Recall that a subset A if a metric space X is said to be bounded if there exists a constant K with $d(x, y) \leq K$ for all x, $y \in A$, and if A is bounded then the diameter of A is defined by $\operatorname{diam}(A) = \sup_{x,\, y \in A} d(x, y)$.

Proposition. *If A is a compact subset of a metric space X then A is bounded.*

Proof. Choose any point x_0 in X—the result is clearly trivial if $X = \emptyset$—and consider the family of all open balls $B(x_0, n)$, for positive integers n. For each $a \in A$ the distance $d(x_0, a)$ is some real number, and we may choose a positive integer k such that $d(x_0, a) < k$. Then $a \in B(x_0, k) \subseteq \bigcup_{n=1}^{\infty} B(x_0, n)$, and since this holds for all $a \in A$ it follows that $A \subseteq \bigcup_{n=1}^{\infty} B(x_0, n)$. Since A is compact it follows that there exists a finite subset J of $\mathbb{Z}^+$ such that $A \subseteq \bigcup_{n \in J} B(x_0, n)$. Now let K be the maximum element of this finite set of numbers J. For all $a \in A$ we have $x \in B(x_0, n)$ for some $n \in J$, and so $d(x_0, a) < n \leq K$. This shows that A is bounded, with diameter at most $2K$, since if a, $b \in A$ then $d(a, b) \leq d(a, x) + d(b, x) < 2K$. □

Our next result is needed for the proof of the Heine-Borel Covering Theorem. It should have really been proved in the section on completeness, since it is not concerned directly with compactness (and completeness is needed).

Cantor's Intersection Theorem. *Let (X, d) be a complete metric space, and let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be an infinite decreasing chain of nonempty, closed, bounded subsets of X. Suppose further that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Then there exists $x \in X$ such that $\bigcap_{n=1}^{\infty} A_n = \{x\}$.*

Proof. The sets A_n are all nonempty; so for each $n \in \mathbb{Z}^+$ we may choose a point $a_n \in A_n$. Our strategy is to show that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence; its limit will be the point x that appears in the theorem statement.

Let $\varepsilon > 0$, and choose $N \in \mathbb{Z}^+$ such that $\operatorname{diam}(A_n) < \varepsilon$ for all $n \geq N$; the hypothesis that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$ guarantees that such an N exists. Now for all m, $n \geq N$ we have

$$a_m \in A_m \subseteq A_N$$
$$a_n \in A_n \subseteq A_N,$$

and therefore $d(a_m, a_n) \leq \operatorname{diam}(A_N) < \varepsilon$. Since ε was arbitrary this shows that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, and since X is complete it follows that $\lim_{n\to\infty} a_n$ exists. Let x be this limit.

Removing a finite number of terms from a sequence does not change its limit; so for all $m \in \mathbb{Z}^+$ the sequence $(a_n)_{n=m}^{\infty}$ has limit x. All the terms of this sequence lie in A_m, since $a_n \in A_n \subseteq A_m$ whenever $n \geq m$. By a proposition we proved in Lecture 8, it follows that the limit x is an element of $\overline{A_m}$, the closure of A_m. But A_m is closed; so $x \in A_m$, and since this holds for all $m \in \mathbb{Z}^+$ it follows that $x \in \bigcap_{m=1}^{\infty} A_m$. Since $\bigcap_{m=1}^{\infty} A_m \subseteq A_n$ for all $n \in \mathbb{Z}^+$, if $y \in \bigcap_{m=1}^{\infty} A_m$ then y, $x \in A_n$ for all $n \in A_n$, and so

$$0 \leq d(x, y) \leq \operatorname{diam}(A_n) \to 0 \quad \text{as } n \to \infty.$$

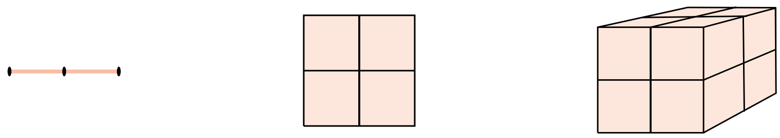
So $d(x, y) = 0$, and so $x = y$. This shows that x is the only point of $\bigcap_{n=1}^{\infty} A_n$, and so $\bigcap_{n=1}^{\infty} A_n = \{x\}$, as required. □

Our next objective is to prove the Heine-Borel Covering Theorem, which says that closed, bounded subsets of $\mathbb{R}^n$ are compact.

