Palindromic automorphisms of right-angled Artin groups

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Abstract

We introduce the palindromic automorphism group and the palindromic Torelli group of a right-angled Artin group A_Γ. The palindromic automorphism group ΠA_Γ is related to the principal congruence subgroups of $\mathrm{GL}(n,\mathbb{Z})$ and to the hyperelliptic mapping class group of an oriented surface, and sits inside the centraliser of a certain hyperelliptic involution in $\mathrm{Aut}(A_\Gamma)$. We obtain finite generating sets for ΠA_Γ and for this centraliser, and determine precisely when these two groups coincide. We also find generators for the palindromic Torelli group.

1 Introduction

Let Γ be a finite simplicial graph, with vertex set $V = \{v_1, \ldots, v_n\}$. Let $E \subset V \times V$ be the edge set of Γ. The graph Γ defines the *right-angled Artin group* A_Γ via the presentation

$$A_\Gamma = \langle v_i \in V \mid [v_i, v_j] = 1 \text{ iff } (v_i, v_j) \in E \rangle.$$

One motivation, among many, for studying right-angled Artin groups and their automorphisms (see Agol [1] and Charney [3] for others) is that the groups A_Γ and $\mathrm{Aut}(A_\Gamma)$ allow us to interpolate between families of groups that are classically well-studied: we may pass between the free group F_n and free abelian group $\mathbb{Z}^n$, between their automorphism groups $\mathrm{Aut}(F_n)$ and $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}(n,\mathbb{Z})$, and even between the mapping class group $\mathrm{Mod}(S_g)$ of the oriented surface S_g of genus g and the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ (this last interpolation is explained in [8]). See Section 2 for background on right-angled Artin groups and their automorphisms.

In this paper, we introduce a new subgroup of $\mathrm{Aut}(A_\Gamma)$ consisting of so-called 'palindromic' automorphisms of A_Γ, which allows us a further interpolation, between certain previously well-studied subgroups of $\mathrm{Aut}(F_n)$ and of $\mathrm{GL}(n,\mathbb{Z})$. An automorphism $\alpha \in \mathrm{Aut}(A_\Gamma)$ is said to be *palindromic* if $\alpha(v) \in A_\Gamma$ is a palindrome for each $v \in V$; that is, each $\alpha(v)$ may be expressed as a word $u_1 \ldots u_k$ on $V^{\pm 1}$ such that $u_1 \ldots u_k$ and its reverse $u_k \ldots u_1$ are identical as words. The collection ΠA_Γ of palindromic automorphisms is, a priori, only a subset of $\mathrm{Aut}(A_\Gamma)$. While it is easy to see that ΠA_Γ is closed under composition, it is not obvious that it is closed under inversion. In Corollary 3.5 we prove that ΠA_Γ is in fact a subgroup of $\mathrm{Aut}(A_\Gamma)$. We thus refer to ΠA_Γ as the *palindromic automorphism group of* A_Γ.

When A_Γ is free, the group ΠA_Γ is equal to the palindromic automorphism group ΠA_n of F_n, which was introduced by Collins [5]. Collins proved that ΠA_n is finitely presented and provided an explicit finite presentation. The group ΠA_n has also been studied by Glover–Jensen [10], who showed, for instance, that it has virtual cohomological dimension $n-1$. At the other extreme, when A_Γ is free abelian, the group ΠA_Γ is the principal level 2 congruence subgroup $\Lambda_n[2]$ of $\mathrm{GL}(n,\mathbb{Z})$. Thus ΠA_Γ enables us to interpolate between these two classes of groups.

Let ι be the automorphism of A_Γ that inverts each $v \in V$. In the case that A_Γ is free, it is easy to verify that the palindromic automorphism group $\Pi A_\Gamma = \Pi A_n$ is equal to the centraliser $C_\Gamma(\iota)$ of ι in $\mathrm{Aut}(A_\Gamma)$ (hence ΠA_n is a group). For a general A_Γ, we prove that ΠA_Γ is a finite index subgroup of $C_\Gamma(\iota)$, by first considering the finite index subgroup of ΠA_Γ consisting of 'pure' palindromic automorphisms; see Theorem 3.3 and Corollary 3.5. The index of ΠA_Γ in $C_\Gamma(\iota)$ depends entirely on connectivity properties of the graph Γ, and we give conditions on Γ that are equivalent to the groups ΠA_Γ and $C_\Gamma(\iota)$ being equal, in Proposition 3.6. In particular, there are non-free A_Γ such that $\Pi A_\Gamma = C_\Gamma(\iota)$.

The order 2 automorphism ι is the obvious analogue in $\mathrm{Aut}(A_\Gamma)$ of the hyperelliptic involution s of an oriented surface S_g, since ι and s act as $-I$ on $H_1(A_\Gamma,\mathbb{Z})$ and $H_1(S_g,\mathbb{Z})$, respectively. The group ΠA_Γ also allows us to generalise a comparison made by the first author in [9, Section 1] between $\Pi A_n \leq \mathrm{Aut}(F_n)$ and the centraliser in $\mathrm{Mod}(S_g)$ of the hyperelliptic involution s, which demonstrated a deep connection between these groups. Our study of ΠA_Γ is thus motivated by its appearance in both algebraic and geometric settings.

The main result of this paper finds a finite generating set for ΠA_Γ. Our generating set includes the so-called *diagram automorphisms* of A_Γ, which are induced by graph symmetries of Γ, and the *inversions* $\iota_j \in \mathrm{Aut}(A_\Gamma)$, with ι_j mapping v_j to ${v_j}^{-1}$ and fixing every $v_k \in V \setminus \{v_j\}$. The function $P_{ij} : V \to A_\Gamma$ sending v_i to $v_j v_i v_j$ and v_k to v_k $(k \neq i)$ induces a well-defined automorphism of A_Γ, also denoted P_{ij}, whenever certain connectivity properties of Γ hold (see Section 3.2). We establish that these three types of palindromic automorphisms suffice to generate ΠA_Γ.

Theorem A. *The group ΠA_Γ is generated by the finite set of diagram automorphisms, inversions and well-defined automorphisms P_{ij}.*

We also obtain a finite generating set for the centraliser $C_\Gamma(\iota)$, in Corollary 3.8, by combining the generating set given by Theorem A with a short exact sequence involving $C_\Gamma(\iota)$ and the pure palindromic automorphism group (see Theorem 3.3). Our generating set for $C_\Gamma(\iota)$ consists of the generators of ΠA_Γ, along with all well-defined automorphisms of A_Γ that map v_i to $v_i v_j$ and fix every $v_k \in V \setminus \{v_i\}$, for some $i \neq j$ with $[v_i, v_j] = 1$ in A_Γ.

Further, for any re-indexing of the vertex set V and each $k = 1, \ldots, n$, we provide a finite generating set for the subgroup $\Pi A_\Gamma(k)$ of ΠA_Γ which fixes the vertices $v_1, \ldots, v_k$, as recorded in Theorem 3.11. The so-called *partial basis complex* of A_Γ, which is an analogue of the curve complex, has as its vertices (conjugacy classes of) the images of members of V under automorphisms of $\mathrm{Aut}(A_\Gamma)$. This complex has not, to our knowledge, appeared in the literature, but its definition is an easy generalisation of the free group version introduced by Day–Putman [6] in order to generate the Torelli subgroup of $\mathrm{Aut}(F_n)$. A 'palindromic' partial basis complex was also used in [9] to approach the study of palindromic automorphisms of F_n. Theorem 3.11 is thus a first step towards understanding stabilisers of simplices in the palindromic partial basis complex of A_Γ.

We prove Theorem A and our other finite generation results in Section 3, using machinery developed by Laurence [16] for his proof that $\mathrm{Aut}(A_\Gamma)$ is finitely generated. The added constraint for us that our automorphisms be expressed as a product of *palindromic* generators forces a more delicate treatment. In addition, our proof uses Servatius' Centraliser Theorem [18], and a generalisation to A_Γ of arguments used by Collins [5, Proposition 2.2] to generate $\Pi\mathrm{A}_n$. Throughout this paper, we employ a decomposition into block matrices of the image of $\mathrm{Aut}(A_\Gamma)$ in $\mathrm{GL}(n,\mathbb{Z})$ under the canonical map induced by abelianising A_Γ; this decomposition was observed by Day [7] and by Wade [19].

We also in this work introduce the *palindromic Torelli group* $\mathcal{PI}_\Gamma$ *of* A_Γ, which we define to consist of the palindromic automorphisms of A_Γ that induce the identity automorphism on $H_1(A_\Gamma) = \mathbb{Z}^n$. The group $\mathcal{PI}_\Gamma$ is the right-angled Artin group analogue of the hyperelliptic Torelli group $\mathcal{SI}_g$ of an oriented surface S_g, which has applications to Burau kernels of braid groups [2] and to the Torelli space quotient of the Teichmüller space of S_g [12]. Analogues of these objects exist for right-angled Artin groups (see, for example, [4]), but are not yet well-developed. We expect that the palindromic Torelli group will play a role in determining their structure.

