has infinitely many ends. In this section we will use some results about graphs of groups, a reference for which is Chapter I of [15].

By [8, Proposition 8.8.2], every Coxeter group W has a tree-of-groups decomposition (not in general unique) in which every vertex group is a spherical or a one-ended special subgroup, and every edge group is a spherical special subgroup. More precisely, the edge groups are of the form W_T where $T \in \mathcal{S}$ is such that the punctured nerve $L - \sigma_T$ is disconnected. Since W has infinitely many ends, by Theorem 4 any such tree-of-groups decomposition has at least one edge. Our analysis will depend upon the nature of the vertex and edge groups in such a decomposition.

The group W is virtually free if and only if all vertex groups in such a tree-of-groups decomposition of W are spherical [8, Proposition 8.8.5]. The Davis complex X is then quasi-isometric to a tree, with $\partial_T X$ homeomorphic to the set of ends of this tree. The topology on $\partial_T X$ is thus totally disconnected and so for each $\xi \in \partial_T X$, we have $C(\xi) = \{\xi\}$.

Next consider the subcase that all vertex groups are spherical and all edge groups are trivial. This is equivalent to W being a free product of spherical special subgroups. The Davis complex X then consists of infinitely many copies of the (finite) Davis complexes for the vertex groups, glued together at certain centres of chambers. Thus all walls are finite subcomplexes of X and so have empty boundary. It follows that for each $\xi \in \partial_T X$, there is a unique equivalence class of infinite reduced words \mathbf{i} such that $\partial_T X(\mathbf{i}) = C(\xi) = \{\xi\}$ is a single point.

We next discuss the subcase that all vertex groups are spherical and there is some nontrivial edge group. Fix a tree-of-groups decomposition of W and let r be a reflection in W . Since reflections have finite order, each reflection is contained in a conjugate of at least one vertex group. If r is not contained in any conjugates of edge groups, then the wall $M = M(r)$ is contained in the finite Davis complex for a unique (conjugate of a) vertex group, and so M has empty boundary. If r is contained in some (conjugate of an) edge group, then the wall $M = M(r)$ may or may not have empty boundary, as shown by the following examples. (We do not provide a complete characterisation of the reflections r such that the corresponding wall M has nonempty boundary.)

Examples. (1) Consider $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = 1 \rangle$. Then W splits as the amalgamated free product $W = W_{\{s,t\}} *_{W_{\{t\}}} W_{\{t,u\}} \cong S_3 *_{C_2} S_3$ and hence has a tree-of-groups decomposition consisting of an edge with vertex groups $W_{\{s,u\}}$ and $W_{\{t,u\}}$ and edge group $W_{\{t\}}$. The reflection t is obviously contained in an edge group, but the wall $M(t)$ is finite hence has empty boundary. In fact all walls in this example have empty boundary.

(2) Let $W = W_{T_1} *_{W_T} W_{T_2}$ where T_1 and T_2 are spherical and $T = T_1 \cap T_2$ is nonempty. Let r be a reflection in W_T . Suppose that for $i = 1, 2$ there is a reflection $r_i \in W_{T_i} \setminus W_T$ which commutes with r . Then for $i = 1, 2$ and every $w \in \langle r_1, r_2 \rangle \cong D_\infty$, the reflection r is contained in $W_{T_i}^w$. It follows from the fact that there are infinitely many distinct such conjugates $W_{T_i}^w$ and the construction of X that the wall $M = M(r)$ has nonempty boundary, consisting of at least two points.

Continuing the analysis of the subcase in which all vertex groups are spherical and there is a non-trivial edge group W_T , let $M = M(r)$ be a wall with nonempty boundary, where the reflection r is without loss of generality in W_T . For each $\xi \in \partial_T X$ such that $M \in \mathcal{M}(\xi)$, there is then a nontrivial partial order on the set of equivalence classes of infinite reduced words \mathbf{i} such that $\partial_T X(\mathbf{i}) = \xi$. Since X does not contain an isometrically embedded Euclidean plane, Corollary 28 implies that $\mathcal{M}(\xi)$ is finite. Hence if $M' \neq M$ is also in $\mathcal{M}(\xi)$, with r' the reflection fixing M' , then r and r' generate a finite subgroup of W . Thus $\langle r, r' \rangle$ fixes a point p in X , and we obtain a geodesic ray $[p, \xi)$ which is contained in both M and M' . It follows that r' must be in W_T as well. Applying this argument to every wall in $\mathcal{M}(\xi)$, we conclude that every reflection in $R(\xi)$ is contained in the edge group W_T . There may however be reflections in W_T which are not in $R(\xi)$.

