To end our section on completeness, we give one more application of the Contraction Mapping Theorem.

As we mentioned in Lecture 9, it is a straightforward matter to use the completeness of $\mathbb{R}$ to deduce that $\mathbb{R}^n$ is also complete, relative to any of the metrics $d_p$. For the present example we shall identify $\mathbb{R}^n$ with the set of all $n$-component column vectors over $\mathbb{R}$, and we shall use the metric $d = d_1$; thus, for all $x$, $y \in \mathbb{R}^n$,

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$$

where $x_i$, $y_i$ are the $i$-th components of the vectors $x$, $y$ (for $1 \leq i \leq n$).

Let $C$ be an $n \times n$ matrix over $\mathbb{R}$, and let $C_{ij}$ denote the $(i, j)$-entry of $C$. Suppose that $C$ satisfies the condition

$$\sum_{i=1}^{n} |C_{ij}| < 1 \qquad \text{for all } j \in \{1, 2, \ldots, n\}. \tag{1}$$

Observe that this is a condition on the columns of $C$; the example in Choo's notes uses instead a condition on the rows of $C$, giving a similar appearing but different result to that which we are about to prove. The only difference between the proofs is the choice of metric.

Put $K = \max_{1 \leq j \leq n}(\sum_{i=1}^{n} |C_{ij}|$, and note that $K < 1$, in view of (1). We shall show that, for any $b \in \mathbb{R}^n$, the mapping $f \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x) = Cx + b \qquad (\text{for all } x \in \mathbb{R}^n)$$

is a contraction mapping. Indeed, $d(f(x), f(y)) \leq K d(x, y)$ for all $x$, $y \in \mathbb{R}^n$, where $K$ is the constant defined above. By the Contraction Mapping Theorem it follows that $f$ has a unique fixed point in $\mathbb{R}^n$; that is, the system of linear equations $x = Cx + b$ has a unique solution $x$, irrespective of the value of $b$. Since this linear system can be rewritten as $(I - C)x = b$, we can deduce that the coefficient matrix $I - C$ must be invertible.†

Let $x$, $y \in \mathbb{R}^n$, and put $z = f(x) = Cx + b$ and $w = f(y) = Cy + b$. Then

$$d(f(x), f(y)) = d(z, w) = \sum_{i=1}^{n} |z_i - w_i|.$$

Now $z - w = (Cx + b) - (Cy + b) = C(x - y)$, and the $i$-th component of this vector equation tells us that

$$z_i - w_i = \sum_{j=1}^{n} C_{ij}(x_j - y_j) \qquad (\text{for all } i \in \{1, 2, \ldots, n\}).$$

† Similarly, the result proved is Choo's notes implies that $I - C$ is invertible if the transpose of $C$ satisfies condition (1). Since the matrix $I - C$ is invertible if and only if its transpose is invertible, the result Choo proves can be deduced from the one we shall prove, and vice versa. A condition for the invertibility of $I - C$ which is not obviously equivalent to these two can be obtained by using instead the Euclidean metric on $\mathbb{R}^n$; see Tutorial 10.

Therefore $|z_i - w_i| \leq \sum_{j=1}^{n} |C_{ij}|\,|x_j - y_j|$, and since $\sum_{j=1}^{n} |C_{ij}| \leq K$ (by the definition of $K$) we deduce that

$$\begin{aligned} d((f(x), f(y)) = \sum_{i=1}^{n} |z_i - w_i| &\leq \sum_{i=1}^{n} \Bigl( \sum_{j=1}^{n} |C_{ij}|\,|x_j - y_j| \Bigr) \\ &= \sum_{j=1}^{n} \Bigl( \sum_{i=1}^{n} |C_{ij}|\,|x_j - y_j| \Bigr) = \sum_{j=1}^{n} \Bigl( |x_j - y_j| \sum_{i=1}^{n} |C_{ij}| \Bigr) \\ &\leq \sum_{j=1}^{n} |x_j - y_j| K = K d(x, y), \end{aligned}$$

showing, as required, that $f$ is a contraction mapping.

## Compactness

There is a famous theorem of real analysis, known as the Heine-Borel Covering Theorem, which says that if $C$ is any closed and bounded subset of Euclidean space $\mathbb{R}^n$, and if $(V_i)_{i \in I}$ is any family of open sets such that

$$C \subseteq \bigcup_{i \in I} V_i,$$

then there is a finite subset $\{i_1, i_2, \ldots, i_k\}$ of the indexing set $I$ such that

$$C \subseteq V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}.$$

This is certainly a rather technical statement, and as such its usefulness may not be immediately apparent. It is, in fact, a very powerful result, and in due course we shall show how other basic results in real analysis may be derived as corollaries of the Heine-Borel theorem.