Even in the free group case, where $\mathcal{PI}_\Gamma$ is denoted by $\mathcal{PI}_n$, little seems to be known about the palindromic Torelli group. Collins [5] observed that $\mathcal{PI}_n$ is non-trivial, and Jensen–McCammond–Meier [14, Corollary 6.3] proved that $\mathcal{PI}_n$ is not homologically finite if $n \geq 3$. An infinite generating set for $\mathcal{PI}_n$ was obtained recently in [9, Theorem A], and this is made up of so-called *doubled commutator transvections* and *separating π-twists*. In Section 4 we recall and then generalise the definitions of these two classes of free group automorphisms, to give two classes of palindromic automorphisms of a general A_Γ, which we refer to by the same names. As a first step towards understanding the structure of $\mathcal{PI}_\Gamma$, we obtain an explicit generating set as follows.

Theorem B. *The group $\mathcal{PI}_\Gamma$ is generated by the set of all well-defined doubled commutator transvections and separating π-twists in $\Pi\mathrm{A}_\Gamma$.*

The generating set we obtain in Theorem B compares favourably with the generators obtained in [9] in the case that A_Γ is free. Specifically, the generators given by Theorem B are the images in $\mathrm{Aut}(A_\Gamma)$ of those generators of $\mathcal{PI}_n$ that descend to well-defined automorphisms of A_Γ (viewing A_Γ as a quotient of the free group F_n on the set V).

The proof of Theorem B in Section 4 combines our results from Section 3 with results for $\mathcal{PI}_n$ from [9]. More precisely, as a key step towards the proof of Theorem A, we find a finite generating set for the pure palindromic subgroup of $\Pi\mathrm{A}_\Gamma$ (Theorem 3.7). We then use these generators to determine a finite presentation for the image Θ of this subgroup under the canonical map $\mathrm{Aut}(A_\Gamma) \to \mathrm{GL}(n,\mathbb{Z})$ (Theorem 4.2). In order to find this finite presentation for $\Theta \leq \mathrm{GL}(n,\mathbb{Z})$, we also need Corollary 1.1 from [9], which leverages the generating set for $\mathcal{PI}_n$ from [9] to obtain a finite presentation for the principal level 2 congruence subgroup $\Lambda_n[2] \leq \mathrm{GL}(n,\mathbb{Z})$. Finally, using a standard argument, we lift the relators of Θ to obtain a normal generating set for $\mathcal{PI}_\Gamma$.

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2 Preliminaries

In this section we give definitions and some brief background on right-angled Artin groups and their automorphisms. Throughout this section and the rest of the paper, we continue to use the notation introduced in Section 1. We will also frequently use $v_i \in V$ to denote both a vertex of the graph Γ and a generator of A_Γ, and when discussing a single generator we may omit the index i. Section 2.1 recalls definitions related to the graph Γ and Section 2.2 recalls some useful combinatorial results about words in the group A_Γ. In Section 2.3 we recall a finite generating set for $\mathrm{Aut}(A_\Gamma)$ and some important subgroups of $\mathrm{Aut}(A_\Gamma)$, and in Section 2.4 we recall a matrix block decomposition for the image of $\mathrm{Aut}(A_\Gamma)$ in $\mathrm{GL}(n, \mathbb{Z})$.

2.1 Graph-theoretic notions

We briefly recall some graph-theoretic definitions, in particular the domination relation on vertices of Γ.

The *link* of a vertex $v \in V$, denoted $\mathrm{lk}(v)$, consists of all vertices adjacent to v, and the *star* of $v \in V$, denoted $\mathrm{st}(v)$, is defined to be $\mathrm{lk}(v) \cup \{v\}$. We define a relation $\leq$ on V, with $u \leq v$ if and only if $\mathrm{lk}(u) \subset \mathrm{st}(v)$. In this case, we say v *dominates* u, and refer to $\leq$ as the *domination* relation [15], [16]. Figure 1 demonstrates the link of one vertex being contained in the star of another. Note that when $u \leq v$, the vertices u and v may be adjacent in Γ, but need not be. To distinguish these two cases, we will refer to *adjacent* and *non-adjacent* domination.

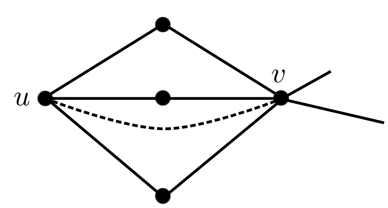

Figure 1: *An example of a vertex u being dominated by a vertex v. The dashed edge is meant to emphasise that u and v may be adjacent, but need not be.*

Domination in the graph Γ may be used to define an equivalence relation $\sim$ on the vertex set V, as follows. We say $v_i \sim v_j$ if and only if $v_i \leq v_j$ and $v_j \leq v_i$, and write $[v_i]$ for the equivalence class of $v_i \in V$ under $\sim$. We also define an equivalence relation $\sim'$ by $v_i \sim' v_j$ if and only if $[v_i] = [v_j]$ and $v_i v_j = v_j v_i$, writing $[v_i]'$ for the equivalence class of $v_i \in V$ under $\sim'$. We refer to $[v_i]$ as the *domination class of* v_i and to $[v_i]'$ as the *adjacent domination class of* v_i. Note that the vertices in $[v_i]$ necessarily span either an edgeless or a complete subgraph of Γ; in the former case, we will call $[v_i]$ a *free domination class*, while in the latter, where $[v_i] = [v_i]'$, we will call $[v_i]$ an *abelian domination class.*

2.2 Word combinatorics in right-angled Artin groups

In this section we recall some useful properties of words on $V^{\pm 1}$, which give us a measure of control over how we express group elements of A_Γ. We include the statement of Servatius' Centraliser Theorem [18] and of a useful proposition of Laurence from [16].

First, a word on $V^{\pm 1}$ is *reduced* if there is no shorter word representing the same element of A_Γ. Unless otherwise stated, we shall always use reduced words when representing members of A_Γ. Now let w and w' be words on $V^{\pm 1}$. We say that w and w' are *shuffle-equivalent* if we can obtain one from the other via repeatedly exchanging subwords of the form uv for vu when u and v are adjacent vertices in Γ. Hermiller–Meier [13] proved that two reduced words w and w' are equal in A_Γ if and only if w and w' are shuffle-equivalent, and also showed that any word can be made reduced by a sequence of these shuffles and cancellations of subwords of the form $u^\epsilon u^{-\epsilon}$ ($u \in V$, $\epsilon \in \{\pm 1\}$). This allows us to define the *length* of a group element $w \in A_\Gamma$ to be the number of letters in a reduced word representing w, and the *support* of $w \in A_\Gamma$, denoted $\mathrm{supp}(w)$, to be the set of vertices $v \in V$ such that v or v^{-1} appears in a reduced word representing w. We say $w \in A_\Gamma$ is *cyclically reduced* if it cannot be written in reduced form as $vw'v^{-1}$, for some $v \in V^{\pm 1}$, $w' \in A_\Gamma$.

Servatius [18, Section III] analysed centralisers of elements in arbitrary A_Γ, showing that the centraliser of any $w \in A_\Gamma$ is again a (well-defined) right-angled Artin group, say A_Δ. Laurence [16] defined the *rank* of $w \in A_\Gamma$ to be the number of vertices in the graph Δ defining A_Δ. We denote the rank of $w \in A_\Gamma$ by $\mathrm{rk}(w)$.

In order to state his theorem on centralisers in A_Γ, Servatius [18] introduced a canonical form for any cyclically reduced $w \in A_\Gamma$, which Laurence [16] calls a *basic form* of w. For this, partition the support of w into its connected components in Γ^c, the complement graph of Γ, writing

$$\mathrm{supp}(w) = V_1 \sqcup \cdots \sqcup V_k,$$

where each V_i is such a connected component. Then we write

$$w = {w_1}^{r_1} \dots {w_k}^{r_k},$$

where each $r_i \in \mathbb{Z}$ and each $w_i \in \langle V_i \rangle$ is not a proper power in A_Γ (that is, each $|r_i|$ is maximal). Note that by construction, $[w_i, w_j] = 1$ for $1 \le i < j \le k$. Thus the basic form of w is unique up to permuting the order of the w_i, and shuffling within each w_i. With this terminology in place, we now state Servatius' 'Centraliser Theorem' for later use.

Theorem 2.1 (Servatius, [18])**.** *Let w be a cyclically-reduced word on $V^{\pm 1}$ representing an element of A_Γ. Writing $w = {w_1}^{r_1} \dots {w_k}^{r_k}$ in basic form, the centraliser of w in A_Γ is isomorphic to*

$$\langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times \langle \mathrm{lk}(w) \rangle,$$

where $\mathrm{lk}(w)$ *denotes the subset of V of vertices which are adjacent to each vertex in* $\mathrm{supp}(w)$*.*

We will also make frequent use of the following result, due to Laurence [16], and so state it now for reference.