We finally consider the case that W has infinitely many ends and is not virtually free, equivalently there is at least one one-ended vertex group in the tree-of-groups decomposition of W . Then an infinite reduced word \mathbf{i} either eventually stays within a fixed conjugate W_T^w of a one-ended vertex group W_T , or \mathbf{i} visits infinitely many distinct conjugates of vertex groups. For each $w \in W$ and each one-ended vertex group W_T , the partial order on the set of equivalence classes of infinite reduced words that eventually stay in W_T^w is the same as the partial order on the set of equivalence classes of infinite reduced words in W_T . In a similar manner to the analysis when all vertex groups are spherical, if there is some nontrivial edge group W_T , then there may also be a nontrivial partial order on equivalence classes of certain infinite reduced words \mathbf{i} which visit infinitely many distinct conjugates of vertex groups.

6.5. Word hyperbolic Coxeter groups. We use the following characterisation of word hyperbolic Coxeter groups, which appears in [8, Corollary 12.6.3]. See [2] for the definition and basic properties of δ -hyperbolic spaces.

Theorem 41 (Moussong). *A Coxeter group (W, S) is word hyperbolic if and only if there is no subset T of S satisfying either of the two conditions:*

- (1) W_T is affine of rank at least 2.
- (2) (W_T, T) decomposes as $(W_T, T) = (W_{T'} \times W_{T''}, T' \cup T'')$ with both $W_{T'}$ and $W_{T''}$ infinite.

Proposition 42. *W is word hyperbolic if and only if for all $\xi \in \partial_T X$, the set $\mathcal{M}(\xi)$ is finite.*

Proof. Suppose there is a $\xi \in \partial_T X$ such that $\mathcal{M}(\xi)$ is infinite. Then by Propositions 26 and 27, the space X contains an isometrically embedded Euclidean plane. By the Flat Plane Theorem [2, III.H.1.5], this implies X is not δ -hyperbolic, hence W is not word hyperbolic.

For the converse, let T be a subset of S satisfying either of the conditions in Theorem 41. We denote by X_T the Davis complex for (W_T, T) , which embeds isometrically in the Davis complex X for (W, S) . It suffices to prove there is a ξ in the Tits boundary of X_T such that ξ is in the boundary of infinitely many walls in X_T .

This is immediate when W_T is affine, by taking ξ in the boundary of a family of parallel walls in X_T . If $W_T = W_{T'} \times W_{T''}$ as in (2) of Theorem 41, let ξ be any point in the boundary of $X_{T''} \subset X_T$ (since $W_{T''}$ is infinite, $X_{T''}$ has nonempty boundary). Now as $W_{T'}$ is infinite, $W_{T'}$ contains infinitely many distinct reflections r_i . The r_i are also reflections in W_T , with corresponding walls say M_i in X_T . Since $W_{T''}$ commutes with $W_{T'}$, the point ξ is fixed by each of the reflections r_i , and so $\xi \in \partial M_i$ for all i . \square

Corollary 43. *W is word hyperbolic if and only if for all $\xi \in \partial_T X$, the group $W(\xi)$ is finite.*

Corollary 44. *If W is word hyperbolic and one-ended then for all $\xi \in \partial_T X$, we have $|\partial \mathcal{M}(\xi)| = |\mathcal{M}(\xi)| < \infty$.*

Proof. Since $|\partial \mathcal{M}(\xi)| \leq |\mathcal{M}(\xi)| < \infty$, it suffices to show that if M, M' are distinct walls in $\mathcal{M}(\xi)$ then $\partial M \neq \partial M'$. By Lemma 35, the intersection $M \cap M'$ is nonempty. By Proposition 5, since W is one-ended X has the geodesic extension property. Thus by Lemma 29, $\partial M \neq \partial M'$. \square

We do not know how to characterise those W such that $\partial \mathcal{M}(\xi)$ is finite for all $\xi \in \partial_T X$ (noting that, by results above, if W is affine or word hyperbolic then all $\partial \mathcal{M}(\xi)$ are finite). The following result gives a case in which there is an infinite $\partial \mathcal{M}(\xi)$.