Let $d = d_\infty$ be the sup metric on $\mathbb{R}^n$. Then for any any point $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}$ the set H of all points $x \in \mathbb{R}^n$ of distance at most r from a is the Cartesian product of the closed intervals $[a_i - r, a_i + r]$ in $\mathbb{R}$:

$$\begin{aligned} H = \{\, x \in \mathbb{R}^n \mid d(x,a) \le r \,\} &= [a_1 - r, a_1 + r] \times [a_2 - r, a_2 + r] \times \cdots \times [a_n - r, a_n + r] \\ &= \{\, (x_1, x_2, \ldots, x_n \mid a_i - r \le x_i \le a_i + r \text{ for all } i \,\}. \end{aligned}$$

This is a line segment if $n = 1$, a square if $n = 2$ and a cube if $n = 3$. For general n we shall use the term "hypercube". Observe that H can be written as a union of 2^n hypercubes of diameter $\frac{1}{2}\operatorname{diam}(H)$; the cases $n = 1$, 2 and 3 are illustrated in the following diagram:



To be specific, if for each i we define $A_i^1 = [a_i - r, a_i]$ and $A_i^2 = [a_i, a_i + r]$, then

$$\begin{aligned} H &= (A_1^1 \cup A_1^2) \times (A_2^1 \cup A_2^2) \times \cdots \times (A_n^1 \cup A_n^2) \\ &= \bigcup_{\varepsilon_1 \in \{1,2\}} \bigcup_{\varepsilon_2 \in \{1,2\}} \cdots \bigcup_{\varepsilon_n \in \{1,2\}} A_1^{\varepsilon_1} \times A_2^{\varepsilon_2} \times \cdots \times A_n^{\varepsilon_n}. \end{aligned}$$

(There are two possible values for each ε_i, and so 2^n terms altogether in this union.)

Heine-Borel Covering Theorem. *Let C be a subset of $\mathbb{R}^n$ that is bounded and closed (with respect to the usual topology). Then C is compact.*

Proof. Suppose, for a contradiction, that C is closed and bounded but not compact. Then there is some open covering of C with no finite subcovering. Choose such a covering: $(V_i)_{i \in I}$ is a family of open sets such that

(i) $C \subseteq \bigcup_{i \in I} V_i$, and

(ii) there is no finite subset J of I with $C \subseteq \bigcup_{i \in J} V_i$.

Of course, (ii) implies that C is nonempty, for otherwise $C \subseteq \bigcup_{i \in J} V_i$ would hold with $J = \emptyset$. Note also that since C is bounded we may choose a closed hypercube H with the property that $C \subseteq H$: choose any $a \in C$, and let H consist of points of distance at most $\operatorname{diam}(C)$ from a. Let $D = \operatorname{diam}(H)$. (Recall that we are using the sup metric.)

Write $C = C_0$. Our strategy is to produce an infinite decreasing chain of closed, bounded, nonempty sets C_k, each covered by $(V_i)_{i \in I}$ but by no finite subfamily of this family. They will be chosen in such a way that $\operatorname{diam}(C_k) \to 0$ as $k \to \infty$, so that Cantor's Intersection Theorem will be applicable. Indeed, the following properties will hold for all $k \in \mathbb{Z}^+$.

(a) $C_k \subseteq \bigcup_{i \in I} V_i$;

(b) there is no finite subset J of I with $C_k \subseteq \bigcup_{i \in J} V_i$;

(c) C_k is closed and nonempty, and $C_k \subseteq C_{k-1}$;

(d) $C_k \subseteq H_k$, for some closed hypercube H_k of diameter $\frac{1}{2^k} D$.

The final contradiction will then arise as follows. Cantor's theorem yields a point x that lies in each C_k, and hence in some V_i. Since V_i is open there must be an $\varepsilon > 0$ such that all points whose distance from x is less than ε are in V_i, and since the diameters of the C_k approach 0 this implies that $C_k \subseteq V_i$ for k large enough. But this contradicts (b) above.