Proposition 2.2 (Proposition 3.5, Laurence [16])**.** *Let $w \in A_\Gamma$ be cyclically reduced, and write $w = {w_1}^{r_1} \dots {w_k}^{r_k}$ in basic form, with $V_i := \mathrm{supp}(w_i)$. Then:*

1. $\mathrm{rk}(v) \geq \mathrm{rk}(w)$ *for all* $v \in \mathrm{supp}(w)$*; and*

2. if $\mathrm{rk}(v) = \mathrm{rk}(w)$ *for some* $v \in V_i$*, then:*

   (a) $v \leq u$ *for all* $u \in \mathrm{supp}(w)$*;*

   (b) each V_j *is a singleton* ($j \neq i$)*; and*

   (c) v *does not commute with any vertex of* $V_i \setminus \{v\}$*.*

Recall that a *clique* in a graph Γ is a complete subgraph. If Δ is a clique in Γ then A_Δ is free abelian of rank equal to the number of vertices of Δ, so any word supported on Δ can be written in only finitely many reduced ways. The set of cliques in Δ is partially ordered by inclusion, giving rise to the notion of a maximal clique in a graph Γ.

2.3 Automorphisms of right-angled Artin groups

In this section we recall a finite generating set for $\mathrm{Aut}(A_\Gamma)$. This generating set was obtained by Laurence [16], confirming a conjecture of Servatius [18], who had verified that the set generates $\mathrm{Aut}(A_\Gamma)$ in certain special cases.

In the following list, the action of each generator of $\mathrm{Aut}(A_\Gamma)$ is given on $v \in V$, with the convention that if a vertex is omitted from discussion, it is fixed by the automorphism. There are four types of generators:

1. *Diagram automorphisms* ϕ: each $\phi \in \mathrm{Aut}(\Gamma)$ induces an automorphism of A_Γ, which we also denote by ϕ, mapping $v \in V$ to $\phi(v)$.
2. *Inversions* ι_j: for each $v_j \in V$, ι_j maps v_j to ${v_j}^{-1}$.
3. *Dominated transvections* τ_{ij}: for $v_i, v_j \in V$, whenever v_i is dominated by v_j, there is an automorphism τ_{ij} mapping v_i to $v_i v_j$. We refer to a (well-defined) dominated transvection τ_{ij} as an *adjacent transvection* if $[v_i, v_j] = 1$; otherwise, we say τ_{ij} is a *non-adjacent transvection*.
4. *Partial conjugations* $\gamma_{i,D}$: fix $v_i \in V$, and select a connected component D of $\Gamma \setminus \mathrm{st}(v_i)$ (see Figure 2). The partial conjugation $\gamma_{v_i,D}$ maps every $d \in D$ to $v_i d {v_i}^{-1}$.

We denote by D_Γ, I_Γ and $\mathrm{PC}(A_\Gamma)$ the subgroups of $\mathrm{Aut}(A_\Gamma)$ generated by diagram automorphisms, inversions and partial conjugations, respectively, and by $\mathrm{Aut}^0(A_\Gamma)$ the subgroup of $\mathrm{Aut}(A_\Gamma)$ generated by all inversions, dominated transvections and partial conjugations.

2.4 A matrix block decomposition

Now we recall a useful decomposition into block matrices of an image of $\mathrm{Aut}(A_\Gamma)$ inside $\mathrm{GL}(n, \mathbb{Z})$. This decomposition was observed by Day [7] and by Wade [19].

Let $\Phi : \mathrm{Aut}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z})$ be the canonical homomorphism induced by abelianising A_Γ. Note that since D_Γ normalises $\mathrm{Aut}^0(A_\Gamma)$, any $\phi \in \mathrm{Aut}(A_\Gamma)$ may be written (non-uniquely, in general), as $\phi = \delta\beta$, where $\delta \in D_\Gamma$ and $\beta \in \mathrm{Aut}^0(A_\Gamma)$.

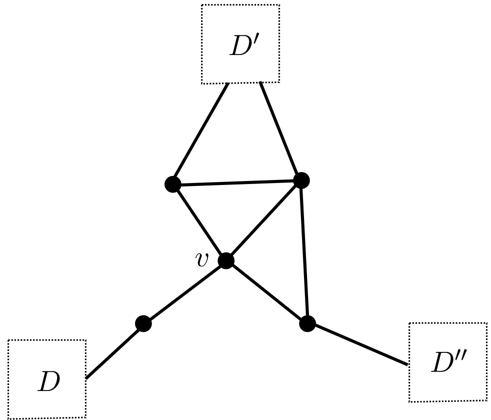

Figure 2: *When we remove the star of v, we leave three connected components D, D' and D''.*

By ordering the vertices of Γ appropriately, matrices in $\Phi(\mathrm{Aut}^0(A_\Gamma)) \leq \mathrm{GL}(n, \mathbb{Z})$ will have a particularly tractable lower block-triangular decomposition, which we now describe. The domination relation $\leq$ on V descends to a partial order, also denoted $\leq$, on the set of domination classes $V/\sim$, which we (arbitrarily) extend to a total order,

$$[u_1] < \cdots < [u_k]$$

where $[u_i] \in V/\sim$. This total order may be lifted back up to V by specifying an arbitrary total order on each domination class $[u_i] \in V/\sim$. We reindex the vertices of Γ if necessary so that the ordering $v_1, v_2, \ldots, v_n$ is this specified total order on V. Let n_i denote the size of the domination class $[u_i] \in V/\sim$. Under this ordering, any matrix $M \in \Phi(\mathrm{Aut}^0(A_\Gamma))$ has block decomposition:

$$\begin{pmatrix} M_1 & 0 & 0 & \ldots & 0 \\ * & M_2 & 0 & \ldots & 0 \\ * & * & M_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & M_k \end{pmatrix},$$

where $M_i \in \mathrm{GL}(n_i, \mathbb{Z})$ and the (i, j) block $*$ $(j < i)$ may only be non-zero if u_j is dominated by u_i in Γ. This triangular decomposition becomes apparent when the images of the generators of $\mathrm{Aut}^0(A_\Gamma)$ are considered inside $\mathrm{GL}(n, \mathbb{Z})$. The diagonal blocks may be any $M_i \in \mathrm{GL}(n_i, \mathbb{Z})$, as by definition each domination class gives rise to all $n_i(n_i - 1)$ transvections in $\mathrm{GL}(n_i, \mathbb{Z})$, which, together with the appropriate inversions, generate $\mathrm{GL}(n_i, \mathbb{Z})$. A diagonal block corresponding to a free domination class will also be called *free*, and a diagonal block corresponding to an abelian domination class will be called *abelian*.

This block decomposition descends to an analogous decomposition of the image of $\mathrm{Aut}^0(A_\Gamma)$ under the canonical map Φ_2 to $\mathrm{GL}(n, \mathbb{Z}/2)$, as this map factors through the homomorphism $\mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z}/2)$ that reduces matrix entries mod 2.

3 Palindromic automorphisms

Our main goal in this section is to prove Theorem A, which gives a finite generating set for the group of palindromic automorphisms ΠA_Γ. First of all, in Section 3.1, we derive a normal form for group elements $\alpha(v) \in A_\Gamma$ where $v \in V$ and α lies in the centraliser $C_\Gamma(\iota)$. In Section 3.2 we introduce the pure palindromic automorphisms $\mathrm{P\Pi A}_\Gamma$, and prove that $\mathrm{P\Pi A}_\Gamma$ is a group by showing that it is a kernel inside $C_\Gamma(\iota)$. We then show that $\Pi\mathrm{A}_\Gamma$ is a group, and determine when the groups $C_\Gamma(\iota)$ and $\Pi\mathrm{A}_\Gamma$ are equal. The proof of Theorem A is carried out in Section 3.3, where the main step is to find a finite generating set for $\mathrm{P\Pi A}_\Gamma$. We also provide finite generating sets for $C_\Gamma(\iota)$ and for certain stabiliser subgroups of $\Pi\mathrm{A}_\Gamma$.

3.1 The centraliser $\mathbf{C}_\Gamma(\iota)$ and a clique-palindromic normal form

In this section we prove Proposition 3.1, which provides a normal form for reduced words $w = u_1 \dots u_k$ ($u_i \in V^{\pm 1}$) that are equal (in the group A_Γ) to their *reverse*,

$$w^{\mathrm{rev}} := u_k \dots u_1.$$

We then in Corollary 3.2 derive implications for the diagonal blocks in the matrix decomposition discussed in Section 2.4. The results of this section will be used in Section 3.2 below.

Green, in her thesis [11], established a normal form for elements of A_Γ, by iterating an algorithm that takes a word w_0 on $V^{\pm 1}$ and rewrites it as $w_0 = p w_1$ in A_Γ, where p is a word consisting of all the letters of w_0 that may be shuffled (as in Section 2.2) to be the initial letter of w_0, and w_1 is the word remaining after shuffling each of these letters into the initial segment p. We now use a similar idea for palindromes.

Let ι denote the automorphism of A_Γ that inverts each $v \in V$. We refer to ι as the *(preferred) hyperelliptic involution of* A_Γ. Denote by $C_\Gamma(\iota)$ the centraliser in $\mathrm{Aut}(A_\Gamma)$ of ι. Note that this centraliser is far from trivial: it contains all diagram automorphisms, inversions and adjacent transvections in $\mathrm{Aut}(A_\Gamma)$, and also contains all palindromic automorphisms. The following proposition gives a normal form for the image of $v \in V$ under the action of some $\alpha \in C_\Gamma(\iota)$.