Lemma 45. *Suppose there is a subset $T \subset S$ such that:*

- (1) (W_T, T) decomposes as $(W_T, T) = (W_{T'} \times W_{T''}, T' \cup T'')$ with both $W_{T'}$ and $W_{T''}$ infinite; and
- (2) there are infinitely many distinct reflections r_i in $W_{T'}$ with corresponding walls M_i in X such that for all $i \neq j$, $\partial M_i \neq \partial M_j$.

Then there is a ξ such that $\partial\mathcal{M}(\xi)$ is infinite.

Proof. Let ξ be any point in the boundary of $X_{T''} \subset X_T$. Then the ∂M_i are infinitely many distinct elements of $\partial\mathcal{M}(\xi)$. \square

An example where (2) of Lemma 45 holds is $W_{T'}$ one-ended and word hyperbolic. Take the M_i to be an infinite family of pairwise disjoint walls. If $\partial M_i = \partial M_j$ then by Proposition 30, $X_{T'}$ contains a flat, which contradicts the hyperbolicity of $W_{T'}$.

Proposition 46. *W is word hyperbolic if and only if for all $\xi \in \partial_T X$ the equivalence class $C(\xi)$ is reduced to $\{\xi\}$.*

Proof. Assume W is word hyperbolic. Since W is quasi-isometric to the Davis complex X (equipped with its piecewise Euclidean metric), we have that X is δ -hyperbolic for some $\delta > 0$. Suppose that there are $\xi \neq \xi' \in \partial_T X$ with $\xi' \in C(\xi)$. Fix a basepoint $p \in X$. Then by Proposition 18 and Lemma 19, the geodesic rays $c = [p, \xi)$ and $c' = [p, \xi')$ cross the same set of walls.

For each wall M crossed by c (and thus by c'), denote by γ_M the geodesic segment in M which connects the points of intersection of c and c' with M , and let Δ_M be the geodesic triangle in X with one vertex at p , with γ_M its side not containing p , and with its other two sides initial subsegments of c and c' . Then by δ -hyperbolicity of X , the geodesic segment γ_M is contained in the δ -neighbourhood of the union of c and c' .

Since $\xi \neq \xi'$, the δ -neighbourhoods of c and c' have bounded intersection. Thus the geodesic segment $\gamma_M \subset M$ passes within bounded distance of p . By the local finiteness of the collection of walls, there can be only finitely many distinct walls M such that a geodesic segment contained in this wall is within bounded distance of p . But c (and thus c') crosses infinitely many walls M , so we obtain a contradiction. Thus c and c' are equivalent, and moreover their distance from each other is bounded in terms of δ . Hence $\xi = \xi'$.

Now assume that W is not word hyperbolic, and let W_T be an affine or reducible special subgroup which is an obstruction to the hyperbolicity of W , as in Theorem 41 above. If $C(\xi) = \{\xi\}$ then the closure $\overline{C(\xi)}$ equals $C(\xi)$, so by Proposition 18 and Theorem 23 above, it suffices to find infinite reduced words \mathbf{i} and \mathbf{j} such that $\mathbf{j} < \mathbf{i}$ (strict inequality) and $\partial_T X(\mathbf{i}) \neq \partial_T X(\mathbf{j})$. The existence of suitable pairs \mathbf{i} and \mathbf{j} in such W_T follows from our discussion of the affine and reducible cases above. \square

Summarising, we have the following result.

Theorem 47. *Let W be word hyperbolic. Then*

- (1) *For any infinite reduced word \mathbf{i} we have that $\partial_T X(\mathbf{i})$ consists of a single point $\xi = \xi(\mathbf{i})$.*
- (2) *Two infinite reduced words \mathbf{i} and \mathbf{j} are comparable only if \mathbf{i} and \mathbf{j} are in the same block.*
- (3) *The limit weak order restricted to a block $B(\mathbf{i})$ is isomorphic to the weak order of the (possibly trivial) finite Coxeter group $W(\xi(\mathbf{i}))$.*

Example 2. Let P be a regular dodecahedron in 3-dimensional (real) hyperbolic space with all dihedral angles right angles, and let W be the Coxeter group generated by reflections in the codimension one faces of P . Then W is word hyperbolic and the Davis complex X may be equipped with a piecewise hyperbolic metric so that X is isometric to the induced tessellation of hyperbolic 3-space by copies of P . This tessellation is depicted in Figure 2.

