Birational geometry of Moduli space of curves

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Objects in algebraic geometry

Affine algebraic variety

$$V(f_1, f_2, \dots, f_s) = \{x = (x_1, \dots, x_n) \in \mathbb{C}^n \mid f_1(x) = f_2(x) = \cdots = f_s(x) = 0\}$$

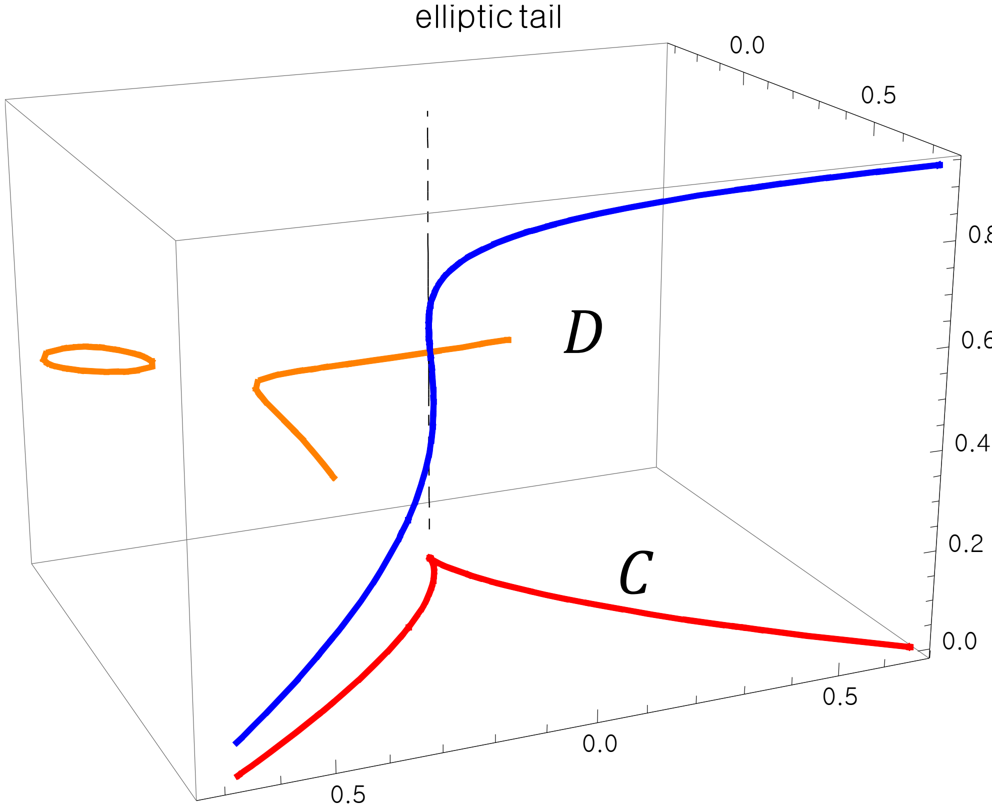
$$f_i(x) \in \mathbb{C}[x_1, \dots, x_n]$$

Example

$$f(x, y, z) = (y - \sqrt{3})^2 - (x-1)(x-2)(x-3)$$

$$D = V(f, z-1) \cup V(x - (z-1)^3, y - (z-1)^2)$$

$$C = V(y^2 - x^3, z) \subset \mathbb{C}^3$$



Objects in algebraic geometry

- Projective algebraic variety

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$$

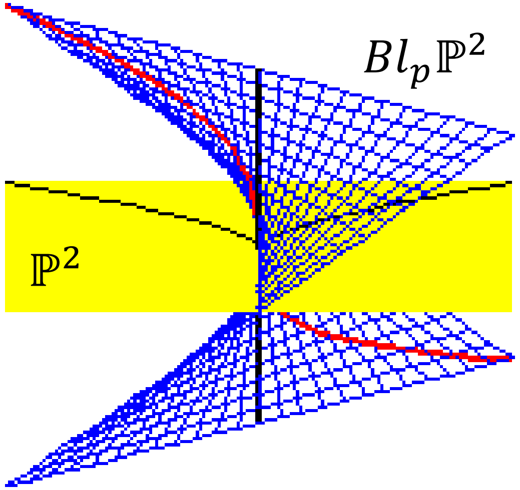
$$V(f_1, f_2, \dots, f_s) = \{x = (x_0, \dots, x_n) \in \mathbb{P}^n \mid f_1(x) = \cdots = f_s(x) = 0\}$$

$f_i(x) \in \mathbb{C}[x_0, \dots, x_n]$ (homogeneous)

- Example

$$p = (1,0,0) \in \mathbb{P}^2$$

$$Bl_p\mathbb{P}^2 = \{(x, y, z; s, t) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yt = zs\}$$



Source: D. Arapura's website

Moduli theory

- *A* ***moduli space*** (of curves, of surfaces, of vector bundles over a curve etc) is
  - the set of isomorphism classes (equipped with a structure of algebraic variety)
  - and certain universal properties
- The term *moduli* was first used by B. Riemann (1826-1866).

“This depends on $3g-3$ *modulun*”

Slit & Sew

- $g+1$ Riemann spheres → Genus g Riemann surface

Plane isometries

Distance preserving map

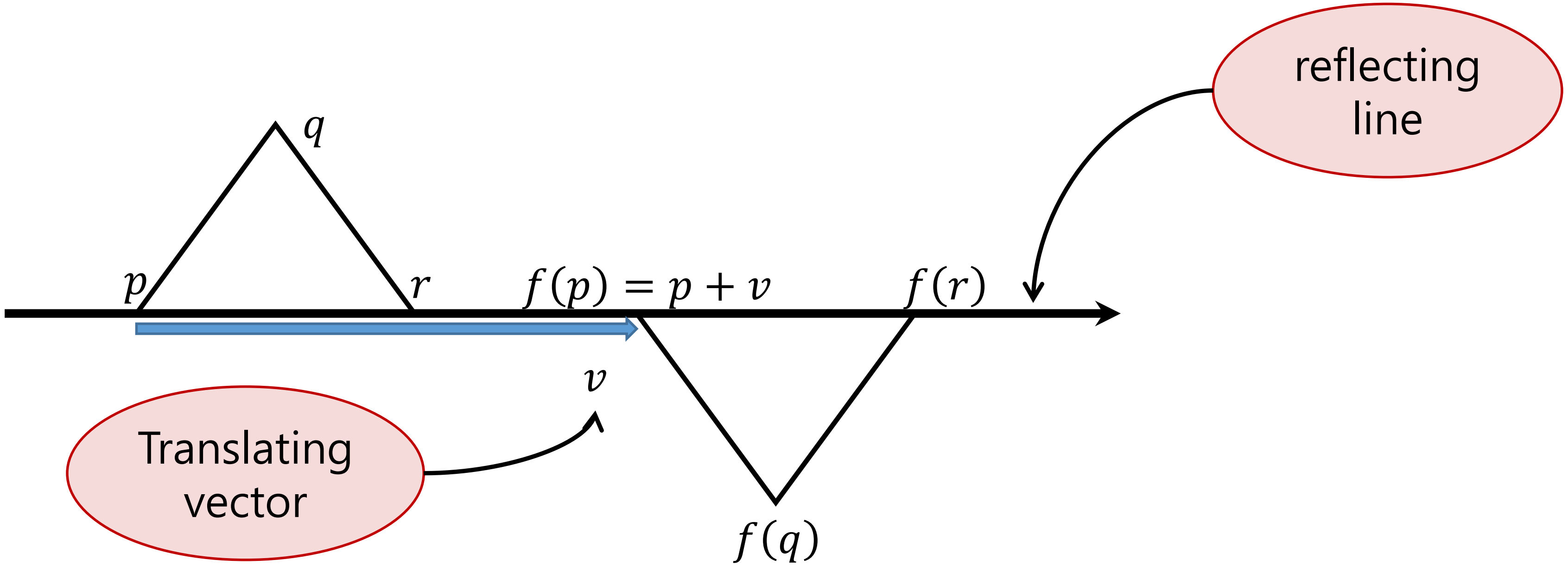
- Plane isometries ↔ (translation)∘(orthogonal transformation)
  - $f(z) = az + b \ (a \in S^1, b \in \mathbb{C})$ is a rotation unless $a = 1$.

Rotation by $\mathrm{Arg}(a)$

around $b(1-a)^{-1}$

  - $f(z) = a\overline{z} + b \ (a \in S^1, b \in \mathbb{C})$ is a glide unless $b + a\overline{b} = 0$.

reflection

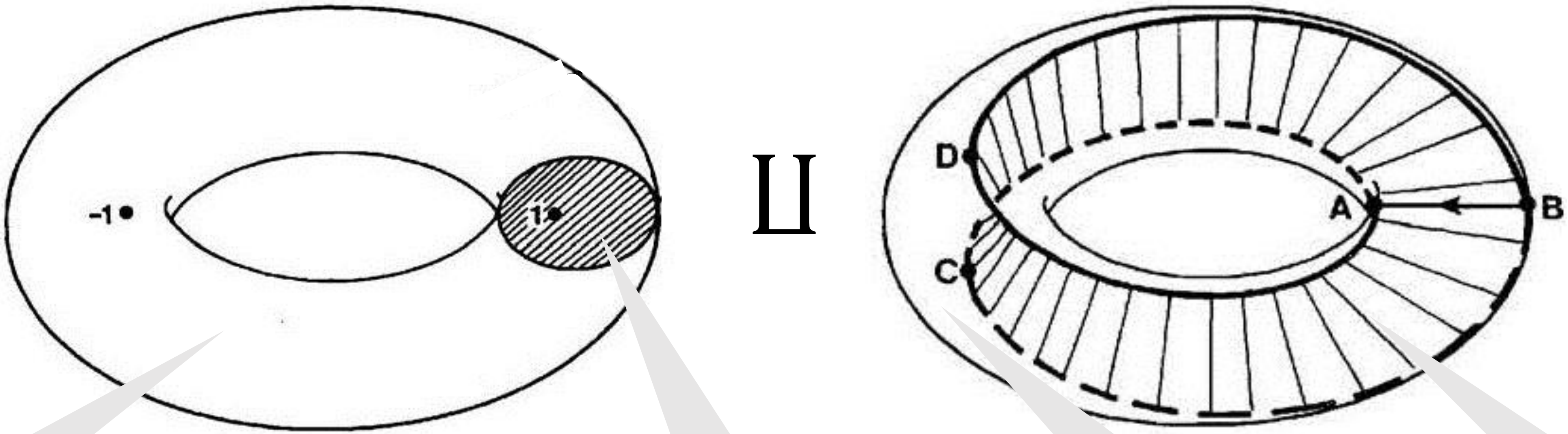


Moduli of plane isometries

- Moduli of plane isometries $\coloneqq O(2) \times \mathbb{R}^2 \simeq (S^1 \times \mathbb{R}^2) \coprod (S^1 \times \mathbb{R}^2)$

$$\begin{array}{cc} (a,b) & (a,b) \\ \updownarrow & \updownarrow \\ az+b & a\bar{z}+b \end{array}$$