Proposition 3.1 (Clique-palindromic normal form)**.** *Let $\alpha \in C_\Gamma(\iota)$ and $v \in V$. Then we may write*

$$\alpha(v) = w_1 \dots w_{k-1} w_k w_{k-1} \dots w_1,$$

where w_i is a word supported on a clique in Γ ($1 \leq i \leq k$), and if $k \geq 3$ then $[w_i, w_{i+1}] \neq 1$ ($1 \leq i \leq k-2$). Moreover, this expression for $\alpha(v)$ is unique up to the finitely many rewritings of each word w_i in A_Γ.

We refer to this normal form as *clique-palindromic* because the words under consideration, while equal to their reverses in the group A_Γ as genuine palindromes are, need only be palindromic 'up to cliques', as in the expression in the statement of the proposition.

Proof. Suppose $\alpha \in C_\Gamma(\iota)$ and $v \in V$. Write $\alpha(v) = u_1 \dots u_r$ in reduced form, where each u_i is in $V^{\pm 1}$. Since $\alpha\iota(v) = \iota\alpha(v)$, we have that

$$u_1 \dots u_r = u_r \dots u_1 \tag{1}$$

in A_Γ. If $\alpha(v)$ is supported on a clique, then there is nothing to show. Otherwise, put $A_1 = \alpha(v)$ and let Z_1 be the (possibly empty) subset of V consisting of the vertices in supp(A_1) which commute with every vertex in supp(A_1). We note that Z_1 is supported on a clique, and that Z_1 is, by assumption, a proper subset of supp(A_1).

We now rewrite $A_1 = u_1 \dots u_r$ as $w_1 u_1{}' \dots u_s{}'$, where $u_j{}' \in V^{\pm 1}$ $(1 \leq j \leq s)$, and $w_1 \in A_\Gamma$ is the word consisting of all the u_i which are not in $Z_1^{\pm 1}$ and which may be shuffled to the start of $u_1 \dots u_r$. That is, w_1 consists of all letters $u_i \notin Z_1^{\pm 1}$ so that if $i \geq 1$, the letter u_i commutes with each of $u_1, \dots, u_{i-1}$. Notice that w_1 is nonempty since the first u_i which is not in Z_1 will be in w_1. By construction, w_1 is supported on a clique in Γ.

Now any u_i that may be shuffled to the start of $u_1 \dots u_r$ may also be shuffled to the end of $u_r \dots u_1$, by (1). Hence we may also rewrite A_1 as $u_1'' \dots u_s'' w_1$ for the same word w_1. Since the support of w_1 is disjoint from Z_1, the letters of A_1 used in the copy of w_1 at the start of $w_1 u_1{}' \dots u_s{}'$ are disjoint from the letters of A_1 used in the copy of w_1 at the end of $u_1'' \dots u_s'' w_1$. We thus obtain that

$$A_1 = \alpha(v) = w_1 u_1{}'' \dots u_t{}'' w_1$$

in A_Γ, with $u_i{}'' \in V^{\pm 1}$. Since $\alpha\iota(v) = \iota\alpha(v)$, it must be the case that $u_1{}'' \dots u_t{}'' = u_t{}'' \dots u_1{}''$ in A_Γ.

Now put $A_2 = u_1{}'' \dots u_t{}''$, so that $A_1 = w_1 A_2 w_1$. Note that supp(A_2) contains Z_1. If A_2 is supported on a clique, for example if supp(A_2) $= Z_1$, then we put $w_2 = A_2$ and are done. (In this case, supp(A_2) $= Z_1$ if and only if w_1 and w_2 commute.) If A_2 is not supported on a clique, we define Z_2 to be the vertices in supp(A_2) which commute with the entire support of A_2, and iterate the process described above. Since each word w_i constructed by this process is nonempty, the word A_{i+1} is shorter than A_i, hence the process terminates after finitely many steps. Notice also that $Z_1 \subseteq Z_2 \subseteq \dots \subseteq Z_i \subseteq \text{supp}(A_{i+1})$, so any letters of A_i which lie in Z_i become part of the word A_{i+1}. In particular, any letter of $A_1 = \alpha(v)$ which is in some Z_i, for example a letter in $Z(A_\Gamma)$, will end up in the word w_k when the process terminates.

By construction, each w_i is supported on a clique in Γ. Now the word A_{i+1} is not supported on a clique if and only if a further iteration is needed, which occurs if and only if $i \leq k-2$. In this case, Z_i must be a proper subset of supp(A_{i+1}) and so w_{i+1} does not commute with w_i (the word w_k may or may not commute with w_{k-1}). Thus the expression obtained for $\alpha(v)$ when this process terminates is as in the statement of the proposition. Moreover, this expression is unique up to rewriting each of the w_i, as they were defined in a canonical manner. This completes the proof. □

This normal form gives us the following corollary regarding the structure of diagonal blocks in the lower block-triangular decomposition of the image of $\alpha \in C_\Gamma(\iota)$ under the canonical map $\Phi : \text{Aut}(A_\Gamma) \to \text{GL}(n, \mathbb{Z})$, discussed in Section 2.4. Recall that $\Lambda_k[2]$ denotes the principal level 2 congruence subgroup of $\text{GL}(k, \mathbb{Z})$.

Corollary 3.2. *Write $\alpha \in C_\Gamma(\iota)$ as $\alpha = \delta\beta$, for some $\beta \in \text{Aut}^0(A_\Gamma)$ and $\delta \in D_\Gamma$. Let M be the matrix appearing in a diagonal block of rank k in the lower block-triangular decomposition of $\Phi(\beta) \in \text{GL}(n, \mathbb{Z})$. Then:*

1. *if the diagonal block is abelian, then M may be any matrix in $\text{GL}(k, \mathbb{Z})$; and*

2. if the diagonal block is free then M must lie in $\Lambda_k[2]$, up to permuting columns.

Proof. First, note that since $D_\Gamma \leq C_\Gamma(\iota)$, we must have that $\beta \in C_\Gamma(\iota)$. We deal with the abelian block case first. The group $C_\Gamma(\iota) \cap \mathrm{Aut}^0(A_\Gamma)$ contains all the adjacent transvections and inversions necessary to generate $\mathrm{GL}(k, \mathbb{Z})$ under Φ, so the matrix M in this diagonal block may be any member of $\mathrm{GL}(k, \mathbb{Z})$.

Now, suppose that the diagonal block is free. Suppose the column of M corresponding to $v \in V$ contains two odd entries, in turn corresponding to vertices $u_1, u_2 \in [v]$, say. This implies that $\beta(v)$ has odd exponent sum of u_1 and of u_2. Use Proposition 3.1 to write

$$\beta(v) = w_1 \dots w_k \dots w_1$$

in normal form, with each $w_i \in A_\Gamma$ being supported on some clique in Γ. It must be the case that w_k has odd exponent sum of u_1 and of u_2, since all other w_i $(i \neq k)$ appear twice in the normal form expression. Thus u_1 and u_2 commute. This contradicts the assumption that the diagonal block is free, so there must be precisely one odd entry in each column of M. Hence up to permuting columns, we have $M \in \Lambda_k[2]$. □

3.2 Pure palindromic automorphisms

In this section we introduce the pure palindromic automorphisms $\mathrm{P\Pi A}_\Gamma$, which we will see form an important finite index subgroup of $\Pi\mathrm{A}_\Gamma$. In Theorem 3.3 we prove that $\mathrm{P\Pi A}_\Gamma$ is a group, by showing that it is the kernel of the map from the centraliser $C_\Gamma(\iota)$ to $\mathrm{GL}(n, \mathbb{Z}/2)$ induced by mod 2 abelianisation. Proposition 3.4 then says that any element of $\Pi\mathrm{A}_\Gamma$ can be expressed as a product of an element of $\mathrm{P\Pi A}_\Gamma$ with a diagram automorphism, and as Corollary 3.5 we obtain that the collection of palindromic automorphisms $\Pi\mathrm{A}_\Gamma$ is in fact a group. This section concludes by establishing a necessary and sufficient condition on the graph Γ for the groups $\Pi\mathrm{A}_\Gamma$ and $C_\Gamma(\iota)$ to be equal, in Proposition 3.6.

We define $\mathrm{P\Pi A}_\Gamma \subset \Pi\mathrm{A}_\Gamma$ be the subset of palindromic automorphisms of A_Γ such that for each $v \in V$, the word $\alpha(v)$ may be expressed as a palindrome whose middle letter is either v or v^{-1}. For instance, $I_\Gamma \subset \mathrm{P\Pi A}_\Gamma$ but $D_\Gamma \cap \mathrm{P\Pi A}_\Gamma$ is trivial. If $v_i \leq v_j$, there is a well-defined pure palindromic automorphism $P_{ij} := (\iota\tau_{ij})^2$, which sends v_i to $v_j v_i v_j$ and fixes every other vertex in V. We refer to P_{ij} as a *dominated elementary palindromic automorphism of* A_Γ.