The Tits boundary $\partial_T X$ is a 2-dimensional sphere and the distance between any two points in $\partial_T X$ is infinite. Each wall of X is a hyperplane tessellated by right-angled pentagons, and may be viewed as a copy of the Davis complex from Example 1 in the introduction. Each $\xi \in \partial_T X$ is thus contained in the boundaries of at most 2 walls of X , with these walls perpendicular to each other.

Suppose \mathbf{i} is an infinite reduced word with $\xi = \xi(\mathbf{i})$. If ξ is in the boundary of two walls $M = M(r)$ and $M' = M'(r')$ then the block $B(\mathbf{i})$ consists of four equivalence classes of infinite reduced words,

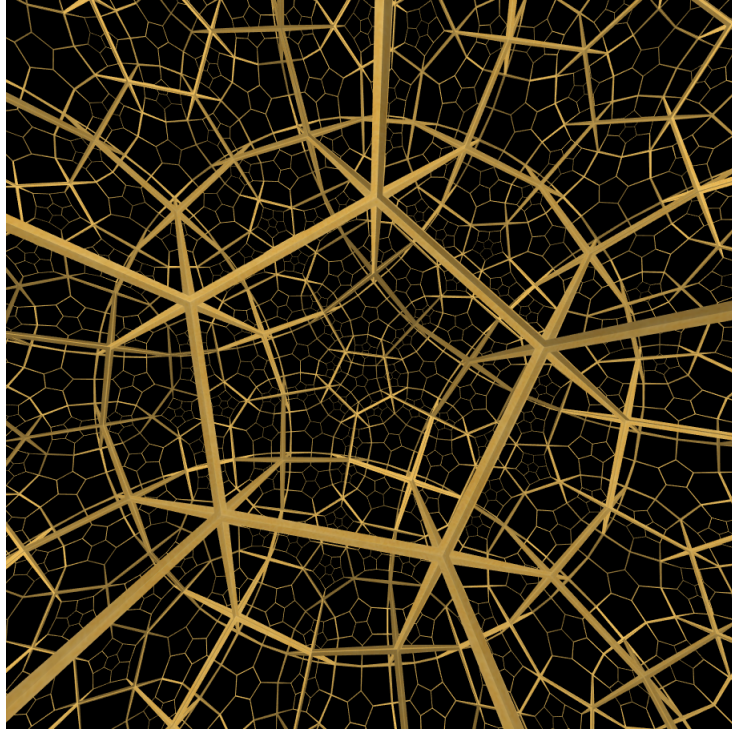


FIGURE 2. The Davis complex for Example 2

corresponding to the four elements of $W(\xi) = \langle r, r' \rangle \cong C_2 \times C_2$. If ξ is in the boundary of a unique wall $M = M(r)$ then the block $B(\mathbf{i})$ consists of two equivalence classes of infinite reduced words, corresponding to the two elements of the finite Coxeter group $W(\xi(\mathbf{i})) = \langle r \rangle$. If ξ is not in the boundary of any wall, then $B(\mathbf{i})$ contains a single equivalence class of infinite reduced words and $W(\xi(\mathbf{i}))$ is trivial.

6.6. Isolated flats. In this section we use results of Caprace [4] to discuss some W for which X is a space with isolated flats, in the sense studied by Hruska–Kleiner in [12]. In this section, a *flat* in X is defined to be a subset which is isometric to Euclidean space of dimension ≥ 2 .

Given $T \subset S$ and a chamber wK of X , the *residue of type T* containing wK is the subcomplex of X consisting of all chambers $w'K$ such that $w^{-1}w' \in W_T$. We denote by R_T the residue of type T containing the base chamber K . The stabiliser in W of the residue R_T is W_T , and for any $w \in W$ the stabiliser of wR_T is $W_T^w = wW_Tw^{-1}$. Therefore every parabolic subgroup of W stabilises some collection of residues.

By the construction and metrisation of X , there is a canonical isometric embedding of the Davis complex for (W_T, T) , denoted X_T , into R_T . (In [4], the sets X_T and R_T appear to have been identified, although in [5] and elsewhere a residue is defined as a union of chambers.) Hence for every $w \in W$, there is a canonical isometrically embedded copy of X_T in the residue wR_T . The action of W_T^w on wR_T preserves this copy of X_T .

Assume that W is not word hyperbolic. Then by Theorem 41 above, either there is some $T \subset S$ such that W_T is affine of rank ≥ 2 , or there are non-spherical $T', T'' \subset S$ such that $[T', T''] = 1$. If W_T is affine of rank ≥ 2 then X_T is isometric to Euclidean space of dimension ≥ 2 . Hence if W_T is affine of rank ≥ 2 , then every residue wR_T contains a canonical flat which is preserved by the action of the affine parabolic subgroup W_T^w .

