The University of Sydney

MATH2008 Introduction to Modern Algebra

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2008/)

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Assignment 1

1. If V is any inner product space then the *length* of a vector $\underset{\sim}{v} \in V$ is the quantity $\|\underset{\sim}{v}\|$ defined by $\|\underset{\sim}{v}\| = \sqrt{(\underset{\sim}{v}, \underset{\sim}{v})}$.

Use the axioms IP1, IP2, IP3 and IP4 to prove that

$$\|\underset{\sim}{x} - \underset{\sim}{y}\|^2 + \|\underset{\sim}{x} + \underset{\sim}{y}\|^2 = 2\|\underset{\sim}{x}\|^2 + 2\|\underset{\sim}{y}\|^2$$

for all $\underset{\sim}{x}$, $\underset{\sim}{y} \in V$.

Solution.

The axioms IP1 to IP4 say that the inner product is commutative (that is, $(\underset{\sim}{x}, \underset{\sim}{y}) = (\underset{\sim}{y}, \underset{\sim}{x})$ for all vectors $\underset{\sim}{x}$ and $\underset{\sim}{y}$ in the space V), linear in the first variable $((\underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{z}) = (\underset{\sim}{x}, \underset{\sim}{z}) + (\underset{\sim}{y}, \underset{\sim}{z})$ and $(\lambda\underset{\sim}{x}, \underset{\sim}{y}) = \lambda(\underset{\sim}{x}, \underset{\sim}{y})$ for all vectors $\underset{\sim}{x}$, $\underset{\sim}{y}$ and $\underset{\sim}{z}$ and all scalars λ) and positive definite $((\underset{\sim}{x}, \underset{\sim}{x}) \geq 0$, with equality only if $\underset{\sim}{x} = \underset{\sim}{0})$. It follows readily from the commutativity and linearity in the first variable that it is also linear in the second variable $((\underset{\sim}{x}, \underset{\sim}{y} + \underset{\sim}{z}) = (\underset{\sim}{x}, \underset{\sim}{y}) + (\underset{\sim}{x}, \underset{\sim}{z})$ and $(\underset{\sim}{x}, \lambda\underset{\sim}{y}) = \lambda(\underset{\sim}{x}, \underset{\sim}{y}))$. For the purposes of questions like this one, students are permitted to use this as though it were an axiom, although strictly speaking it is not. Similarly it is easy to show that $(\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{z}) = (\underset{\sim}{x}, \underset{\sim}{z}) - (\underset{\sim}{y}, \underset{\sim}{z})$, and students may use this also as though it were an axiom.

Now for all $\underset{\sim}{x}$, $\underset{\sim}{y} \in V$,

$$\begin{aligned}
\|\underset{\sim}{x} - \underset{\sim}{y}\|^2 + \|\underset{\sim}{x} + \underset{\sim}{y}\|^2 &= (\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{x} - \underset{\sim}{y}) + (\underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y}) \\
&= (\underset{\sim}{x}, \underset{\sim}{x} - \underset{\sim}{y}) - (\underset{\sim}{y}, \underset{\sim}{x} - \underset{\sim}{y}) + (\underset{\sim}{x}, \underset{\sim}{x} + \underset{\sim}{y}) + (\underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y}) \\
&= (\underset{\sim}{x}, \underset{\sim}{x}) - (\underset{\sim}{x}, \underset{\sim}{y}) - (\underset{\sim}{y}, \underset{\sim}{x}) + (\underset{\sim}{y}, \underset{\sim}{y}) \\
&\qquad + (\underset{\sim}{x}, \underset{\sim}{x}) + (\underset{\sim}{x}, \underset{\sim}{y}) + (\underset{\sim}{y}, \underset{\sim}{x}) + (\underset{\sim}{y}, \underset{\sim}{y}) \\
&= 2(\underset{\sim}{x}, \underset{\sim}{x}) + 2(\underset{\sim}{y}, \underset{\sim}{y}) \\
&= 2\|\underset{\sim}{x}\|^2 + 2\|\underset{\sim}{y}\|^2
\end{aligned}$$

as required.