The following theorem shows that $\mathrm{P\Pi A}_\Gamma$ is a group, by establishing that it is a kernel inside $C_\Gamma(\iota)$. We will thus refer to $\mathrm{P\Pi A}_\Gamma$ as the *pure palindromic automorphism group of* A_Γ.

Theorem 3.3. *There is an exact sequence*

$$1 \longrightarrow \mathrm{P\Pi A}_\Gamma \longrightarrow C_\Gamma(\iota) \longrightarrow \mathrm{GL}(n, \mathbb{Z}/2). \tag{2}$$

Moreover, the image of $C_\Gamma(\iota)$ in $\mathrm{GL}(n, \mathbb{Z}/2)$ is generated by the images of all diagram automorphisms and adjacent dominated transvections in $\mathrm{Aut}(A_\Gamma)$.

Proof. Let $\Phi_2 : \mathrm{Aut}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z}/2)$ be the map induced by the mod 2 abelianisation map $A_\Gamma \to (\mathbb{Z}/2)^n$. We will show that $\mathrm{P\Pi A}_\Gamma$ is the kernel of the restriction of Φ_2 to $C_\Gamma(\iota)$.

Let $\alpha \in C_\Gamma(\iota)$. Note that for each $v \in V$, the element $\alpha(v)$ necessarily has odd length, since $\alpha(v)$ must survive under the mod 2 abelianisation map $A_\Gamma \to (\mathbb{Z}/2)^n$. Now for each $v \in V$, write $\alpha(v)$ in clique-palindromic normal form $w_1 \dots w_k \dots w_1$, as in Proposition 3.1. Both the index k and the word w_k here depend upon v, so we write $w(v)$ for the central clique word in the clique-palindromic normal form for $\alpha(v)$. Then each word $w(v)$ is a palindrome of odd length which is supported on a clique in Γ. It follows that the automorphism α lies in $\mathrm{P\Pi A}_\Gamma$ if and only if for each $v \in V$, the exponent sum of v in the word $w(v)$ is odd, and every other exponent sum is even. Thus $\mathrm{P\Pi A}_\Gamma$ is precisely the kernel of the restriction of Φ_2.

We now derive the generating set for $\Phi_2(C_\Gamma(\iota))$ in the statement of the theorem. Given $\alpha \in C_\Gamma(\iota)$, write $\alpha = \delta\beta$, where $\delta \in D_\Gamma$ and $\beta \in \mathrm{Aut}^0(A_\Gamma)$. We map β into $\mathrm{GL}(n, \mathbb{Z}/2)$ using the canonical map Φ_2, and give $\Phi_2(\beta)$ the lower block-triangular decomposition discussed in Section 2.4.

By Corollary 3.2, we can reduce each diagonal block of $\Phi_2(\beta)$ to an identity matrix by composing $\Phi_2(\beta)$ with appropriate members of $\Phi_2(C_\Gamma(\iota))$: permutation matrices (in the case of a free block), or images of adjacent transvections (in the case of an abelian block). The resulting matrix $N \in \Phi_2(C_\Gamma(\iota))$ lifts to some $\alpha' \in C_\Gamma(\iota)$.

If N has an off-diagonal 1 in its ith column, this corresponds to $\alpha'(v_i)$ having odd exponent sum of both v_i and v_j, say. Writing $\alpha'(v_i)$ in clique-palindromic normal form $w_1 \dots w_k \dots w_1$, we must have that v_i and v_j both have odd exponent sum in w_k, and hence commute, by Proposition 3.1. The presence of the 1 in the (j, i) entry of N implies that $v_i \leq v_j$, and so we can use the image of the (adjacent) transvection τ_{ij} to clear it.

Thus we conclude that $\Phi_2(\beta)$ may be written as a product of images of diagram automorphisms and adjacent transvections. Hence $\Phi_2(C_\Gamma(\iota))$ is also generated by these automorphisms. □

We now use Theorem 3.3 to prove that the collection of palindromic automorphisms $\Pi\mathrm{A}_\Gamma$ is a subgroup of $\mathrm{Aut}(A_\Gamma)$. We will require the following result.

Proposition 3.4. *Let $\alpha \in \mathrm{Aut}(A_\Gamma)$ be palindromic. Then α can be expressed as $\alpha = \delta\gamma$ where $\gamma \in \mathrm{P\Pi A}_\Gamma$ and $\delta \in D_\Gamma$.*

Proof. Let $\alpha \in \Pi\mathrm{A}_\Gamma$. Define a function $\delta : V \to V$ by letting $\delta(v)$ be the middle letter of a reduced palindromic word representing $\alpha(v)$. Note that δ is well-defined, because all reduced expressions for $\alpha(v)$ are shuffle-equivalent, and in any such reduced expression there is exactly one letter with odd exponent sum. The map δ must be bijective, otherwise the image of α in $\mathrm{GL}(n, \mathbb{Z}/2)$ would have two identical columns. We now show that δ induces a diagram automorphism of A_Γ, which by abuse of notation we also denote δ.

Since $\delta : V \to V$ is a bijection and Γ is simplicial, it suffices to show that δ induces a graph endomorphism of Γ. Suppose that $u, v \in V$ are joined by an edge in Γ. Then $[\alpha(v), \alpha(u)] = 1$, and so we apply Servatius' Centraliser Theorem (Theorem 2.1). Write $\alpha(u)$ in basic form ${w_1}^{r_1} \dots {w_s}^{r_s}$ (see Section 2.2). Since $\alpha(u)$ is a palindrome, all but one of these w_i will be an even length palindrome, and exactly one will be an odd length palindrome, with odd exponent sum of $\delta(u)$. We know by the Centraliser Theorem that

$\alpha(v)$ lies in

$$\langle w_1 \rangle \times \cdots \times \langle w_s \rangle \times \langle \mathrm{lk}(\alpha(u)) \rangle.$$

Since $\delta(v) \neq \delta(u)$, the only way $\alpha(v)$ can have an odd exponent of $\delta(v)$ is if $\delta(v) \in \mathrm{lk}(\alpha(u))$. In particular, $[\delta(v), \delta(u)] = 1$. Thus δ preserves adjacency in Γ and hence induces a diagram automorphism.

The proposition now follows, setting $\gamma = \delta^{-1}\alpha \in \mathrm{P\Pi A}_\Gamma$. □

The following corollary is immediate.

Corollary 3.5. *The set* $\Pi\mathrm{A}_\Gamma$ *forms a group. Moreover, this group splits as* $\mathrm{P\Pi A}_\Gamma \rtimes D_\Gamma$*.*

We are now able to determine precisely when the groups $\Pi\mathrm{A}_\Gamma$ and $C_\Gamma(\iota)$ appearing in the exact sequence (2) in the statement of Theorem 3.3 are equal.

Proposition 3.6. *The groups* $\Pi\mathrm{A}_\Gamma$ *and* $C_\Gamma(\iota)$ *are equal if and only if* Γ *has no adjacent domination classes.*

Proof. If Γ has an adjacent domination class, then the adjacent transvections to which it gives rise are in $C_\Gamma(\iota)$ but not in $\Pi\mathrm{A}_\Gamma$.

For the converse, suppose $\alpha \in C_\Gamma(\iota) \setminus \Pi\mathrm{A}_\Gamma$. Write $\alpha = \delta\beta$, where $\delta \in D_\Gamma$ and $\beta \in \mathrm{Aut}^0(A_\Gamma)$, as in the proof of Theorem 3.3. Note that since $D_\Gamma \leq C_\Gamma(\iota)$ we have that $\beta \in C_\Gamma(\iota)$. There must be a $v \in V$ such that $\beta(v)$ has at least two letters of odd exponent sum, say u_1 and u_2, as otherwise α would lie in $\Pi\mathrm{A}_\Gamma$. Recall that u_1 and u_2 must commute, as they both must appear in the central clique word of the clique-palindromic normal form of $\beta(v)$, in order to have odd exponent.

Consider $\Phi(\beta)$ in $\mathrm{GL}(n, \mathbb{Z})$ under our usual lower block-triangular matrix decomposition, discussed in Section 2.4. It must be the case that both u_1 and u_2 dominate v. This is because the odd entries in the column of $\Phi(\beta)$ corresponding to v that arise due to u_1 and u_2 either lie in the diagonal block containing v, or below this block. In the former case, this gives $u_1, u_2 \in [v]$, while in the latter, the presence of non-zero entries below the diagonal block of v forces $u_1, u_2 \geq v$ (as discussed in Section 2.4). If v dominates u_1, say, in return, then we obtain $u_1 \leq v \leq u_2$, and so by transitivity u_1 is (adjacently) dominated by u_2, proving the proposition in this case.