Picture: Notes on geometry by E. Rees



rotation

translation
a=1

glide

reflection
$b + a\bar{b} = 0$

Moduli of plane cubics

- Plane cubics (elliptic curves)

$$\begin{tikzcd}
\{(x,y,z) \in \mathbb{P}^2 \,|\, F(x,y,z) = \sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0\} \arrow[d, leftrightarrow] \\
(a_{300}, a_{210}, a_{201}, \dots, a_{021}, a_{012}, a_{003}) \in \mathbb{P}^9
\end{tikzcd}$$

$$\begin{tikzcd}
y^2 z - x^2(x-z) = 0 \\
x^0y^2z^1 - x^3y^0z^0 + x^2y^0z^1 = 0 \arrow[d, leftrightarrow] \\
(-1,0,1,0,0,0,0,1,0,0)
\end{tikzcd}$$

- {plane cubics}**/**(coordinate change) $= \mathbb{P}^9 / SL_3(\mathbb{C})$

Equals isomorphism

- Moduli space = (parameter space)**/**(group action)
- Construction of quotient in algebraic geometry is *NOT* automatic!

Bad degeneration of elliptic curves

$$\begin{tikzcd}
C_t\colon y^2z = x(x-t^2z)(x-2t^3z) \arrow[r, squiggly, "t \to 0"'] \arrow[d, squiggly, "x' = t^2x"', "y' = t^3y"] \arrow[d, bend right, leftrightarrow] & y^2z = x^3 \arrow[d, bend left, leftrightarrow, "\times"] \\
C'_t\colon y'^2z = x'(x'-z)(x'-2tz) \arrow[r, squiggly, "t \to 0"'] & y'^2z = x'^2(x'-z)
\end{tikzcd}$$

Isomorphic when $t \neq 0$

Not isomorphic at $t = 0$:
Two different possible limits!

$[C_t] = [C'_t]$ in a moduli space

For $\{[C_t] = [C'_t]\}_{t \neq 0}$ to have a unique limit, we need to keep one and discard the other.

Which curve should we pick? (GIT)

- Need a reliable system to pick the correct limits.

- Mumford's Geometric Invariant Theory (GIT)

To understand a quotient space

- Question. *What are the (regular) functions on the quotient space?*

constant
on
G-orbits

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\pi^*h"'] & X/G \arrow[d, "h"] \\ & \mathbb{C} \end{tikzcd}$$

Regular function
h on X/G

↔

G-invariant function
π^*h on X

Ring of invariants

- Ring of regular functions on X/G ⟷ Subring $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ of G-invariant regular functions on X

- **Theorem** (Hilbert-Weyl-Haboush) If G is reductive, the ring of invariants is finitely generated.

- $\{\sigma_0, \dots, \sigma_s\}$: generators of $\mathbb{C}[X]^G$ ⟶ $X/G = \left[\begin{matrix} \text{(image of } \phi\text{)} \subset \mathbb{P}^s \\ \phi(x) = \big(\sigma_0(x), \dots, \sigma_s(x)\big) \end{matrix}\right]$

- If $\sigma_0, \dots, \sigma_s$ all vanish at x, then ϕ is NOT defined at x!

Back to moduli of elliptic curves

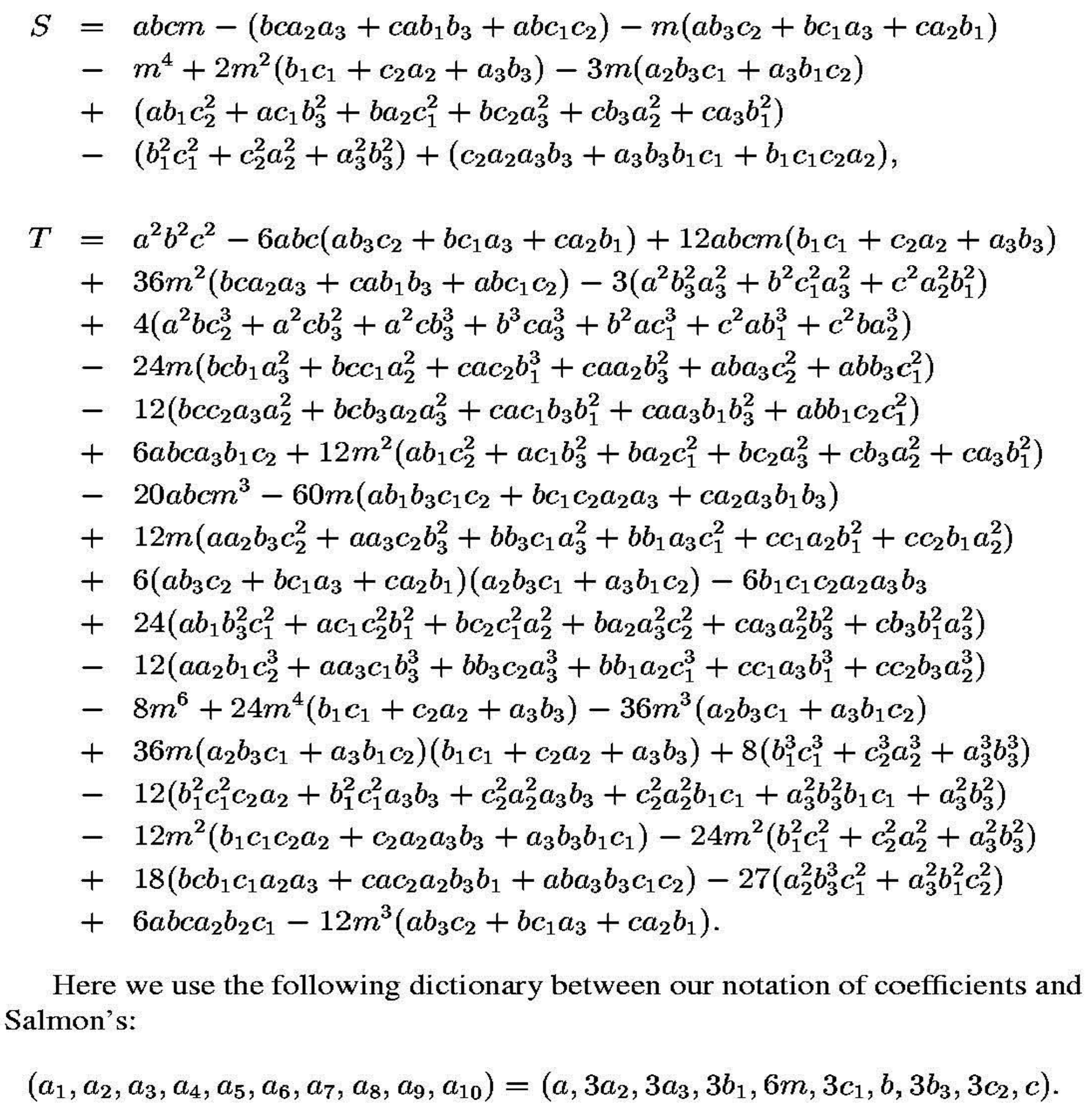
- space of cubic polynomials = $\mathbb{C}[a_{300}, a_{210}, \dots, a_{003}]$

- $\mathbb{C}[a_{ijk}]^{SL_3(\mathbb{C})} = \mathbb{C}[S, T]$, $\deg(S) = 4, \deg(T) = 6$ (Aronhold, 1850)

- $J = \frac{16S^3}{T^2+64S^3}$ (characteristic $\neq$ 2,3)

- $S = T = 0$
  for $y^2z - x^3 = 0$ → $y^2z - x^3 = 0$ is NOT defined in $\mathbb{P}^1$ with homogeneous coordinates S, T

Back to moduli of elliptic curves



$$\begin{aligned}
S &= abcm - (bca_2a_3 + cab_1b_3 + abc_1c_2) - m(ab_3c_2 + bc_1a_3 + ca_2b_1)\\
&- m^4 + 2m^2(b_1c_1 + c_2a_2 + a_3b_3) - 3m(a_2b_3c_1 + a_3b_1c_2)\\
&+ (ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2)\\
&- (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) + (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2),
\end{aligned}$$