Write $H = H_0$ as the union of 2^n hypercubes of diameter half $\operatorname{diam}(H)$, in the manner described above. Thus $H_0 = \bigcup_{j=1}^{2^n} H_j^{(0)}$, where each $H_j^{(0)}$ is a closed hypercube of diameter $\frac{1}{2}D$. Then since $C \subseteq H$,

$$C = C \cap H_0 = \bigcup_{1 \leq j \leq 2^n} (C \cap H_j^{(0)}).$$

Suppose that for each $j \in \{1, 2, \ldots, 2^n\}$ a finite subset J_j of I exists with the property that $C \cap H_j^{(0)} \subseteq \bigcup_{i \in J_j} V_i$. Then

$$C = \bigcup_{1 \leq j \leq 2^n} (C \cap H_j^{(0)}) \subseteq \bigcup_{1 \leq j \leq 2^n} \Big(\bigcup_{i \in J_j} V_i \Big) = \bigcup_{i \in J_1 \cup \cdots \cup J_{2^n}} V_i,$$

contradicting (ii), since the set $J = J_1 \cup \cdots \cup J_{2^n}$ is a finite union of finite sets, and hence finite. So for at least one $j \in \{1, 2, \ldots, 2^n\}$ there is no finite subset J of I such that $C \cap H_j^{(0)} \subseteq \bigcup_{i \in J} V_i$. Now if we define $H_1 = H_j^{(0)}$ and $C_1 = C \cap H_1$ then the properties (a), (b), (c) and (d) above are satisfied for $k = 1$. Property (a) holds since $C_1 \subseteq C$, and $C \subseteq \bigcup_{i \in I} V_i$ by (i). Property (b) holds by the choice of the j in the definition of C_1. Property (b) implies that $C_1 \neq \emptyset$, and since C_1 is defined as the intersection of two closed sets, one of which is $C_0 = C$, it follows that C_1 is closed and $C_1 \subseteq C_0$. Thus Property (c) holds. And Property (d) holds since $C_1 = C \cap H_1$, and $H_1 = H_j^{(0)}$ has diameter $\frac{1}{2} \operatorname{diam}(H) = \frac{1}{2}D$.

We simply repeat this argument to establish (a), (b), (c) and (d) for all values of k. Proceeding inductively, we assume that (a), (b), (c) and (d) hold with $k-1$ in place of k. Write $H_{k-1} = \bigcup_{j=0}^{2^n} H_j^{(k-1)}$, where each $H_j^{(k-1)}$ is a hypercube of diameter $\frac{1}{2} \operatorname{diam}(H_{k-1}) = \frac{1}{2}(\frac{D}{2^{k-1}}) = \frac{1}{2^k} D$. Now

$$C_{k-1} = C_{k-1} \cap H_{k-1} = \bigcup_{1 \leq j \leq 2^n} (C_{k-1} \cap H_j^{(k-1)}),$$

and since C_{k-1} is not covered by any finite collection of the sets V_i, it follows that at least one of the sets $C_{k-1} \cap H_j^{(k-1)}$ is not covered by any finite collection of the V_i's. Choose j accordingly, and define $H_k = H_j^{(k-1)}$ and $C_k = C_{k-1} \cap H_k$. As above, we se that (a), (b), (c) and (d) are satisfied. By induction, they hold for all $k \in \mathbb{Z}^+$.

Since $C_k \subseteq H_k$ for all k it follows that $0 \leq \operatorname{diam}(C_k) \leq \operatorname{diam}(H_k) \to 0$ as $k \to \infty$. Since $\mathbb{R}^n$ is complete, and since each C_k is closed, bounded and nonempty, and satisfies $C_k \subseteq C_{k-1}$, it follows from Cantor's Intersection Theorem that there exists a point x with $x \in C_k$ for all k. As $\bigcup_i V_i \supseteq C \supseteq C_1 \supseteq C_2 \supseteq \cdots$, we have $x \in \bigcup_i V_i$, and so $x \in V_j$ for some $j \in I$. Since V_j is open there exists an $\varepsilon > 0$ with $B(x, \varepsilon) \subseteq V_j$. Since $\operatorname{diam}(C_k) \to 0$ as $k \to \infty$ there exists a $k \in \mathbb{Z}^+$ with $\operatorname{diam}(C_k) < \varepsilon$. Note that $x \in C_k$ (since $x \in C_m$ for all m). Now for all $y \in C_k$ we have $d(y, x) \leq \operatorname{diam}(C_k) < \varepsilon$, and so

$$y \in B(x, \varepsilon) \subseteq V_j.$$

Thus $C_k \subseteq V_j$; so if we put $J = \{j\}$ then J is a finite subset of I and $C_k \subseteq V_j = \bigcup_{i \in J} V_i$. This contradicts Property (b) for C_k, thereby completing the proof of the Heine-Borel Theorem. □