Now consider the case that neither u_1 nor u_2 is dominated by v. By Corollary 3.2, we may carry out some sequence of row operations to $\Phi(\beta)$ corresponding to the images of inversions, adjacent transvections, or P_{ij} in $\Phi(C_\Gamma(\iota))$, to reduce the diagonal block corresponding to $[v]$ to the identity matrix. The resulting matrix lifts to some $\beta' \in C_\Gamma(\iota)$, such that $\beta'(v)$ has exponent sum 1 of v, and odd exponent sums of u_1 and of u_2. As we argued in the proof of Corollary 3.2, this means u_1, u_2 and v pairwise commute, and so v is adjacently dominated by u_1 (and u_2). This completes the proof. □

3.3 Finite generating sets

In this section we prove Theorem A of the introduction, which gives a finite generating set for the palindromic automorphism group $\Pi\mathrm{A}_\Gamma$. The main step is Theorem 3.7, where we determine a finite set of generators for the pure palindromic automorphism group $\mathrm{P\Pi A}_\Gamma$. We also obtain finite generating sets for the centraliser $C_\Gamma(\iota)$ in Corollary 3.8, and for certain stabiliser subgroups of $\Pi\mathrm{A}_\Gamma$ in Theorem 3.11.

Theorem 3.7. *The group* $\mathrm{P\Pi A}_\Gamma$ *is generated by the finite set comprising the inversions and the dominated elementary palindromic automorphisms.*

Before proving Theorem 3.7, we state a corollary obtained by combining Theorems 3.3 and 3.7.

Corollary 3.8. *The group* $C_\Gamma(\iota)$ *is generated by diagram automorphisms, adjacent dominated transvections and the generators of* $\mathrm{P\Pi A}_\Gamma$*.*

Our proof of Theorem 3.7 is an adaptation of Laurence's proof [16] of finite generation of $\mathrm{Aut}(A_\Gamma)$. First, in Lemma 3.9 below, we show that any $\alpha \in \mathrm{P\Pi A}_\Gamma$ may be precomposed with suitable products of our proposed generators to yield what we refer to as a 'simple' automorphism of A_Γ (defined below). The simple palindromic automorphisms may then be understood by considering subgroups of $\mathrm{P\Pi A}_\Gamma$ that fix certain free product subgroups inside A_Γ; we define and obtain generating sets for these subgroups in Lemma 3.10. Combining these results, we complete our proof of Theorem 3.7.

For each $v \in V$, we define $\alpha \in \mathrm{P\Pi A}_\Gamma$ to be v-*simple* if $\mathrm{supp}(\alpha(v))$ is connected in Γ^c. We say that $\alpha \in \mathrm{P\Pi A}_\Gamma$ is *simple* if α is v-simple for all $v \in V$. Laurence's definition of a v-simple automorphism $\phi \in \mathrm{Aut}(A_\Gamma)$ is more general and differs from ours, however the two definitions are equivalent when $\phi \in \mathrm{P\Pi A}_\Gamma$.

Let S denote the set of inversions and dominated elementary palindromic automorphisms in $\Pi\mathrm{A}_\Gamma$ (that is, the generating set for $\mathrm{P\Pi A}_\Gamma$ proposed by Theorem 3.7). We say that $\alpha, \beta \in \mathrm{P\Pi A}_\Gamma$ are π-*equivalent* if there exists $\theta \in \langle S \rangle$ such that $\alpha = \beta\theta$. In other words, $\alpha, \beta \in \mathrm{P\Pi A}_\Gamma$ are π-equivalent if $\beta^{-1}\alpha \in \langle S \rangle$.

Lemma 3.9. *Every* $\alpha \in \mathrm{P\Pi A}_\Gamma$ *is* π*-equivalent to some simple automorphism* $\chi \in \mathrm{P\Pi A}_\Gamma$*.*

Proof. Suppose $\alpha \in \mathrm{P\Pi A}_\Gamma$. We note once and for all that the palindromic word $\alpha(u)$ is cyclically reduced, for any $u \in V$.

Select a vertex $v \in V$ of maximal rank for which $\alpha(v)$ is not v-simple. Now write

$$\alpha(v) = {w_1}^{r_1} \dots {w_s}^{r_s}$$

in basic form, reindexing if necessary so that $v \in \mathrm{supp}(w_1)$. The ranks of v and $\alpha(v)$ are equal, since α induces an isomorphism from the centraliser in A_Γ of v to that of $\alpha(v)$. Hence by Proposition 2.2, parts 2(b) and 2(a) respectively, each $w_i \in A_\Gamma$ (for $i > 1$) is some vertex generator in V, and $w_i \geq' v$. Moreover, for $i > 1$, each r_i is even, since $\alpha(v)$ is palindromic.

Now, for $i > 1$, suppose $w_i \geq' v$ but $[v]' \neq [w_i]'$. By Servatius' Centraliser Theorem (Theorem 2.1), we know that the centraliser of a vertex is generated by its star, and hence conclude that $\mathrm{rk}(w_i) > \mathrm{rk}(v)$. This gives that α is w_i-simple, by our assumption on the maximality of the rank of v. In basic form, then,

$$\alpha(w_i) = p^\ell,$$

where $\ell \in \mathbb{Z}$, $p \in A_\Gamma$, and $\mathrm{supp}(p)$ is connected in Γ^c. Note also that $\mathrm{supp}(p)$ contains w_i, since $\alpha \in \mathrm{P\Pi A}_\Gamma$.

Suppose there exists $t \in \mathrm{supp}(p) \setminus \{w_i\}$. As for v before, by Proposition 2.2, we have $t \geq w_i$, since $\mathrm{rk}(\alpha(w_i)) = \mathrm{rk}(w_i)$. We know $w_i \geq' v$, and so $t \geq v$. Since w_i, v and t are pairwise distinct, this forces w_i and t to be adjacent, which contradicts Proposition 2.2, part 2(c). So

$$\alpha(w_i) = {w_i}^\ell,$$

and necessarily $\ell = \pm 1$. Knowing this, we replace α with $\alpha\beta_i$ where $\beta_i \in \langle S \rangle$ is the palindromic automorphism of the form

$$v \mapsto w_i^{\frac{\ell r_i}{2}} v w_i^{\frac{\ell r_i}{2}}.$$

By doing this for each such w_i, we ensure that any w_i that strictly dominates v is not in the support of $\alpha\beta_i(v)$. Note $\alpha(v') = \alpha\beta_i(v')$ for all $v' \neq v$.

If $s = 1$, then α is v-simple, so by our assumption on v, we must have $s > 1$. Because we have reduced to the case where $w_i \in [v]'$ for $i > 1$, we must have $w_1 = v^{\pm 1}$, otherwise we get a similar adjacency contradiction as in the previous paragraph: if there exists $t \in \mathrm{supp}(w_1) \setminus \{v\}$, then, as before, $t \geq v$, and since $[w_i]' = [v]'$, this would force t and v to be adjacent. Thus $\alpha(v) \in \langle [v]' \rangle$. Indeed, the discussion in the previous two paragraphs goes through for any $u \in [v]'$, so we may assume that $\alpha(u) \in \langle [v]' \rangle$ for any $u \in [v]'$. Thus $\alpha\langle [v]' \rangle \leq \langle [v]' \rangle$, with equality holding by [16, Proposition 6.1].

The group $\langle [v]' \rangle$ is free abelian, and by considering exponent sums, we see that the restriction of α to the group $\langle [v]' \rangle$ is a member of the level 2 congruence subgroup $\Lambda_k[2]$, where $k = |[v]'|$. We know that Theorem 3.7 holds in the special case of these congruence groups (see [9, Lemma 2.4], for example), so we can precompose α with the appropriate automorphisms in the set S so that the new automorphism obtained, α', is the identity on $\langle [v]' \rangle$, and acts the same as α on all other vertices in V. The automorphisms α and α' are π-equivalent, and α' is v-simple (indeed: $\alpha'(v) = v$).

From here, we iterate this procedure, selecting a vertex $u \in V \setminus \{v\}$ of maximal rank for which α' is not u-simple, and so on, until we have exhausted the vertices of Γ preventing α from being simple. □

Now, for each $v \in V$, define Γ^v be the set of vertices that dominate v but are not adjacent to v. Further define $X_v := \{v = v_1, \ldots, v_r\} \subseteq \Gamma^v$ to be the vertices of Γ^v that are also dominated by v. Partition Γ^v into its connected components in the graph $\Gamma \setminus \mathrm{lk}(v)$. This partition is of the form

$$\left(\bigsqcup_{i=1}^{t} \Gamma_i \right) \sqcup \left(\bigsqcup_{i=1}^{r} \{v_i\} \right),$$

where $\bigsqcup_{i=1}^{t} \Gamma_i = \Gamma^v \setminus X_v$. Letting $H_i = \langle \Gamma_i \rangle$, we see that

$$H := \langle \Gamma^v \rangle = H_1 * \cdots * H_t * \langle X_v \rangle, \tag{3}$$

where $F_r := \langle X_v \rangle$ is a free group of rank r. Notice that H is itself a right-angled Artin group.