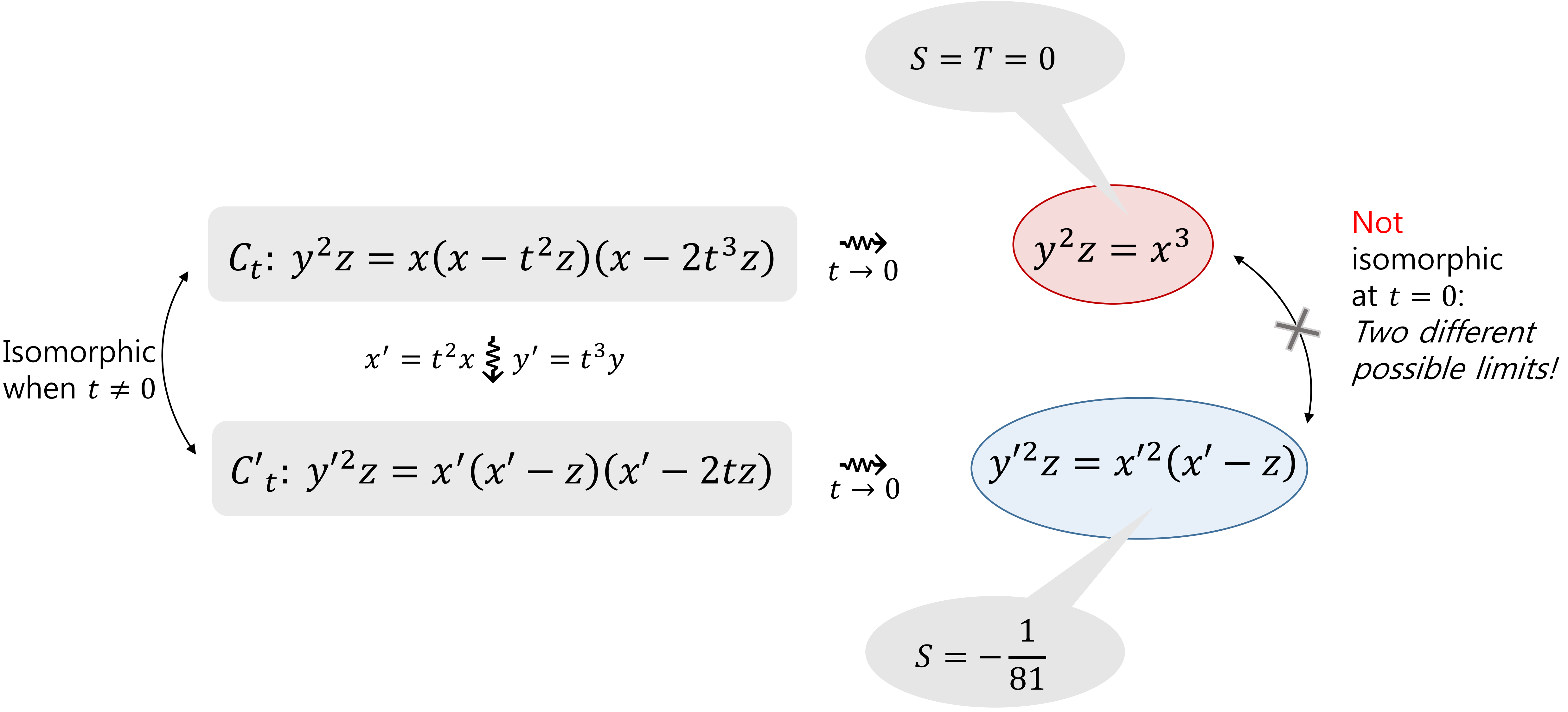
$$\begin{aligned}
T &= a^2b^2c^2 - 6abc(ab_3c_2 + bc_1a_3 + ca_2b_1) + 12abcm(b_1c_1 + c_2a_2 + a_3b_3)\\
&+ 36m^2(bca_2a_3 + cab_1b_3 + abc_1c_2) - 3(a^2b_3^2a_3^2 + b^2c_1^2a_3^2 + c^2a_2^2b_1^2)\\
&+ 4(a^2bc_2^3 + a^2cb_3^2 + a^2cb_3^3 + b^3ca_3^3 + b^2ac_1^3 + c^2ab_1^3 + c^2ba_2^3)\\
&- 24m(bcb_1a_3^2 + bcc_1a_2^2 + cac_2b_1^3 + caa_2b_3^2 + aba_3c_2^2 + abb_3c_1^2)\\
&- 12(bcc_2a_3a_2^2 + bcb_3a_2a_3^2 + cac_1b_3b_1^2 + caa_3b_1b_3^2 + abb_1c_2c_1^2)\\
&+ 6abca_3b_1c_2 + 12m^2(ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2)\\
&- 20abcm^3 - 60m(ab_1b_3c_1c_2 + bc_1c_2a_2a_3 + ca_2a_3b_1b_3)\\
&+ 12m(aa_2b_3c_2^2 + aa_3c_2b_3^2 + bb_3c_1a_3^2 + bb_1a_3c_1^2 + cc_1a_2b_1^2 + cc_2b_1a_2^2)\\
&+ 6(ab_3c_2 + bc_1a_3 + ca_2b_1)(a_2b_3c_1 + a_3b_1c_2) - 6b_1c_1c_2a_2a_3b_3\\
&+ 24(ab_1b_3^2c_1^2 + ac_1c_2^2b_1^2 + bc_2c_1^2a_2^2 + ba_2a_3^2c_2^2 + ca_3a_2^2b_3^2 + cb_3b_1^2a_3^2)\\
&- 12(aa_2b_1c_2^3 + aa_3c_1b_3^3 + bb_3c_2a_3^3 + bb_1a_2c_1^3 + cc_1a_3b_1^3 + cc_2b_3a_2^3)\\
&- 8m^6 + 24m^4(b_1c_1 + c_2a_2 + a_3b_3) - 36m^3(a_2b_3c_1 + a_3b_1c_2)\\
&+ 36m(a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + c_2a_2 + a_3b_3) + 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3)\\
&- 12(b_1^2c_1^2c_2a_2 + b_1^2c_1^2a_3b_3 + c_2^2a_2^2a_3b_3 + c_2^2a_2^2b_1c_1 + a_3^2b_3^2b_1c_1 + a_3^2b_3^2)\\
&- 12m^2(b_1c_1c_2a_2 + c_2a_2a_3b_3 + a_3b_3b_1c_1) - 24m^2(b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2)\\
&+ 18(bcb_1c_1a_2a_3 + cac_2a_2b_3b_1 + aba_3b_3c_1c_2) - 27(a_2^2b_3^3c_1^2 + a_3^2b_1^2c_2^2)\\
&+ 6abca_2b_2c_1 - 12m^3(ab_3c_2 + bc_1a_3 + ca_2b_1).
\end{aligned}$$

Here we use the following dictionary between our notation of coefficients and Salmon's:

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) = (a, 3a_2, 3a_3, 3b_1, 6m, 3c_1, b, 3b_3, 3c_2, c).$$

Source: Lectures on invariant theory by I. Dolgachev

Bad degeneration of elliptic curves



$S = T = 0$

$C_t\colon y^2 z = x(x - t^2 z)(x - 2t^3 z)$ $\underset{t \to 0}{\rightsquigarrow}$ $y^2 z = x^3$

Isomorphic when $t \neq 0$

$x' = t^2 x \downarrow y' = t^3 y$

$C'_t\colon y'^2 z = x'(x' - z)(x' - 2tz)$ $\underset{t \to 0}{\rightsquigarrow}$ $y'^2 z = x'^2(x' - z)$

Not isomorphic at $t = 0$: *Two different possible limits!*

$S = -\frac{1}{81}$

Construction of quotients (GIT)

- X = projective variety
- G = algebraic group $\curvearrowright X$
- X^{ss} = open locus of **semistable points** at which an invariant function does not vanish
- Theorem (Mumford) *There exists a projective quotient*

$$\begin{tikzcd}
X^{ss} \arrow[r] & X^{ss}/G \\
U \arrow[u, phantom, "\cup" description] \arrow[r] & Spec\ \mathbb{C}[U]^G \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

Usually denoted by $X//G$

Recall: Hypersurface case (single equation)

Plane cubics (elliptic curves)

$$\{(x, y, z) \in \mathbb{P}^2 \,|\, F(x, y, z) = \sum_{i+j+k=3} a_{ijk} x^i y^j z^k = 0\}$$

$$\updownarrow$$

$$(a_{300}, a_{210}, a_{201}, \ldots, a_{021}, a_{012}, a_{003}) \in \mathbb{P}^9$$

$$y^2 z - x^2(x - z) = 0$$

$$x^0 y^2 z^1 - x^3 y^0 z^0 + x^2 y^0 z^1 = 0$$

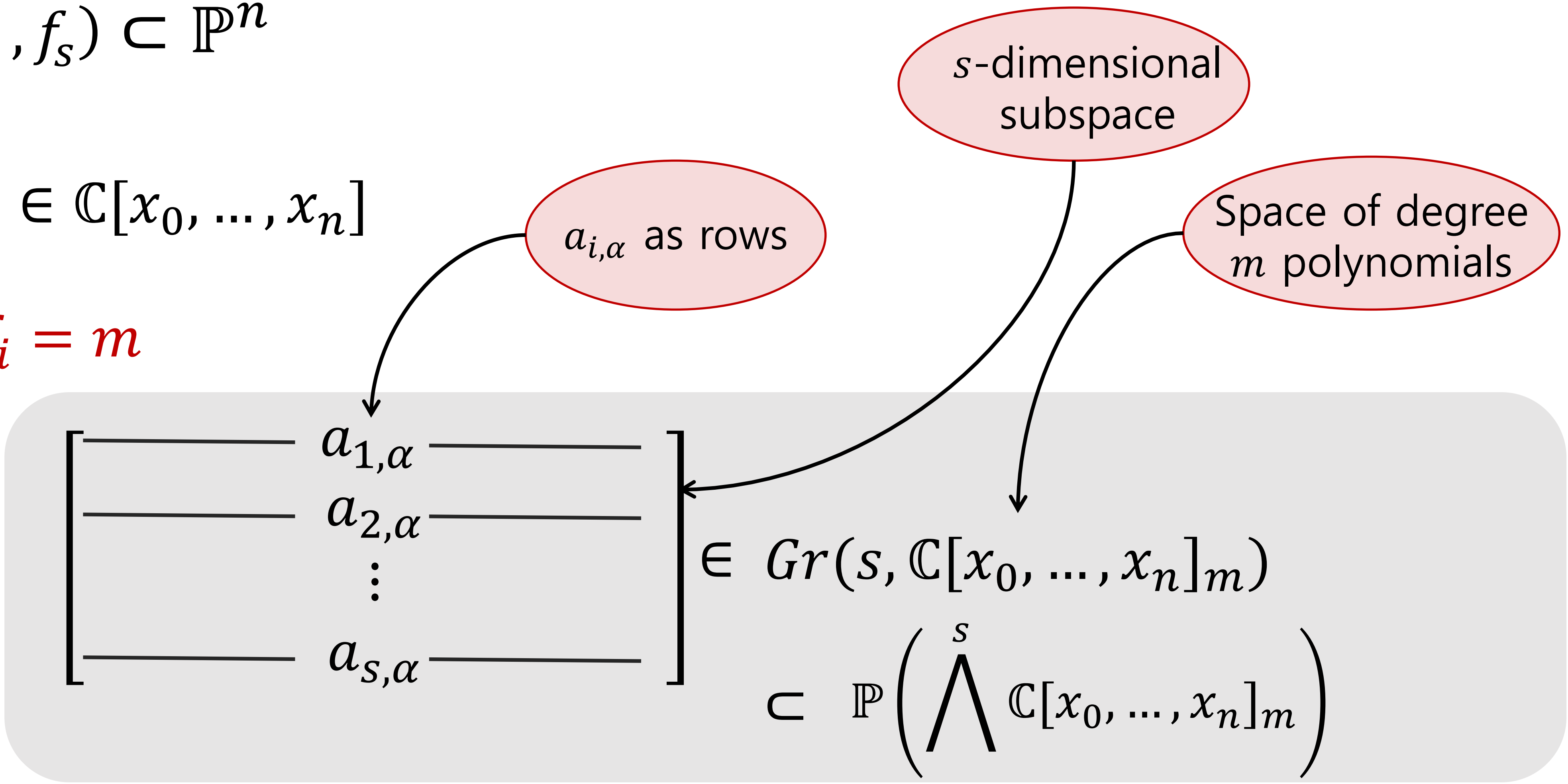
$$\updownarrow$$

$$(-1,0,1,0,0,0,0,1,0,0)$$

Construction of moduli of curves

- Need multiple equations to define a curve $C \subset \mathbb{P}^n$
- $C = V(f_1, f_2, \dots, f_s) \subset \mathbb{P}^n$
- $f_i = \sum a_{i,\alpha}\, x^\alpha \in \mathbb{C}[x_0, \dots, x_n]$