Observe that in the statement of the Heine-Borel theorem there is no restriction on the size of the set $I$. This makes it relatively easy to satisfy the hypotheses of the theorem. For example, suppose that $C$ is a closed and bounded set in $\mathbb{R}^n$ (such as a closed interval in $\mathbb{R}$) and suppose that for each point $x \in C$ we choose some extremely tiny open ball $B(x, \varepsilon_x)$. The set $C$ has quite likely got uncountably many points; so we have got an extremely large number of these tiny open balls. The balls cover the set $C$, in the sense that $C \subseteq \bigcup_{x \in C} B(x, \varepsilon_x)$, since if $x_0$ is any element of $C$ then $x_0 \in B(x_0, \varepsilon_{x_0})$, which certainly implies that $x_0 \in \bigcup_{x \in C} B(x, \varepsilon_x)$. We have covered $C$ with a probably uncountable number of tiny open sets. Not a hard thing to do. The Heine-Borel Theorem can now be applied, and it says that we do not need to use all these sets in order cover $C$. Some finite number of them will suffice. It is possible to throw away all but a finite number of the open balls $B(x, \varepsilon_x)$ and still be left with a covering of $C$. Finiteness conditions like this, guaranteeing that some finite subset of a given set will be adequate for some purpose, are potentially very handy tools for proving theorems. So it is with Heine-Borel.

When trying to generalize standard facts from real analysis to spaces other than $\mathbb{R}^n$, some replacement for the Heine-Borel theorem may be needed. In particular, results about closed intervals in $\mathbb{R}$ can often be extended to those subsets $C$ of a topological space that possess the Heine-Borel property: for every open covering of $C$ there is a finite subcovering.

**Definition.** Let $X$ be a topological space. A subset $C$ of $X$ is said to be *compact* if every open covering of $C$ has a finite subcovering. That is, $C$ is compact if and only if the following holds: for every set $I$ and indexed family $(V_i)_{i\in I}$ of open subsets of $X$, if $C \subseteq \bigcup_{i\in I} V_i$ then there exists a finite subset $J$ of $I$ such that $C \subseteq \bigcup_{i\in J} V_i$.

In $\mathbb{R}^n$ a set is compact if and only if it is closed and bounded. In other more complicated spaces, such as spaces of functions, closed and bounded is not usually enough to give compactness. However, it is usually true that compact sets are closed and bounded. (Boundedness, of course, is defined in terms of distance, and thus only makes sense in metric spaces.)

Recall that a topological space $X$ is *Hausdorff* if for all $a$, $b \in X$, if $a \neq b$ then there exist open sets $U$, $V$ such that $a \in U$ and $b \in V$, and $U \cap V = \emptyset$.†

**Proposition.** *Compact subsets of Hausdorff spaces are closed.*

*Proof.* Let $X$ be Hausdorff and $C \subseteq X$ with $C$ compact. We shall prove that $X \setminus C$ is open, by proving that each point of $X \setminus C$ is an interior point of $X \setminus C$.

Let $a \in X \setminus C$ be arbitrary. It will suffice to show that there exists an open set $U$ with $a \in U \subseteq X \setminus C$, for this implies $a \in \mathrm{Int}(X \setminus C)$. Now for each point $b \in C$ we certainly have $b \neq a$, and since $X$ is Hausdorff there exist open sets $U_b$, $V_b$ such that $a \in U_b$ and $b \in V_b$, and $U_b \cap V_b = \emptyset$. For each $c \in C$ we have

$$c \in V_c \subseteq \bigcup_{b\in C} V_b,$$

and thus $C \subseteq \bigcup_{b\in C} V_b$. In other words, since the sets $V_b$ are open, the indexed family $(V_b)_{b\in C}$ constitutes an open covering of $C$. Since $C$ is compact it follows that there exists a finite subcover. That is, there exists a finite set $Q = \{b_1, b_2, \dots, b_n\}$ such that

$$C \subseteq \bigcup_{b\in Q} V_b = V_{b_1} \cup V_{b_2} \cup \cdots \cup V_{b_n}.$$

We put $U = \bigcap_{b\in Q} U_b = U_{b_1} \cup U_{b_2} \cup \cdots \cup U_{b_n}$, and proceed to show that $U$ has the desired properties: $U$ is open, $a \in U$ and $U \subseteq X \setminus C$.

Since $a \in U_b$ for all $b$, it follows immediately that $a \in \bigcap_{b\in Q} U_b = U$. It is an axiom of topology that the intersection of any finite collection of open sets is open; so since each $U_b$ is open and the set $Q$ is finite, it follows that $\bigcap_{b\in Q} U_b = U$ is open. The intersection of a collection of sets is contained in each of those sets; so $U \subseteq U_b$ for each $b \in Q$, and so, by the way that $U_b$ and $V_b$ were chosen,

$$U \cap V_b \subseteq U_b \cap V_b = \emptyset$$

for all $b \in Q$. Thus, since $C \subseteq \bigcup_{b\in Q} V_b$,

$$U \cap C \subseteq U \cap \bigcup_{b\in Q} V_b = \bigcup_{b\in Q} U \cap V_b = \emptyset.$$

Thus $U \subseteq X \setminus C$, which was the last property we had to establish. □

† The standard weak pun is that $a$ and $b$ can be housed off from each other.