If one wished to strictly use just IP1 to IP4, the expansion of $(\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{x} - \underset{\sim}{y})$ could be done as follows:

$$\begin{aligned}
(\underset{\sim}{x} - \underset{\sim}{y}, \underset{\sim}{x} - \underset{\sim}{y}) &= (\underset{\sim}{x} + (-\underset{\sim}{y}), \underset{\sim}{x} + (-\underset{\sim}{y})) \\
&= (\underset{\sim}{x} + (-1)\underset{\sim}{y}, \underset{\sim}{x} + (-1)\underset{\sim}{y}) \\
&= (\underset{\sim}{x}, \underset{\sim}{x} + (-1)\underset{\sim}{y}) + ((-1)\underset{\sim}{y}, \underset{\sim}{x} + (-1)\underset{\sim}{y}) \\
&= (\underset{\sim}{x} + (-1)\underset{\sim}{y}, \underset{\sim}{x}) + (-1)(\underset{\sim}{y}, \underset{\sim}{x} + (-1)\underset{\sim}{y}) \\
&= (\underset{\sim}{x}, \underset{\sim}{x}) + ((-1)\underset{\sim}{y}, \underset{\sim}{x}) + (-1)(\underset{\sim}{x} + (-1)\underset{\sim}{y}, \underset{\sim}{y}) \\
&= (\underset{\sim}{x}, \underset{\sim}{x}) + (-1)(\underset{\sim}{y}, \underset{\sim}{x}) + (-1)((\underset{\sim}{x}, \underset{\sim}{y}) + ((-1)\underset{\sim}{y}, \underset{\sim}{y})) \\
&= (\underset{\sim}{x}, \underset{\sim}{x}) - (\underset{\sim}{y}, \underset{\sim}{x}) - ((\underset{\sim}{x}, \underset{\sim}{y}) + (-1)(\underset{\sim}{y}, \underset{\sim}{y})) \\
&= (\underset{\sim}{x}, \underset{\sim}{x}) - (\underset{\sim}{y}, \underset{\sim}{x}) - (\underset{\sim}{x}, \underset{\sim}{y}) + (\underset{\sim}{y}, \underset{\sim}{y}))
\end{aligned}$$

as expected.

2. Let A be the following 4×4 matrix:

$$A = \begin{pmatrix} 1 & 3 & 3 & 3 \\ 3 & 1 & 3 & 3 \\ 3 & 3 & 1 & 3 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$

(i) The characteristic equation of A is $(\lambda + 2)^3(\lambda - 10) = 0$. (You are not required to prove this.) Find a basis for the -2-eigenspace of A. That is, find the general solution of the system of linear equations $(A+2I)\underset{\sim}{v} = \underset{\sim}{0}$. (You should find that the basis consists of three vectors.)

(ii) Find a 10-eigenvector of A, and show that it is orthogonal to all three basis vectors that you found in Part (i).

(iii) Apply the Gram-Schmidt process to the basis you found in Part (i), and hence find an orthogonal basis for the -2-eigenspace of A.

(iv) Use Parts (ii) and (iii) to write down an orthogonal basis of $\mathbb{R}^4$ made up of three -2-eigenvectors of A and a 10-eigenvector of A. Normalize this basis to obtain an orthonormal basis of $\mathbb{R}^4$.

(v) Let P be the matrix whose columns constitute the orthonormal basis of $\mathbb{R}^4$ that you found in Part (iv). Check that P is an orthogonal matrix and that P^TAP is diagonal.

Solution.

(i) The -2-eigenspace is the solution space of the homogeneous system of equations

$$\begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Applying row operations to the augmented matrix quickly leads to the following echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

There is only one leading entry, and it occurs in the column that corresponds to the variable x. So y, z and w are free. So the general solution is obtained by letting $y = r$, $z = s$ and $w = t$, where r, s and t are arbitrary parameters. The equation corresponding to the nonzero row of the echelon matrix yields $x = -r - s - t$. So the general solution is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -r-s-t \\ r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and we deduce that

$$\left\{ v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the -2-eigenspace.