  Assume: $\deg f_i = m$



$a_{i,\alpha}$ as rows

s-dimensional subspace

Space of degree m polynomials

- $$C \subset \mathbb{P}^n \longleftrightarrow \begin{bmatrix} \rule{2cm}{0.4pt} & a_{1,\alpha} & \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} & a_{2,\alpha} & \rule{2cm}{0.4pt} \\ & \vdots & \\ \rule{2cm}{0.4pt} & a_{s,\alpha} & \rule{2cm}{0.4pt} \end{bmatrix} \in Gr(s, \mathbb{C}[x_0, \dots, x_n]_m) \subset \mathbb{P}\left(\bigwedge^{s} \mathbb{C}[x_0, \dots, x_n]_m\right)$$

Construction of moduli of curves

- Use a *"canonical embedding"* → Uniform number s and degree m of the defining equations

- $C =$ nonsingular projective curve of genus $g = \dim H^0(\omega_C)$

- $\omega_C =$ sheaf of holomorphic 1-forms

Does NOT depend on C

- $C \hookrightarrow \mathbb{P}\left(H^0\left(\omega_C{}^{\otimes \nu}\right)\right) \cong \mathbb{P}^{(2\nu-1)(g-1)-1}$ is cut out by s degree m equations!

Hilbert points

- $$C \hookrightarrow \mathbb{P}\left(H^0\left({\omega_C}^{\otimes \nu}\right)\right) \longleftrightarrow \begin{bmatrix} \text{———} & a_{1,\alpha} & \text{———} \\ \text{———} & a_{2,\alpha} & \text{———} \\ & \vdots & \\ \text{———} & a_{s,\alpha} & \text{———} \end{bmatrix} \in Gr(s, \mathbb{C}[x_0, \dots, x_n]_m)$$

$$\Vert$$

$[C]_m$ = mth Hilbert point of C

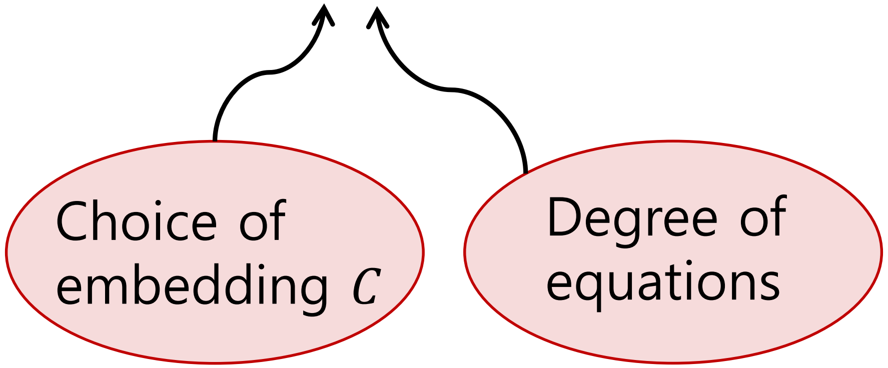
Vector space of Dimension $\binom{n+m}{m}$

- $$SL_{n+1}(\mathbb{C}) \curvearrowright \boldsymbol{H}_{\nu,m} = \overline{\{[C]_m \mid C \text{ nonsingular genus } g\}} \subset Gr(s, \mathbb{C}[x_0, \dots, x_n]_m)$$

- $\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C})$ = moduli space of curves of genus g

Moduli of curves

- (Moduli space of curves of genus g) = $\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C})$



Choice of embedding C

Degree of equations

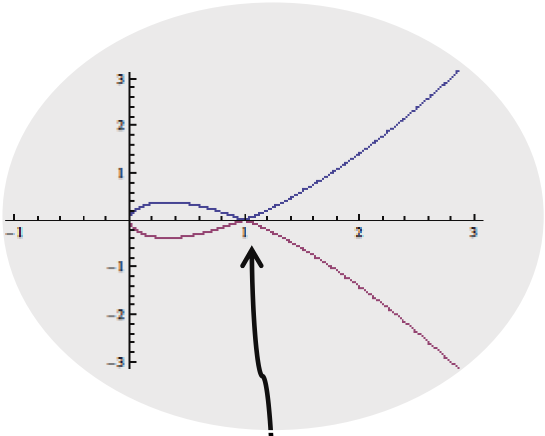
- Semistability depends on the choice of (ν, m) → Moduli space $\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C})$ depends on (ν, m)

- **Problem** *Describe the moduli spaces associated to various* (ν, m).

Deligne-Mumford stable curves

- A complete connected curve (of genus $g \geq 2$) is Deligne-Mumford stable if
  - it has only nodes as singularity, and;
  - it has only finite automorphisms.

A smooth rational component meets the rest of the curve in ≥ 3 points



An ordinary node

- $(\nu, m) = (\geq 5, \gg 0)$

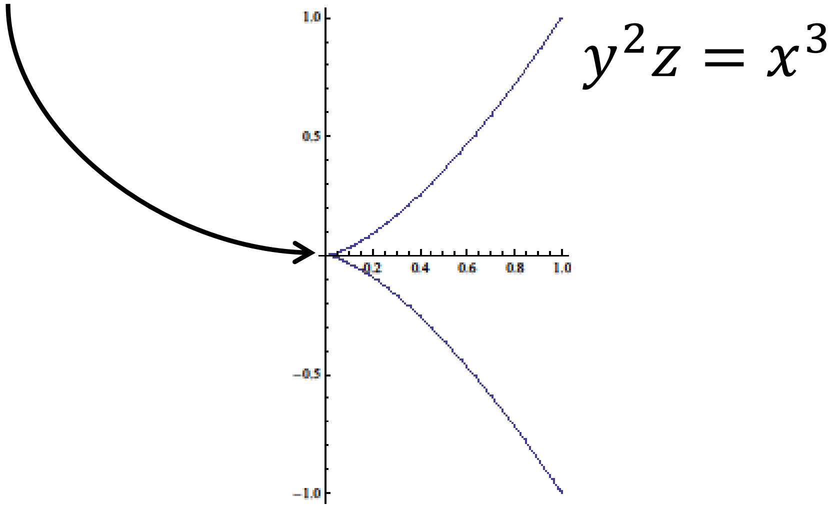
"Mumford and I realized in 1974 that.." (Gieseker's Tata lecture notes)

Theorem (Mumford, Gieseker 1974~80)

$\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C}) \simeq$ Moduli space $\overline{M_g}$ of Deligne-Mumford stable curves.

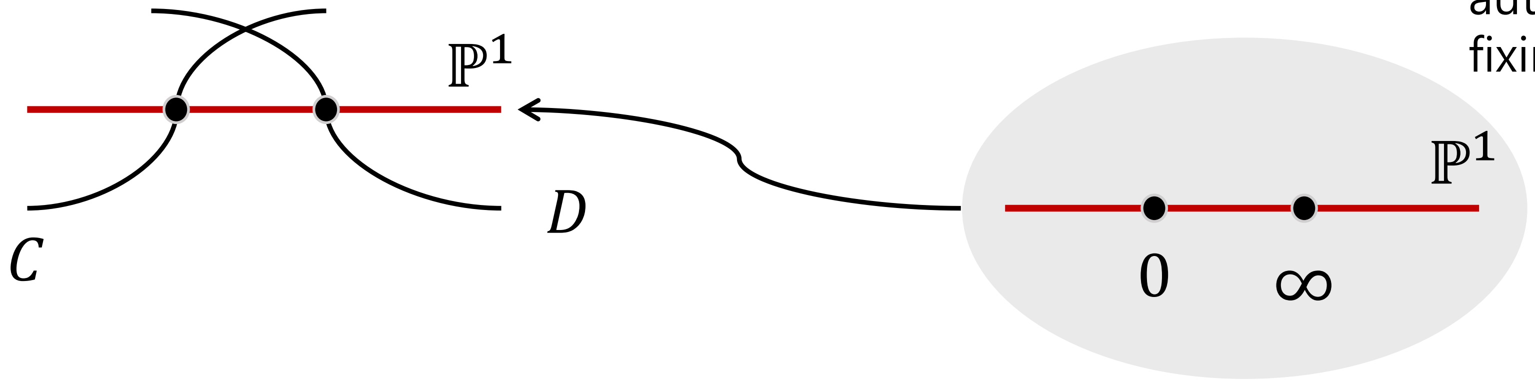
Deligne-Mumford stable curves

- Unstable curves (bad singularity)



- Unstable curves (infinite automorphisms)

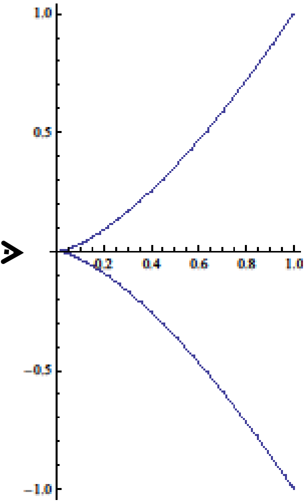
Fix C, D and move the points on $\mathbb{P}^1$ by automorphisms fixing 0 and ∞

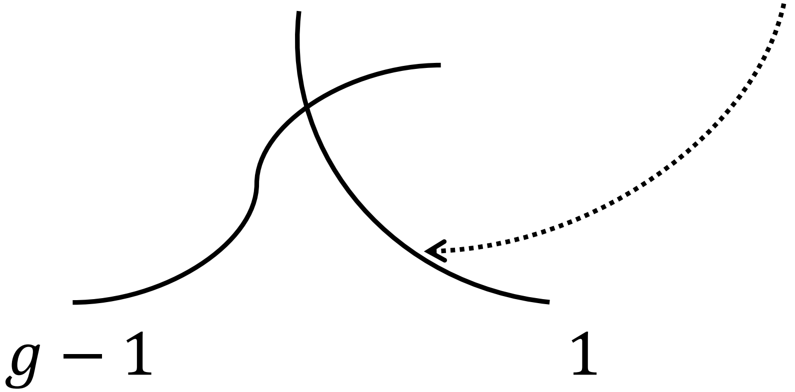


Infinite automorphisms fixing 0 and ∞

Pseudostable curves

- A complete connected curve is *pseudostable* if it has nodes and cusps as singularities, has finite automorphisms, but no elliptic tails.