The final step in proving Theorem 3.7 requires a generating set for a certain subgroup of palindromic automorphisms in $\mathrm{Aut}(H)$, which we now define. Let $\mathcal{Y}$ denote the subgroup of $\mathrm{Aut}(H)$ consisting of the pure palindromic automorphisms of H that restrict to the identity on each H_i. The following lemma says that this group is generated by its intersection with the finite list of generators stated in Theorem 3.7. In the special case when there are no H_i factors in the free product (3) above, this result was established by Collins [5]. Our proof is a generalisation of his.

Lemma 3.10. *The group $\mathcal{Y}$ is generated by the inversions of the free group F_r and the elementary palindromic automorphisms of the form $P(s,t) : s \mapsto tst$, where $t \in \Gamma^v$ and $s \in X_v$.*

Proof. For $\alpha \in \mathcal{Y}$, we define its *length* $l(\alpha)$ to be the sum of the lengths of $\alpha(v_i)$ for each $v_i \in X_v$. We induct on this length. The base case is $l(\alpha) = r$, in which case α is a product of inversions of F_r. From now on, assume $l(\alpha) > r$.

Let $L(w)$ denote the length of a word w in the right-angled Artin group H, with respect to the vertex set Γ^v. Suppose for all $\epsilon_i, \epsilon_j \in \{\pm 1\}$ and distinct $a_i, a_j \in \alpha(\Gamma^v)$ we have

$$L(a_i^{\epsilon_i} a_j^{\epsilon_j}) > L(a_i) + L(a_j) - 2(\lfloor L(a_i)/2 \rfloor + 1), \tag{4}$$

where $\lfloor x \rfloor$ is the integer part of $x \in [0, \infty)$. Conceptually, we are assuming that for every expression $a_i^{\epsilon_i} a_j^{\epsilon_j}$, whatever cancellation occurs between the words $a_i^{\epsilon_i}$ and $a_j^{\epsilon_j}$, more than half of $a_i^{\epsilon_i}$ and more than half of $a_j^{\epsilon_j}$ survives after all cancellation is complete.

Fix $v_i \in X_v$ so that $a_i := \alpha(v_i)$ satisfies $L(a_i) > 1$. Such a vertex v_i must exist, as we are assuming that $l(\alpha) > r$. Notice that since $L(a_i) > 1$, we have $v_i \neq a_i^{\pm 1}$. Now, any reduced word in H of length m with respect to the generating set $\alpha(\Gamma^v)$ has length at least m with respect to the vertex generators Γ^v, due to our cancellation assumption. Since $v_i \neq {a_i}^{\pm 1}$, the generator v_i must have length strictly greater than 1 with respect to $\alpha(\Gamma^v)$, and so v_i must have length strictly greater than 1 with respect to Γ^v. But v_i is an element of Γ^v, which is a contradiction. Therefore, the above inequality (4) fails at least once.

We now argue each case separately. Let $a_i, a_j \in \alpha(\Gamma^v)$ be distinct and write

$$a_i = \alpha(v_i) = w_i {v_i}^{\eta_i} {w_i}^{\mathrm{rev}} \quad \text{and} \quad a_j = \alpha(v_j) = w_j {v_j}^{\eta_j} {w_j}^{\mathrm{rev}},$$

where $v_i, v_j \in \Gamma^v$, w_i, $w_j \in H$ and $\eta_i, \eta_j \in \{\pm 1\}$. Suppose the inequality (4) fails for this pair when $\epsilon_i = \epsilon_j = 1$. Then it must be the case that $w_j = ({w_i}^{\mathrm{rev}})^{-1} v_i^{-\eta_i} z$, for some $z \in H$, since H is a free product. In this case, replacing α with $\alpha P(v_j, v_i) = \alpha P_{ji}$ decreases the length of the automorphism. We reduce the length of α in the remaining cases as follows:

- For $\epsilon_i = \epsilon_j = -1$, replace α with $\alpha \iota_j P(v_j, v_i)^{-1} = \alpha \iota_j P_{ji}^{-1}$.
- For $\epsilon_i = -1$ and $\epsilon_j = 1$, or vice versa, replace α with $\alpha \iota_j P(v_j, v_i) = \alpha \iota_j P_{ji}$.

By induction, we have thus established the proposed generating set for the group $\mathcal{Y}$. □

We now prove Theorem 3.7, obtaining a finite generating set for the group $\mathrm{P\Pi A}_\Gamma$.

Proof of Theorem 3.7. Let S denote the set of inversions and dominated elementary palindromic automorphisms in $\mathrm{P\Pi A}_\Gamma$. By Lemma 3.9, all we need do is write any simple $\alpha \in \mathrm{P\Pi A}_\Gamma$ as a product of members of $S^{\pm 1}$.

Let v be a vertex of maximal rank that is not fixed by α. Define Γ^v, its partition, and the free product it generates using the same notation as in the discussion before the statement of Lemma 3.10. By maximality of the rank of v, any vertex of any Γ_i must be fixed by α (since it has rank higher than that of v). By Lemma 5.5 of Laurence and its corollary [16], we conclude that (for this v we have chosen), $\alpha(H) = H$.

This establishes that α restricted to H lies in the group $\mathcal{Y} \leq \mathrm{Aut}(H)$, for which Lemma 3.10 gives a generating set. Thus we are able to precompose α with the appropriate members of $S^{\pm 1}$ to obtain a new automorphism α' that is the identity on H, and which agrees with α on $\Gamma \setminus \Gamma^v$. In particular, α' fixes v. We now iterate this procedure until all vertices of Γ are fixed, and have thus proved the theorem. □

With Theorem 3.7 established, we are now able to prove our first main result, Theorem A, and so obtain our finite generating set for $\Pi\mathrm{A}_\Gamma$.

Proof of Theorem A. By Corollary 3.5, we have that $\Pi\mathrm{A}_\Gamma$ splits as

$$\Pi\mathrm{A}_\Gamma \cong \mathrm{P\Pi A}_\Gamma \rtimes D_\Gamma,$$

and so to generate $\Pi\mathrm{A}_\Gamma$, it suffices to combine the generating set for $\mathrm{P\Pi A}_\Gamma$ given by Theorem 3.7 with the diagram automorphisms of A_Γ. Thus the group $\Pi\mathrm{A}_\Gamma$ is generated by the set of all diagram automorphisms, inversions and well-defined dominated elementary palindromic automorphisms. □

We end this section by remarking that the proof techniques we used in establishing Theorem A allow us to obtain finite generating sets for a more general class of palindromic automorphism groups of A_Γ. Having chosen an indexing $v_1, \ldots, v_n$ of the vertex set V of Γ, denote by $\Pi\mathrm{A}_\Gamma(k)$ the subgroup of $\Pi\mathrm{A}_\Gamma$ that fixes each of the vertices $v_1, \ldots, v_k$. Note that a reindexing of V will, in general, produce non-isomorphic stabiliser groups. We are able to show that each $\Pi\mathrm{A}_\Gamma(k)$ is generated by its intersection with the finite set S.

Theorem 3.11. *The stabiliser subgroup $\Pi\mathrm{A}_\Gamma(k)$ is generated by the set of diagram automorphisms, inversions and dominated elementary palindromic automorphisms that fix each of $v_1, \ldots, v_k$.*

Throughout the proof of Theorem 3.7, each time that we precomposed some $\alpha \in \mathrm{P\Pi A}_\Gamma$ by an inversion ι_i, an elementary palindromic automorphism P_{ij}, or its inverse P_{ij}^{-1}, it was because the generator v_i was not fixed by α. If $v_j \in V$ was already fixed by α, we had no need to use ι_j or any of the $P_{jk}^{\pm 1}$ $(j \neq k)$ in this way. (That this claim holds in the second-last paragraph of the proof of Lemma 3.9, where we are working in the group $\Lambda_k[2]$, follows from [9, Lemma 3.5].) The same is true when we extend $\mathrm{P\Pi A}_\Gamma$ to $\Pi\mathrm{A}_\Gamma$ using diagram automorphisms, in the proof of Theorem A. Thus by following the same method as in our proof of Theorem A, we are also able to obtain the more general result, Theorem 3.11: our approach had already written $\alpha \in \Pi\mathrm{A}_\Gamma(k)$ as a product of the generators proposed in the statement of Theorem 3.11.

4 The palindromic Torelli group

Recall that we defined the *palindromic Torelli group* $\mathcal{PI}_\Gamma$ to consist of the palindromic automorphisms of A_Γ that act trivially on $H_1(A_\Gamma, \mathbb{Z})$. Our main goal in this section is to prove Theorem B, which gives a generating set for $\mathcal{PI}_\Gamma$. For this, in Section 4.1 we obtain a finite presentation for the image in $\mathrm{GL}(n, \mathbb{Z})$ of the pure palindromic automorphism group. Using the relators from this presentation, we then prove Theorem B in Section 4.2.

4.1 Presenting the image in $\mathrm{GL}(n, \mathbb{Z})$ of the pure palindromic automorphism group

In this section we prove Theorem 4.2, which establishes a finite presentation for the image of the pure palindromic automorphism group $\mathrm{P\Pi A}_\Gamma$ in $\mathrm{GL}(n, \mathbb{Z})$, under the canonical map induced by abelianising A_Γ. Corollary 4.3 then gives a splitting of $\mathrm{P\Pi A}_\Gamma$.