(ii) To find a 10-eigenvector, find a nonzero solution of

$$\begin{pmatrix} -9 & 3 & 3 & 3 \\ 3 & -9 & 3 & 3 \\ 3 & 3 & -9 & 3 \\ 3 & 3 & 3 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Applying row operations to the augmented matrix leads to the following echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This time there is only one free variable, and the general solution is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The question asked for an eigenvector; this means that we should specify a value for t. Any nonzero value will do. So (for example) we can put $t = 1$. Thus the column vector v_4 whose four entries are all equal to 1 is a 10-eigenvector.

We must show that v_4 is orthogonal to each of v_1, v_2 and v_3. That is, we must show that $v_1 \cdot v_4 = v_2 \cdot v_4 = v_3 \cdot v_4 = 0$. Now

$$v_1 \cdot v_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (-1) \times 1 + 1 \times 1 + 0 \times 1 + 0 \times 1 = -1 + 1 = 0;$$

similarly, $v_2 \cdot v_4 = -1 + 0 + 1 + 0 = 0$ and $v_2 \cdot v_4 = -1 + 0 + 0 + 1 = 0$.

(iii) The formulas for the Gram-Schmidt process are

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 \\ u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 \end{aligned}$$

Now we have that

$$v_2 \cdot u_1 = v_2 \cdot v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1,$$

and

$$u_1 \cdot u_1 = v_1 \cdot v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 2,$$

and therefore

$$u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}.$$

We find that $v_3 \cdot u_1 = 1$ and $v_3 \cdot u_2 = 1/2$, while $u_1 \cdot u_1 = 1$ and $u_2 \cdot u_2 = 3/2$. So

$$u_3 = v_3 - \tfrac{1}{2}u_1 - \tfrac{1/2}{3/2}u_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}.$$

(*iv*) The vectors u_1, u_2, u_3 and v_4 comprise an orthogonal basis of $\mathbb{R}^4$, since they are nonzero and orthogonal to each other. (This guarantees that they are linearly independent, and since there are four of them—and four is the dimension of $\mathbb{R}^4$—they must also span $\mathbb{R}^4$.) To get an orthonormal basis we must divide each vector in the orthogonal basis by its length, (the square root of the dot product of the vector with itself). Now $\|u_1\| = \sqrt{u_1 \cdot u_1} = \sqrt{2}$, and similarly $\|u_2\| = \sqrt{3/2}$, $\|u_3\| = \sqrt{4/3}$ and $\|v_4\| = 2$. So the orthonormal basis is

$$\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1/2\sqrt{3} \\ -1/2\sqrt{3} \\ -1/2\sqrt{3} \\ \sqrt{3}/2 \end{pmatrix}, \quad \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

(*v*) The matrix P is

$$P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} & 1/2 \\ 0 & \sqrt{2}/\sqrt{3} & -1/2\sqrt{3} & 1/2 \\ 0 & 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

The (i, j) entry of $P^T P$ is the dot product of the ith and jth columns of P, and since the basis is orthonormal this is 0 if $i \neq j$ and 1 if $i = j$. So $P^T P = I$, which means that P is orthogonal. Since the first three columns of P are -2-eigenvectors of A and the fourth column is a 10-eigenvector it follows that the columns of AP are just the same as the columns of P multiplied by the scalars -2, -2, -2 and 10 (respectively). So $P^T AP = P^T (AP)$ is

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 5 \\ -\sqrt{2} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 5 \\ 0 & \frac{-2\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 5 \\ 0 & 0 & -\sqrt{3} & 5 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

3. (To be done using MAGMA.)

(*i*) Let `A` be the matrix in Question 2 above. Define `W` to be the left nullspace of `A + 2I`, and get MAGMA to print `W`. (You should find that `W` has dimension 3.)

(*ii*) MAGMA's names for the three basis vectors it has found for the space `W` are `W.1`, `W.2` and `W.3`. Print these. Then apply the Gram-Schmidt process to these three vectors, to find an orthogonal basis for `W`. (You may wish to load the file `t3defs.m` before doing this part.)

(*iii*) Use MAGMA to find the 5th degree polynomial of best fit for the following ten points:

$$(-2, 100),\ (-1.5, 40),\ (-1, 12),\ (-0.5, 3),\ (0, 1),\ (0.5, 0.5),$$
$$(1, 0),\ (1.5, -3),\ (2, -14), (3, -125).$$

(Note: This will involve entering a certain 6×10 matrix, whose entries are numbers such as the square of -1.5, etc.. Do not forget the necessary brackets when entering this: `(-1.5)^2`, not `-1.5^2`.)