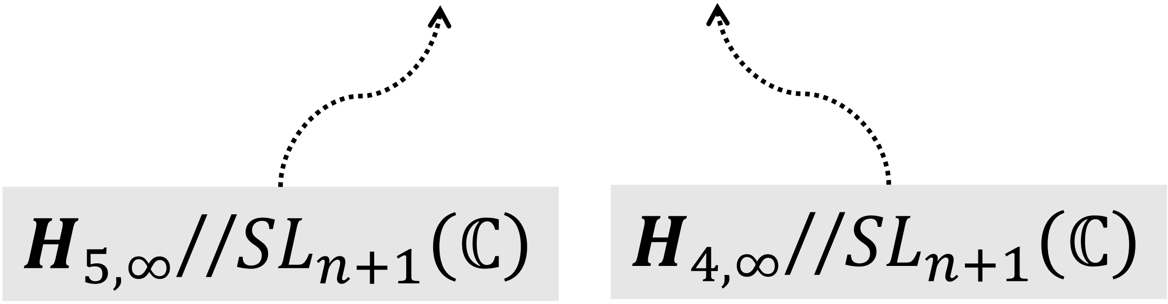
- $(\nu = 4;\ m \gg 0)$

Theorem (Schubert 1991, Morrison-Hyeon 2010)

$\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C}) \simeq$ Moduli space $\overline{M_g}^{ps}$ of *pseudostable* curves.

$\overline{M_g}$ VS $\overline{M_g}^{ps}$

Problem *What is the relation between* $\overline{M_g}$ and $\overline{M_g}^{ps}$?



$\boldsymbol{H}_{5,\infty}//SL_{n+1}(\mathbb{C})$

$\boldsymbol{H}_{4,\infty}//SL_{n+1}(\mathbb{C})$

Problem More generally: *Construct the moduli spaces associated to various* (ν, m) *and describe the relations between them.*

Issues

- In the two "old" moduli spaces, nonisomorphic curves are separated, which is not the case in general due to much more complicated orbit structures.
- No motivation to solve the issue!

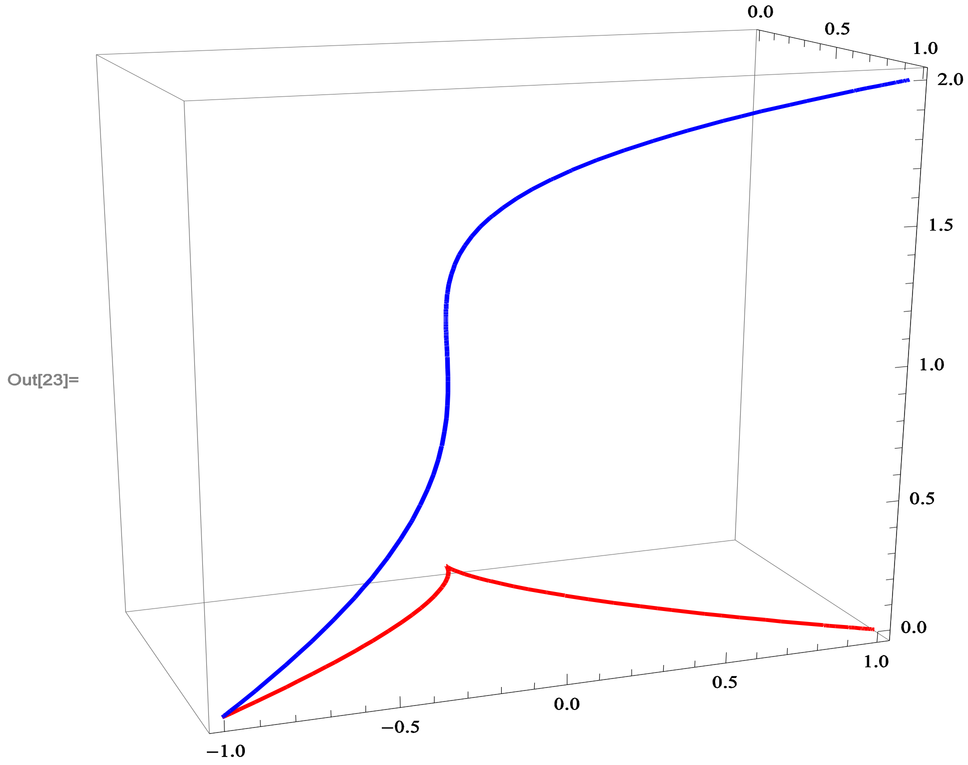
Found in MMP for $\overline{M_g}$

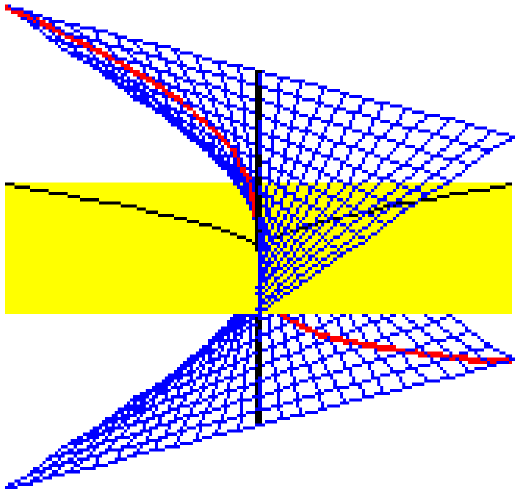
Classification of varieties

- X and Y are **birational** to each other if there exist open subvarieties

$$X \supset U \simeq V \subset Y$$

- e.g. $f : X \to Y$ (a resolution of singularities), blow up





Classification of varieties

Classification up to isomorphism → Moduli theory

- How many varieties (of a given type) are there? : **dimension** of the moduli
- Can a given variety (algebraically) deformed to another? : **connectedness**
- Functoriality ⇝ computation of **invariants** : *new examples of varieties*

Classification up to birational equivalence → Minimal Model Program

- Minimal Model Program : A procedure to find a *canonical representative* in each birational equivalence class

Minimal Model Program

- (Dimension $= 1$) There is a unique nonsingular projective curve X birational to any given curve.

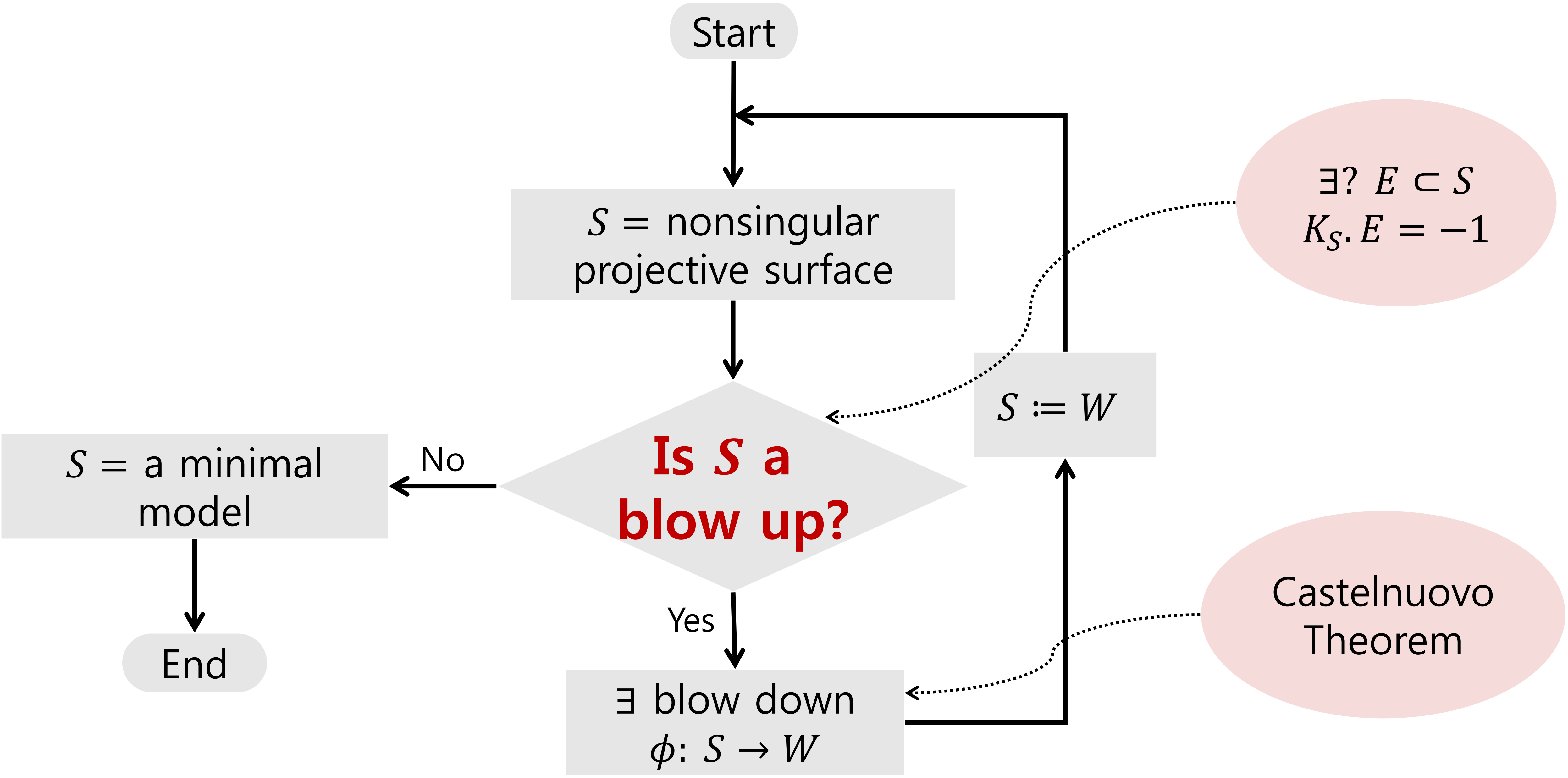
- (Dimension $= 2$) There are many nonsingular projective surfaces that are birational to each other.