Recall that $\Lambda_n[2]$ denotes the principal level 2 congruence subgroup of $\mathrm{GL}(n, \mathbb{Z})$. We start by recalling a finite presentation for $\Lambda_n[2]$ due to the first author. For $1 \leq i \neq j \leq n$, let $S_{ij} \in \Lambda_n[2]$ be the matrix that has 1s on the diagonal and 2 in the (i, j) position, with 0s elsewhere, and let $Z_i \in \Lambda_n[2]$ differ from the identity matrix only in having -1 in the (i, i) position. Theorem 4.1 gives a finite presentation for $\Lambda_n[2]$ in terms of these matrices.

Theorem 4.1 (Fullarton [9])**.** *The principal level 2 congruence group* $\Lambda_n[2]$ *is generated by*

$$\{S_{ij}, Z_i \mid 1 \leq i \neq j \leq n\},$$

subject to the defining relators

1. $Z_i{}^2$
2. $[Z_i, Z_j]$
3. $(Z_i S_{ij})^2$
4. $(Z_j S_{ij})^2$
5. $[Z_i, S_{jk}]$
6. $[S_{ki}, S_{kj}]$
7. $[S_{ij}, S_{kl}]$
8. $[S_{ji}, S_{ki}]$
9. $[S_{kj}, S_{ji}]S_{ki}{}^{-2}$
10. $(S_{ij}S_{ik}{}^{-1}S_{ki}S_{ji}S_{jk}S_{kj}{}^{-1})^2$

where $1 \leq i, j, k, l \leq n$ *are pairwise distinct.*

We will use this presentation of $\Lambda_n[2]$ to obtain a finite presentation of the image of $\mathrm{P\Pi A}_\Gamma$ in $\mathrm{GL}(n, \mathbb{Z})$. Observe that $\iota_j \mapsto Z_j$ and $P_{ij} \mapsto S_{ji}$ $(v_i \leq v_j)$ under the canonical map $\Phi : \mathrm{Aut}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z})$. Let R_Γ be the set of words obtained by taking all the relators in Theorem 4.1 and removing those that include a letter S_{ji} with $v_i \not\leq v_j$.

Theorem 4.2. *The image of* $\mathrm{P\Pi A}_\Gamma$ *in* $\mathrm{GL}(n, \mathbb{Z})$ *is a subgroup of* $\Lambda_n[2]$*, with finite presentation*

$$\langle \{Z_k, S_{ji} : 1 \leq k \leq n, v_i \leq v_j\} \mid R_\Gamma \rangle .$$

Proof. By Theorem 3.7, we know that $\text{P}\Pi\text{A}_\Gamma \leq \text{Aut}^0(A_\Gamma)$, and so matrices in $\Theta := \Phi(\text{P}\Pi\text{A}_\Gamma) \leq \text{GL}(n, \mathbb{Z})$ may be written in the lower-triangular block decomposition discussed in Section 2.4. Moreover, the matrix in a diagonal block of rank k in some $A \in \Theta$ must lie in $\Lambda_k[2]$.

We now use this block decomposition to obtain the presentation of Θ in the statement of the theorem. Observe that we have a forgetful map $\mathcal{F}$ defined on Θ, where we forget the first $k := |[v_1]|$ rows and columns of each matrix. This is a well-defined homomorphism, since the determinant of a lower block-triangular matrix is the product of the determinants of its diagonal blocks. Let $\mathcal{Q}$ denote the image of this forgetful map, and $\mathcal{K}$ its kernel. We have $\mathcal{K} = \Lambda_k[2] \times \mathbb{Z}^t$, where t is the number of dominated transvections that are forgotten under the map $\mathcal{F}$, and the $\Lambda_k[2]$ factor is generated by the images of the inversions and dominated elementary palindromic automorphisms that preserve the subgroup $\langle [v_1] \rangle$.

The group Θ splits as $\mathcal{K} \rtimes \mathcal{Q}$, with the relations corresponding to the semi-direct product action, and those in the obvious presentation of $\mathcal{K}$, all lying in R_Γ. Now, we may define a similar forgetful map on the matrix group $\mathcal{Q}$, so by induction Λ is an iterated semi-direct product, with a complete set of relations given by R_Γ. □

Using the above presentation, we are able to obtain the following corollary, regarding a splitting of the group $\text{P}\Pi\text{A}_\Gamma$. Recall that I_Γ is the subgroup of $\text{Aut}(A_\Gamma)$ generated by inversions. We denote by $\text{E}\Pi\text{A}_\Gamma$ the subgroup of $\text{P}\Pi\text{A}_\Gamma$ generated by all dominated elementary palindromic automorphisms.

Corollary 4.3. *The group* $\text{P}\Pi\text{A}_\Gamma$ *splits as* $\text{E}\Pi\text{A}_\Gamma \rtimes I_\Gamma$.

Proof. The group $\text{P}\Pi\text{A}_\Gamma$ is generated by $\text{E}\Pi\text{A}_\Gamma$ and I_Γ by Theorem 3.7, and I_Γ normalises $\text{E}\Pi\text{A}_\Gamma$. We now establish that $\text{E}\Pi\text{A}_\Gamma \cap I_\Gamma$ is trivial. Suppose $\alpha \in \text{E}\Pi\text{A}_\Gamma \cap I_\Gamma$. By Theorem 4.2, the image of α under the canonical map $\Phi : \text{Aut}(A_\Gamma) \to \text{GL}(n, \mathbb{Z})$ lies in the principal level 2 congruence group $\Lambda_n[2]$. This implies that $\Phi(\alpha)$ is trivial, since $\Lambda_n[2]$ is itself a semi-direct product of groups containing the images of the groups $\text{E}\Pi\text{A}_\Gamma$ and I_Γ, respectively: this is verified by examining the presentation of $\Lambda_n[2]$ given in Theorem 4.1. So the automorphism α must lie in the palindromic Torelli group $\mathcal{PI}_\Gamma$, which has trivial intersection with I_Γ, and hence α is trivial. □

4.2 A generating set for the palindromic Torelli group

Using the relators in the presentation given by Theorem 4.1, we are now able to obtain an explicit generating set for the palindromic Torelli group $\mathcal{PI}_\Gamma$, and so prove Theorem B.

Recall that when A_Γ is a free group, the elementary palindromic automorphism P_{ij} is well-defined for every distinct i and j. The first author defined *doubled commutator transvections* and *separating π-twists* in $\text{Aut}(F_n)$ ($n \geq 3$) to be conjugates in ΠA_n of, respectively, the automorphisms $[P_{12}, P_{13}]$ and $(P_{23}{P_{13}}^{-1}P_{31}P_{32}P_{12}{P_{21}}^{-1})^2$. The latter of these two may seem cumbersome; we refer to [9, Section 2] for a simple, geometric interpretation of separating π-twists.

The definitions of these generators extend easily to the general right-angled Artin groups setting, as follows. Suppose $v_i \in V$ is dominated by v_j and by v_k, for distinct i, j and k. Then

$$\chi_1(i,j,k) := [P_{ij}, P_{ik}] \in \mathrm{Aut}(A_\Gamma)$$

is well-defined, and we define a *doubled commutator transvection* in $\mathrm{Aut}(A_\Gamma)$ to be a conjugate in $\Pi\mathrm{A}_\Gamma$ of any well-defined $\chi_1(i,j,k)$. Similarly, suppose $[v_i] = [v_j] = [v_k]$ for distinct i, j and k. Then

$$\chi_2(i,j,k) := (P_{jk}{P_{ik}}^{-1}P_{ki}P_{kj}P_{ij}{P_{ji}}^{-1})^2 \in \mathrm{Aut}(A_\Gamma)$$

is well-defined, and we define a *separating π-twist* in $\mathrm{Aut}(A_\Gamma)$ to be a conjugate in $\Pi\mathrm{A}_\Gamma$ of any well-defined $\chi_2(i,j,k)$.

We now prove Theorem B, showing that $\mathcal{PI}_\Gamma$ is generated by these two types of automorphisms.

Proof of Theorem B. Recall that $\Theta := \Phi(\mathrm{P}\Pi\mathrm{A}_\Gamma) \leq \mathrm{GL}(n,\mathbb{Z})$. The images in Θ of our generating set for $\mathrm{P}\Pi\mathrm{A}_\Gamma$ (Theorem 3.7) form the generators in the presentation for Θ given in Theorem 4.2. Thus using a standard argument (see, for example, the proof of [17, Theorem 2.1]), we are able to take the obvious lifts of the relators of Θ as a normal generating set of $\mathcal{PI}_\Gamma$ in $\mathrm{P}\Pi\mathrm{A}_\Gamma$, via the short exact sequence

$$1 \longrightarrow \mathcal{PI}_\Gamma \longrightarrow \mathrm{P}\Pi\mathrm{A}_\Gamma \longrightarrow \Theta \longrightarrow 1.$$

The only such lifts and their conjugates that are not trivial in $\mathrm{P}\Pi\mathrm{A}_\Gamma$ are the ones of the form stated in the theorem. □

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