(*iv*) Evaluate this 5th degree polynomial at -7.

Solution.

```
> R:=RealField();
> M:=KMatrixSpace(R,4,4);
> A:=M![1,3,3,3,3,1,3,3,3,3,1,3,3,3,3,1];
> A;
[1 3 3 3]
[3 1 3 3]
[3 3 1 3]
[3 3 3 1]
> I:=M!0;
> for i in [1..4] do
for> I[i,i]:=1;
for> end for;
> I;
[1 0 0 0]
[0 1 0 0]
[0 0 1 0]
[0 0 0 1]
> W:=NullSpace(A+2*I);
> W;
Vector space of degree 4, dimension 3 over Real Field
Echelonized basis:
( 1  0  0 -1)
( 0  1  0 -1)
( 0  0  1 -1)
> u1:=W.1;
> u2:=W.2-(InnerProduct(W.2,u1)/InnerProduct(u1,u1))*u1;
```

```
> u3:=W.3-(InnerProduct(W.3,u1)/InnerProduct(u1,u1))*u1
>         -(InnerProduct(W.3,u2)/InnerProduct(u2,u2))*u2;
> u1;
( 1  0  0 -1)
> u2;
(-1/2    1    0 -1/2)
> u3;
(-1/3 -1/3    1 -1/3)
> MM:=KMatrixSpace(R,6,10);
> B:=MM![1,1,1,1,1,1,1,1,1,1,
> -2,-1.5,-1,-0.5,0,0.5,1,1.5,2,3,
> (-2)^2,(-1.5)^2,(-1)^2,(-0.5)^2,0^2,(0.5)^2,1^2,(1.5)^2,
>   2^2,3^2,
> (-2)^3,(-1.5)^3,(-1)^3,(-0.5)^3,0^3,(0.5)^3,1^3,(1.5)^3,
>   2^3,3^3,
> (-2)^4,(-1.5)^4,(-1)^4,(-0.5)^4,0^4,(0.5)^4,1^4,(1.5)^4,
>   2^4,3^4,
> (-2)^5,(-1.5)^5,(-1)^5,(-0.5)^5,0^5,(0.5)^5,1^5,(1.5)^5,
>   2^5,3^5];
> B;
[1 1 1 1 1 1 1 1 1 1]
[-2 -1.500000000000000 -1 -0.5000000000000000 0
    0.5000000000000000 1 1.500000000000000 2 3]
[4 2.250000000000000 1 0.2500000000000000 0
    0.2500000000000000 1 2.250000000000000 4 9]
[-8 -3.375000000000000 -1 -0.1250000000000000 0
    0.1250000000000000 1 3.375000000000000 8 27]
[16 5.062500000000000 1 0.06250000000000000 0
    0.06250000000000000 1 5.062500000000000 16 81]
[-32 -7.593750000000000 -1 -0.03125000000000000 0
    0.03125000000000000 1 7.593750000000000 32 243]
> V:=VectorSpace(R,10);
> y:=V![100,40,12,3,1,0.5,0,-3,-14,-125];
> y;
(100 40 12 3 1 0.5000000000000000 0 -3 -14 -125)
> x:=Solution(B*Transpose(B),y*Transpose(B));
> x;
(0.49609214315096668037844508432743729396
    -1.9442719045660222130810366104483752030
    4.4834258878376525435348964760729466078
    -3.5976621417797888386123680241327299796
    1.5309886192239133415604003839297957074
    -0.76892911010558069381598793363499246428)
```



```
> P<t>:=PolynomialAlgebra(R);
> f:=P![x[i]:i in [1..6]];
> f;
-0.76892911010558069381598793363499246428*t^5 +
    1.5309886192239133415604003839297957074*t^4 -
    3.5976621417797888386123680241327299796*t^3 +
    4.4834258878376525435348964760729466078*t^2 -
    1.9442719045660222130810366104483752030*t +
    0.49609214315096668037844508432743729396
> Evaluate(f,-7);
18067.087206910736322501028383381324670
> quit;
```