  - E.g. A blow up of X (with a nonsingular center) is nonsingular if X is nonsingular.

- $Bl_pX \simeq \overline{X}\#S^2$ (X is topologically simpler than its blow up!)

- Start with X and blow down (until we can't).

Minimal Model Program (dimension= 2)



Start

$S =$ nonsingular projective surface

Is S a blow up?

No

$S =$ a minimal model

End

Yes

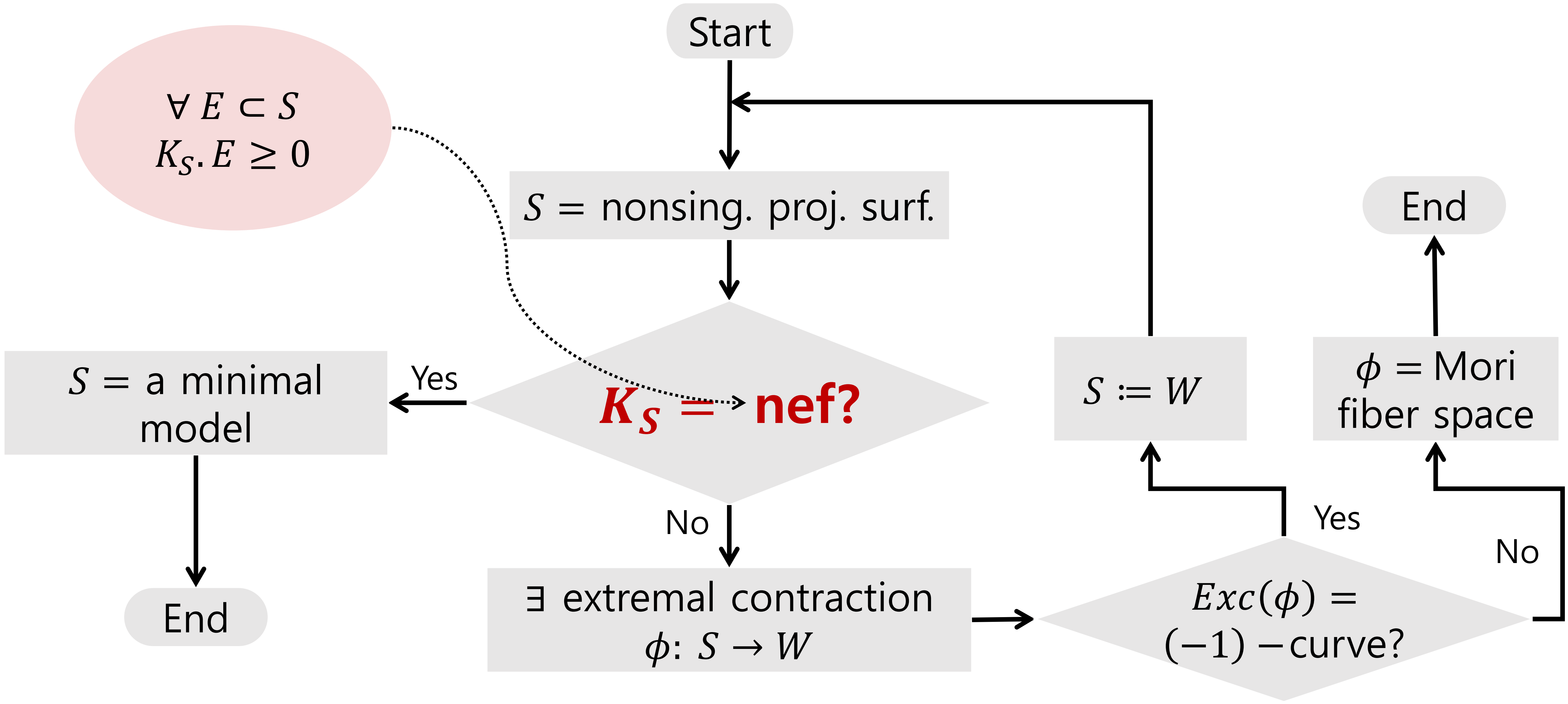
$\exists$ blow down $\phi : S \to W$

$S := W$

$\exists ?\ E \subset S$
$K_S . E = -1$

Castelnuovo Theorem

Minimal Model Program (dimension= 2)



Start

$S =$ nonsing. proj. surf.

$\forall\, E \subset S$
$K_S . E \geq 0$

$K_S =$ nef?

Yes

$S =$ a minimal model

End

No

$\exists$ extremal contraction
$\phi \colon S \to W$

$Exc(\phi) =$
(-1) −curve?

Yes

$S := W$

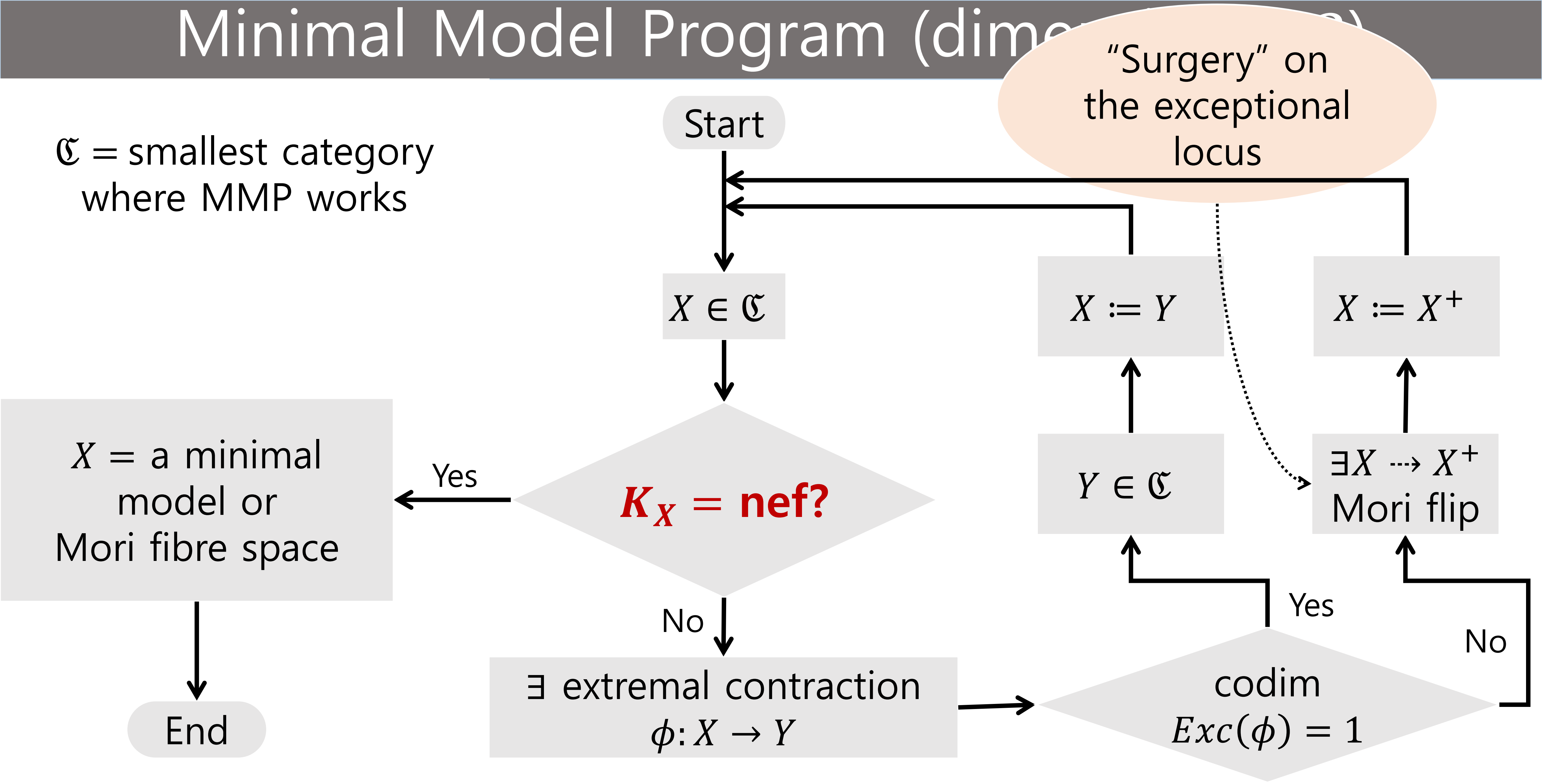
No

$\phi =$ Mori fiber space

End

Minimal Model Program (dime

$\mathfrak{C}$ = smallest category where MMP works



"Surgery" on the exceptional locus

Start

$X \in \mathfrak{C}$

K_X = nef?

Yes

X = a minimal model or Mori fibre space

End

No

$\exists$ extremal contraction $\phi: X \to Y$

codim $Exc(\phi) = 1$

Yes

$Y \in \mathfrak{C}$

$X := Y$

No

$\exists X \dashrightarrow X^+$ Mori flip

$X := X^+$

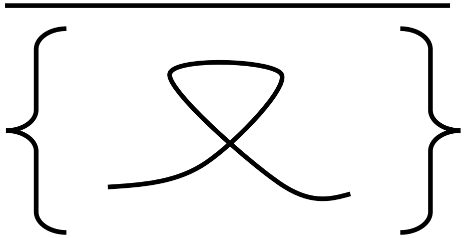
Log MMP for the moduli of curves

- $K_{\overline{M_g}}$ contracts too many divisors! ⟶ Use $\boldsymbol{K_{\overline{M_g}} + \alpha\delta}$ instead

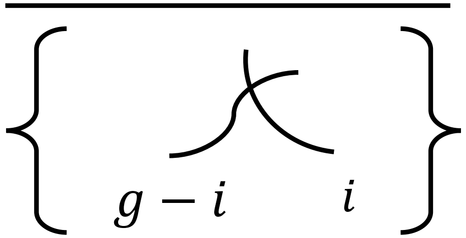
  $\alpha \in [0,1] \cap \mathbb{Q}$

- $\delta = \delta_0 + \delta_1 + \cdots + \delta_{\lfloor g/2 \rfloor}$

$\delta_0 =$



$\delta_i =$



$i = 1, 2, \ldots, \lfloor g/2 \rfloor$

- Replace **$K_{\overline{M_g}}$ nef?** by **$K_{\overline{M_g}} + \alpha\delta$ nef?**

Log MMP for the moduli of curves

- $\boldsymbol{K_{\overline{M_g}} + \alpha\delta}$ **nef?**

Curves
to be
contracted!