Let \mathcal{T} be the collection of all maximal subsets $T \subset S$ such that W_T is affine (possibly reducible) and of rank ≥ 2 . We make the following additional assumptions on the set \mathcal{T} .

- (RH1) For each pair of irreducible non-spherical subsets $T', T'' \subset S$ such that $[T', T''] = 1$, there exists $T \in \mathcal{T}$ such that $T' \cup T'' \subset T$.
- (RH2) For all $T', T'' \in \mathcal{T}$ with $T' \neq T''$, the intersection $T' \cap T''$ is spherical.

Since W is not word hyperbolic, by (RH1) the set \mathcal{T} is nonempty.

As in the proof of Theorem A in [4], for each $T \in \mathcal{T}$, denote by $N(W_T)$ the normaliser of W_T in W . Now define \mathcal{R} to be the collection of all residues wR_T where w runs through a transversal for $N(W_T)$ in W . Note that we are considering representatives w for the cosets of $N(W_T)$ in W , not for the cosets of W_T in W , and so by Proposition 5.5 of [9] the set \mathcal{R} will *not* contain all residues of types $T \in \mathcal{T}$. Let \mathcal{F} be the collection of canonical flats contained in the residues in \mathcal{R} .

By Theorem A, Lemma 4.2 and Corollary D of [4], the Davis complex X then has *isolated flats*, meaning that the following conditions hold:

- (1) there is a constant $D < \infty$ such that every flat of X is in the D -neighbourhood of some element of \mathcal{F} ; and
- (2) for each $0 < \rho < \infty$, there is a constant $\kappa = \kappa(\rho) < \infty$ so that for any two distinct flats $F, F' \in \mathcal{F}$, the intersection of the ρ -neighbourhoods of F and F' has diameter $< \kappa$.

(Also, W is relatively hyperbolic with respect to its collection of maximal affine parabolic subgroups.)

By Theorem 1.2.1 of [12], since X has isolated flats the Tits boundary $\partial_T X$ is a disjoint union of isolated points and standard Euclidean spheres. Note that by condition (1) in the definition of isolated flats, for every flat F in X there is a flat $F' \in \mathcal{F}$ such that $\partial_T F \subset \partial_T F'$. By Theorem 5.2.5 of [12], the spherical components of $\partial_T X$ are precisely the Tits boundaries of the flats in \mathcal{F} . We now analyse these spherical components.

For any flat F in X , define $\mathcal{M}(F)$ to be the set of walls which separate points in F , and let $W(\mathcal{M}(F))$ be the subgroup of W generated by the set of reflections in walls in $\mathcal{M}(F)$. Then by Corollary 3.2 of [4], the group $P := \text{Pc}(W(\mathcal{M}(F)))$ is a direct product of irreducible affine Coxeter groups. (Here, $\text{Pc}(W(\mathcal{M}(F)))$ denotes the parabolic closure of the group $W(\mathcal{M}(F))$.) Moreover, by Proposition 3.3 of [4], the flat F is contained in some residue $R = wR_T$ whose stabiliser is P . Thus in particular if $F \in \mathcal{F}$ then by construction, F is contained in a unique residue $R \in \mathcal{R}$, and since the stabiliser of each residue in \mathcal{R} is a maximal affine parabolic subgroup of W we have that $P = \text{Pc}(W(\mathcal{M}(F)))$ is a maximal affine parabolic. Further, since P preserves R , the canonical flat $F \subset R$ intersects all of the walls corresponding to the reflections in P , and so $P = \text{Pc}(W(\mathcal{M}(F))) = W(\mathcal{M}(F))$.

Proposition 48. *Let R, R' be any two residues in \mathcal{R} which have stabiliser the same maximal affine parabolic P , and let F, F' be the corresponding flats in \mathcal{F} . Then:*

- (1) $F = F'$, hence $R = R'$ is the unique residue in \mathcal{R} which is stabilised by P ;
- (2) $\mathcal{M}(F) = \mathcal{M}(F')$ induce the same arrangement of boundaries of walls in $\partial_T F$; and
- (3) if M is any wall of X such that $M \notin \mathcal{M}(F)$ and ∂M intersects $\partial_T F$, then ∂M contains $\partial_T F$.