- Theorem of Gibney-Keel-Morrison

  \+

  Fulton's conjecture

  →

  $$\left\{E \;\middle|\; (K_{\overline{M_g}}+\alpha\delta).E < 0\right\}$$

Generators
of the cone
of curves

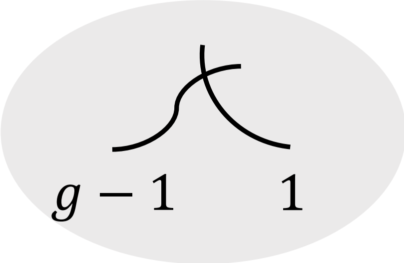
- $\delta_i \simeq \overline{M_{g-i,1}} \times \overline{M_{i,1}}$ → Intersection can be carried out by an inductive argument

Hassett-Keel program

- Hassett-Keel program: Run the log MMP guided by $K_{\overline{M_g}} + \alpha\delta$ as we decrease α from 1 to 0.

Vary the j invariant

Dimension one locus in $\overline{M_g}$



- $(K_{\overline{M_g}} + \alpha\delta)$. < 0 ⟹ $\exists\, T: \overline{M_g} \to Y$ extremal contraction

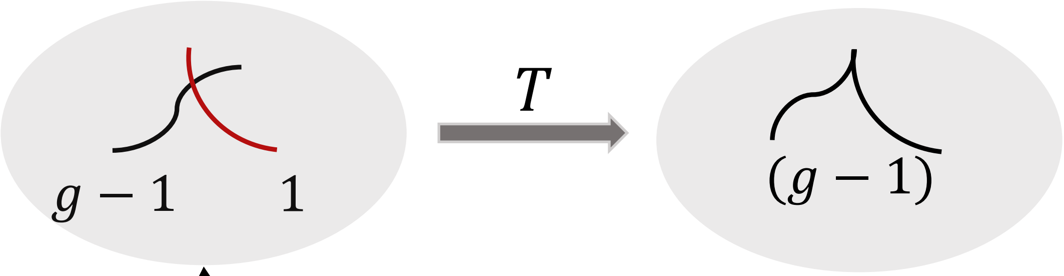
  precisely when $\alpha < 9/11$

Hassett-Hyeon 2009

- $\overline{M_g}$ → MMP w.r.t $K_{\overline{M_g}} + \frac{9}{11}\delta$ → $T: \overline{M_g} \to \overline{M_g}^{ps}$

Hassett-Keel program

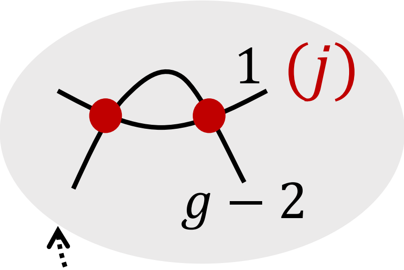
- $T: \overline{M_g} \to \overline{M_g}^{ps}$



$\overline{M_g}^{ps} \in \mathfrak{C}$

Codim one
locus

- $(K_{\overline{M_g}^{ps}} + 7/10\delta^{ps}).$



$= 0$

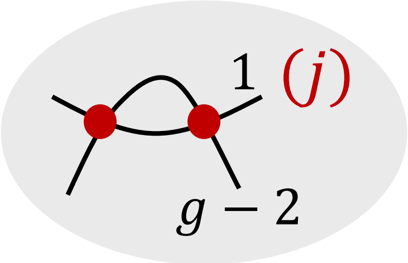
$\exists \Psi: \overline{M_g}^{ps} \to Y$
extremal contraction

Codim two locus
(vary the j-invariant
and attaching points)

Hassett-Keel program

- $\overline{M_g}^{ps}$ → MMP w.r.t $K_{\overline{M_g}^{ps}} + 7/10\delta^{ps}$ → $\Psi: \overline{M_g}^{ps} \to \overline{M_g}^{cs}$

From another parameter space called *Chow variety*

- Codim 

$= 2$ → $\overline{M_g}^{cs} \notin \mathfrak{C}$

- Need a Mori flip: $\overline{M_g}^{ps} \dashrightarrow (\overline{M_g}^{ps})^{+} \in \mathfrak{C}$.

H-semistable curves

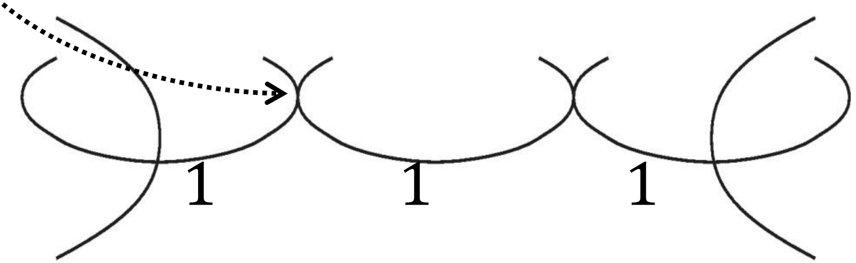
- **Theorem** (Hassett, Lee and Hyeon) $(\nu = 2;\ m \gg 0)$

  (a) $\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C}) \simeq$ Moduli space $\overline{M_g}^{hs}$ of *h-semistable* curves;

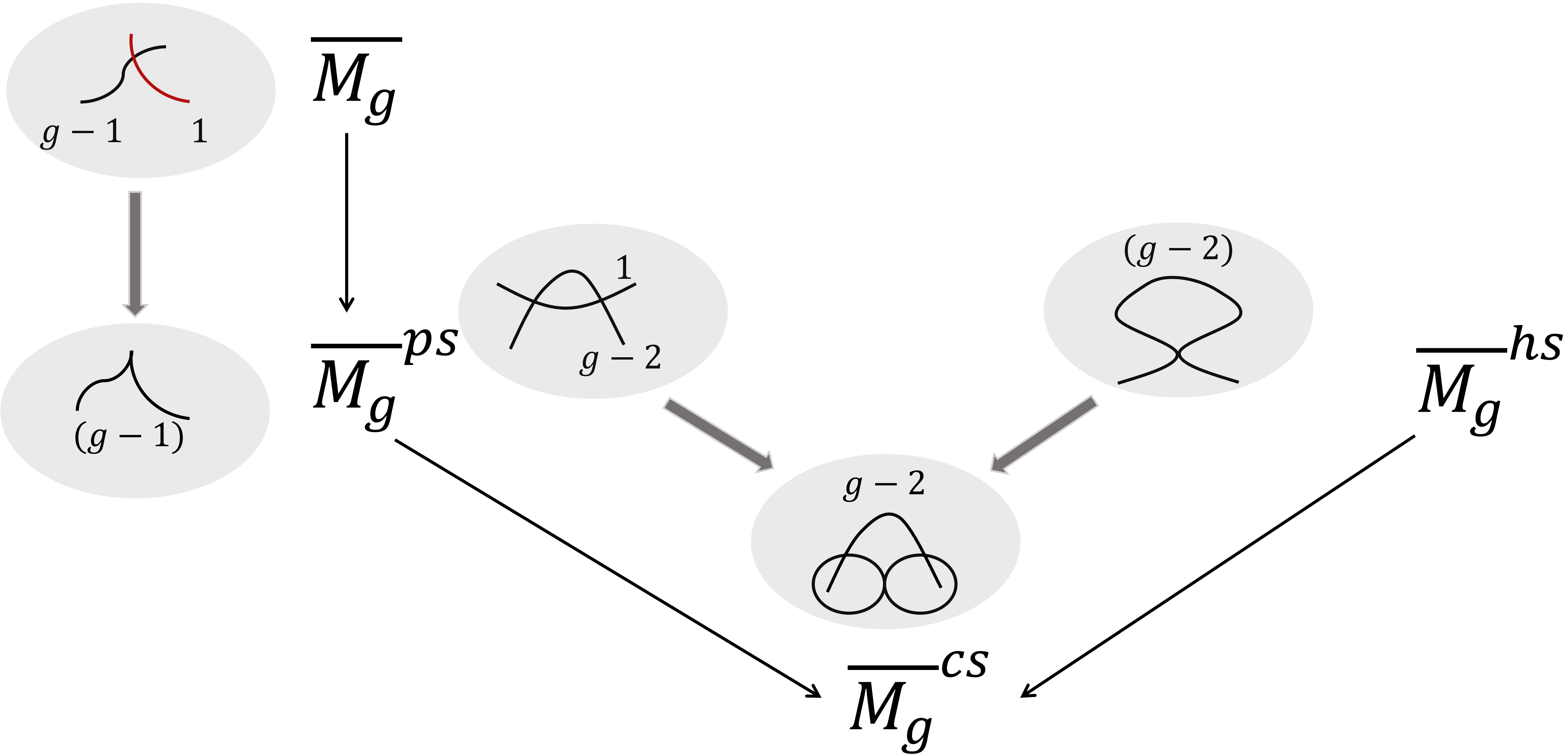
  (b) $\overline{M_g}^{hs} \simeq (\overline{M_g}^{ps})^{+}$ (the Mori flip)

  (c) $\overline{M_g}^{hs} \simeq \overline{M_g}(\alpha)$ for $\alpha \in \left(\frac{7}{10} - \epsilon, \frac{7}{10}\right)$.

- A complete connected curve C is *h-semistable* if
  - it has nodes, cusps and tacnodes as singularities;
  - a smooth rational component of it meets the rest of the curves in ≥ 3 points counting mutiplicity;
  - an elliptic component of it meets the rest of the curves in ≥ 2 points NOT counting mutiplicity;
  - it has no tacnodal elliptic chains.

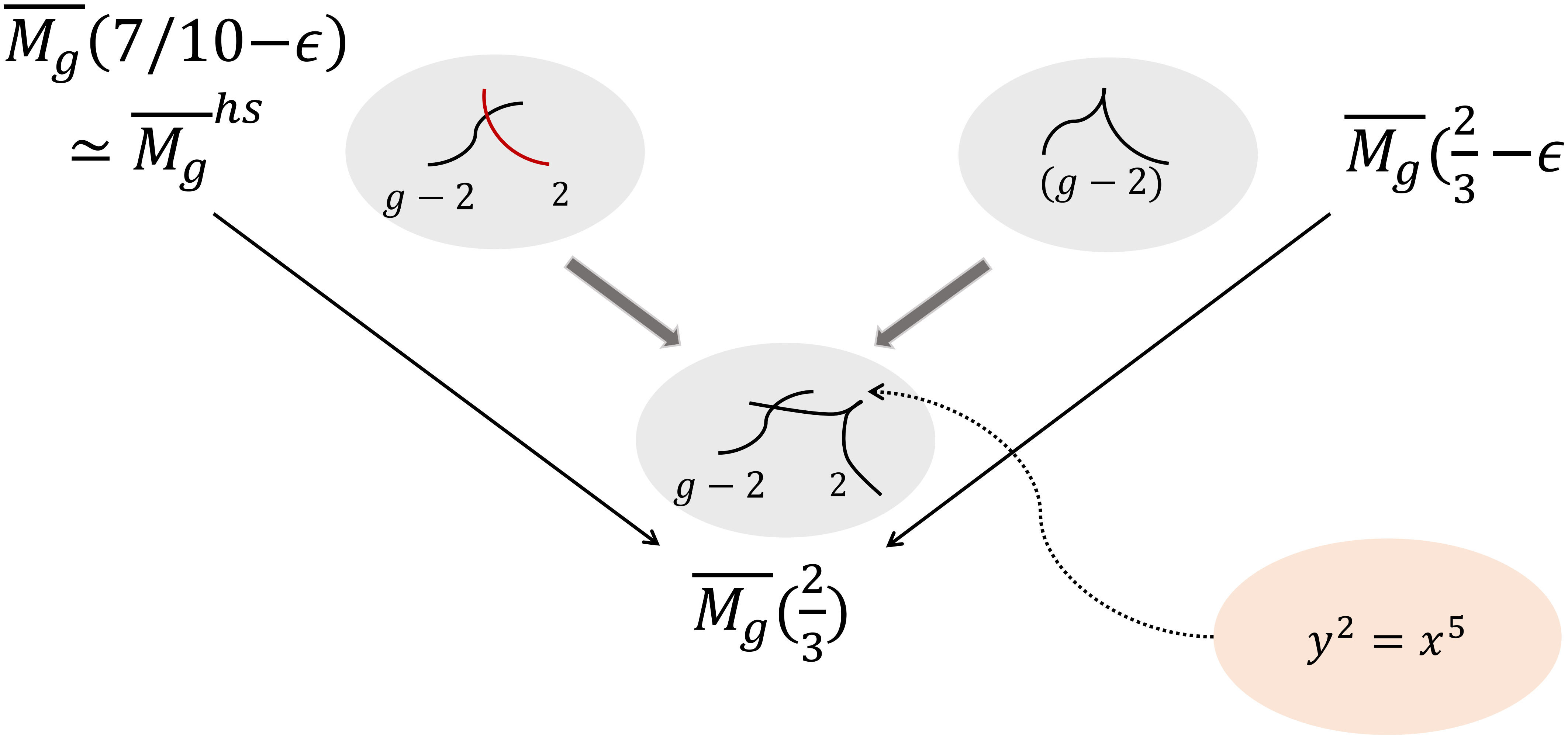


Log MMP for the moduli of curves(2008~13)



$$\begin{tikzcd}
\overline{M_g} \arrow[d] & & \\
\overline{M_g}^{ps} \arrow[dr] & & \overline{M_g}^{hs} \arrow[dl] \\
& \overline{M_g}^{cs} &
\end{tikzcd}$$

Alper-Fedorchuk-Smyth-van der Wyck (2013)



$\overline{M_g}(7/10-\epsilon) \simeq \overline{M_g}^{hs}$

$\overline{M_g}(\frac{2}{3}-\epsilon)$

$\overline{M_g}(\frac{2}{3})$

$y^2 = x^5$

- We are inching toward the canonical model $\overline{M_g}(0)$!

Hassett-Keel Program

- As α gets smaller, $\overline{M_g}(\alpha)$ is expected to be a moduli space of curves with increasingly worse singularities.

GIT quotient $\boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C})$	Moduli space	Singularities
$(2,6)$	$\overline{M_g}(2/3)$	$A_1,\ A_2$
$(2,4.5)$	$\overline{M_g}(19/29)$	$A_1, \dots, A_4,\ A_5{}'$
$(2,1.25)$	$\overline{M_g}(17/28)$	$A_1, \dots, A_5$
$(2,27/14)$	$\overline{M_g}(49/83)$	$A_1, \dots, A_6$
$(2,1.5)$	$\overline{M_g}(5/9)$	$A_1, \dots, A_6,$ $D_4, D_5', D_6{}'$
$(1, \gg 0)$	$\overline{M_g}\left(\dfrac{3g+8}{8g+4} - \epsilon\right)$	ADE, X_9, J_{10}, E_{12},Ribbons etc.

Fedorchuk-Smyth
arXiv 1012.0329

Hilbert points

- $$C \subset \mathbb{P}^n \quad \longleftrightarrow \quad \begin{bmatrix} \text{———} \; a_{1,\alpha} \; \text{———} \\ \text{———} \; a_{2,\alpha} \; \text{———} \\ \vdots \\ \text{———} \; a_{s,\alpha} \; \text{———} \end{bmatrix} \quad \in \quad Gr(s, \mathbb{C}[x_0, \dots, x_n]_m)$$

$$\Vert$$

$[C]_m = m$th Hilbert point of C

Vector space of
Dimension $\binom{n+m}{m}$

- $$\boldsymbol{H}_{v,m} = \overline{\{[\boldsymbol{C}]_{\boldsymbol{m}} \mid C \text{ nonsingular genus } g\}} \subset Gr(s, \mathbb{C}[x_0, \dots, x_n]_m)$$

- $\boldsymbol{H}_{v,m}//SL_{n+1}(\mathbb{C})$ = moduli space of curves of genus g

Toward new moduli spaces

- Alper-Fedorchuk-Smyth-van der Wyck : GIT free approach
- Prediction: $\overline{M_g}\left(\frac{2}{3}\right) \simeq \boldsymbol{H}_{\nu,m}//SL_{n+1}(\mathbb{C})$, with $(\nu, m) = (2,6)$.
- $C \subset \mathbb{P}^n$: defined by an ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$
- Hilbert-Mumford numerical criterion: $[C]_m$ is (semi)stable if and only if

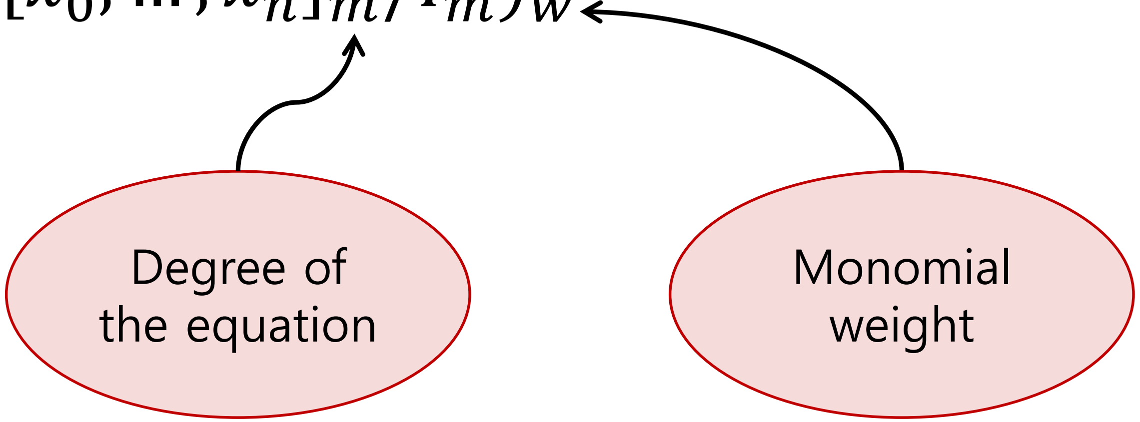
  $\forall$ choice of coordinates and $\forall r = (r_0, \dots, r_n) \in \mathbb{Z}^{n+1}, \sum r_i = 0$,

  $\exists$ a basis $\{x^{a(1)}, \dots, x^{a(l)}\}$ for $\mathbb{C}[x_0, \dots, x_n]_m / I_m$ such that

  $\sum r.a(i) < 0$ (resp. ≤ 0)

Finite Hilbert Stability

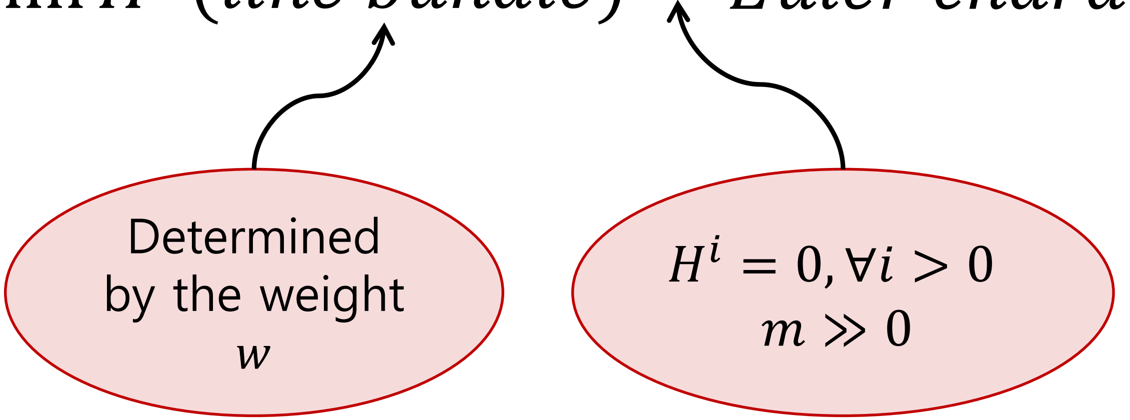
- Key to establishing the stability of $[C]_m$: estimation of the $\dim(\mathbb{C}[x_0,\dots,x_n]_m/I_m)_w$



Degree of the equation

Monomial weight

- Estimation of $\dim(\mathbb{C}[x_0,\dots,x_n]_m/I_m)_w$ → Estimation of $\dim H^0(line\ bundle) = Euler\ characteristic$



Determined by the weight w

$H^i = 0, \forall i > 0$
$m \gg 0$

Finite Hilbert Stability

- Higher cohomologies do NOT vanish for small m. A completely new method should be developed.

- **BIG THEOREM** (Mumford, Gieseker ~**1974**) A smooth ν-canonical curve of genus $g \geq 2$ has stable mth Hilbert point for $\nu \geq 2$ and $m \gg 0$.

- **CONJECTURE** (I. Morrison ~2010) A smooth bicanonical curve of genus $g \geq 3$ has stable mth Hilbert point whenever $(g, m) \neq (3,2)$.