Proof. By condition (2) in the definition of isolated flats above, to show that $F = F'$ it suffices to show that for some $0 < \rho < \infty$, the intersection of the ρ -neighbourhoods of F and F' is unbounded. Using the same argument as in the last paragraph of the proof of Proposition 3.3 of [4], we may obtain R' from R by applying a sequence of reflections which commute with P . So we may without loss of generality assume that $R' = r.R$ for some reflection r which commutes with P . Fix a point $x_0 \in F$ and let $x'_0 = r.x_0 \in F'$. Choose a finite ρ so that $\rho > d(x_0, x'_0)$ and let w be an element of

infinite order in P . Then for each $n \geq 1$, we have

$$\rho > d(x_0, x'_0) = d(w^n . x'_0, w^n . x_0) = d(w^n . r . x_0, t^n . x_0) = d(r . (w^n . x_0), w^n . x_0).$$

Since the set $\{w^n . x_0 \mid n \geq 1\}$ is unbounded, we conclude that $F = F'$.

Part (2) of this result is a formal consequence of part (1). For (3), suppose M is a wall such that $M \notin \mathcal{M}(F)$, ∂M intersects $\partial_T F$ but ∂M does not contain $\partial_T F$. Then ∂M must separate two points say ξ and ξ' of the sphere $\partial_T F$. Let c and c' be geodesic rays from $x_0 \in R$ such that $c(\infty) = \xi$ and $c'(\infty) = \xi'$. Since $M \notin \mathcal{M}(F)$, the rays c and c' lie on the same side of M . But then their limit points ξ and ξ' must lie on the same side of ∂M , a contradiction. \square

Thus we have that for each maximal flat F in X there is a (unique) flat $F' \in \mathcal{F}$ such that $\partial_T F = \partial_T F'$ is a sphere component of the boundary $\partial_T X$, and the arrangement of boundaries of walls in this sphere component is precisely that induced by $\mathcal{M}(F)$.

On the other hand, if $\xi \in \partial_T X$ is an isolated point then $C(\xi) = \{\xi\}$, since the $C(\xi)$ are connected, and we can say little else.

REFERENCES

- [1] A. BJÖRNER AND F. BRENTI, *Combinatorics of Coxeter groups*, vol. 231 of Graduate Texts in Mathematics, Springer, New York, 2005.
- [2] M. R. BRIDSON AND A. HAEFLIGER, *Metric spaces of non-positive curvature*, vol. 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1999.
- [3] B. BRINK AND R. B. HOWLETT, *A finiteness property and an automatic structure for Coxeter groups*, Math. Ann., 296 (1993), pp. 179–190.
- [4] P.-E. CAPRACE, *Buildings with isolated subspaces and relatively hyperbolic Coxeter groups*, Innov. Incidence Geom., 10 (2009), pp. 15–31.
- [5] P.-E. CAPRACE AND F. HAGLUND, *On geometric flats in the CAT(0) realization of Coxeter groups and Tits buildings*, Canad. J. Math., 61 (2009), pp. 740–761.
- [6] P.-E. CAPRACE AND J. LÉCUREUX, *Combinatorial and group-theoretic compactifications of buildings*, Ann. Inst. Fourier (Grenoble), 61 (2011), pp. 619–672.
- [7] P. CELLINI AND P. PAPI, *The structure of total reflection orders in affine root systems*, J. Algebra, 205 (1998), pp. 207–226.
- [8] M. W. DAVIS, *The geometry and topology of Coxeter groups*, vol. 32 of London Mathematical Society Monographs Series, Princeton University Press, Princeton, NJ, 2008.
- [9] V. V. DEODHAR, *On the root system of a Coxeter group*, Comm. Algebra, 10 (1982), pp. 611–630.
- [10] ———, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. (Basel), 53 (1989), pp. 543–546.
- [11] M. DYER, *Reflection subgroups of Coxeter systems*, J. Algebra, 135 (1990), pp. 57–73.
- [12] G. C. HRUSKA AND B. KLEINER, *Hadamard spaces with isolated flats*, Geom. Topol., 9 (2005), pp. 1501–1538. With an appendix by the authors and Mohamad Hindawi.
- [13] K. ITO, *Parameterizations of infinite biconvex sets in affine root systems*, Hiroshima Math. J., 35 (2005), pp. 425–451.
- [14] T. LAM AND P. PYLYAVSKYY, *Total positivity in loop groups II: Chevalley generators*, Transformation Groups, to appear.
- [15] J.-P. SERRE, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

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