In Lecture 10 we described the regular representation of G as the linear representation derived from the permutation representation of G on G corresponding to the left multiplication action of G on itself. That is, to each element $g \in G$ we associate the permutation $\sigma_g: G \to G$ defined by $\sigma_g x = gx$ (for all $x \in G$), and corresponding to this permutation we have a permutation matrix Rg. Then $g \mapsto Rg$ is a matrix representation. A non-matrix version of the regular representation can be obtained by identifying the elements of G with the basis elements of a vector space V—in other words, let V be an $|G|$-dimensional vector space, then choose any basis of V and a one to one correspondence between these basis vectors and the elements of G—and associate with each element $g \in G$ the linear transformation $\rho_g: V \to V$ which permutes the basis according to the permutation σ_g defined above. Then $\rho: g \mapsto \rho_g$ is a representation of G by linear transformations of the space V.

Furthermore, we also noted in Lecture 10 that the set V_G of all complex valued functions on G is a $|G|$-dimensional vector space over $\mathbb{C}$. So we can take the vector space V referred to in the last paragraph to be V_G. If for all $x \in G$ we define the function $f_x \in V_G$ by the formula

$$f_x h = \begin{cases} 1 & \text{if } h = x^{-1}, \\ 0 & \text{if } h \neq x^{-1}, \end{cases}$$

then it is easily seen that the f_x's form a basis of V_G in one to one correspondence with the elements of G. The remarks in the last paragraph thus assert that there is an action of G on V_G such that $gf_x = f_{gx}$ for all $x \in G$ and $g \in G$. An alternative way to describe this action is as follows: for all $g \in G$ and $f \in V_G$ the function $gf \in V_G$ is given by

$$(gf)h = f(hg) \qquad \text{for all } h \in G.$$

The student should check that this formula does indeed yield $gf_x = f_{gx}$, and that the axioms for a linear action of group on a vector space (see Lecture 3) are indeed satisfied.

Lecture 12, 3/9/97

We have defined a *G-module* to be a vector space with a G-action; that is, there must be a function $(g, v) \to gv$ from $G \times V$ to V satisfying (i), (ii) and (iii) of Lecture 3. Strictly speaking, this should be a called *left* G-module, since the G action is on the left. Similarly, a *right G-module* is a vector space V equipped with a function $(v, g) \mapsto vg$ from $V \times G$ to V satisfying

(i) $v1 = v$ for all $v \in V$, where 1 is the identity of G,

(ii) $(vg)h = v(gh)$ for all $v \in V$ and$g$, $h \in G$,

(iii) $(u + v)g = ug + vg$ for all u, $v \in V$ and $g \in G$,

(iv) $(\lambda v)g = \lambda(vg)$ for all $v \in V$ and $g \in G$ and all scalars λ.

We have seen that V_G becomes a left G-module via the (left) action given by $(gf)h = f(hg)$ for all g, $h \in G$ and all functions $f \in V_G$. In fact we can also make V_G into a right G-module by defining $fg: G \to \mathbb{C}$ (whenever $f \in V_G$ and $g \in G$) by

$$(fg)h = f(gh) \qquad \text{for all } h \in G.$$

It is a straightforward matter, which we leave to the reader, to check that (i) to (iv) above are satisfied.

A question which now arises is the following: which functions $f\colon G \to \mathbb{C}$ have the property that $gf = fg$ for all $g \in G$? That is, on which elements $f \in V_G$ do the left and right actions of G agree?

Definition. A function $f\colon G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes. Thus f is a class function if and only if $fx = fy$ whenever x and y are conjugate in G.

Proposition. *A function $f \in V_G$ satisfies $gf = fg$ for all $g \in G$ if and only if it is a class function.*

Proof. Suppose that $gf = fg$ for all $g \in G$ and let x, y be conjugate elements of G. Then there exists $g \in G$ such that $g^{-1}xg = y$, and thus

$$fy = f(g^{-1}xg) = (gf)(g^{-1}x) = (fg)(g^{-1}x) = f(g(g^{-1}x)) = fx,$$

where we have used the definitions of gf and fg and the assumption that $gf = fg$. Thus $f$is a class function.

Conversely, suppose that f is a class function, and let $g \in G$ be arbitrary. Noting that for all $h \in G$ we have

$$g^{-1}(gh)g = hg,$$

so that gh and hg are conjugate, it follows that $f(gh) = f(hg)$ (since f is a class function). Thus

$$(gf)h = f(hg) = f(gh) = (fg)h \qquad \text{for all } h \in G,$$

showing that $gf = fg$ for all $g \in G$, as required. □

For example, the group S_3 has three conjugacy classes. The identity element constitutes a conjugacy class by itself—this is the case in any group—since $\sigma^{-1}(\mathrm{id})\sigma = \mathrm{id}$ for all σ. As $(2\,3)^{-1}(1\,2)(2\,3) = (1\,3)$ and $(1\,3)^{-1}(1\,2)(1\,3) = (2\,3)$ we see that $(1\,2)$, $(1\,3)$ and $(2,\,3)$ are all conjugate, and similarly $(1\,2\,3) = (1\,2)^{-1}(1\,3\,2)(1\,2)$ shows that $(1\,2\,3)$ and $(1\,3\,2)$ are all conjugate. Slightly more work is needed to show that $(1\,2)$ and $(1\,2\,3)$ are not conjugate, but since this is a side issue at present we omit it. The point is that a class function f on S_3 is determined by a triple x, y, z of complex numbers, where

$$\begin{aligned} f1 &= x \\ f(1\,2) = f(1\,3) = f(2\,3) &= y \\ f(1\,2\,3) = f(1\,3\,2) &= z. \end{aligned}$$

Thus we see that the class functions on S_3 form a three dimensional vector space. In general,

Proposition. *The set of all class functions $G \to \mathbb{C}$ form a vector subspace of V_G of dimension equal to the number of conjugacy classes of G.*

Proof. The zero function is clearly constant on conjugacy classes, and so the set of all class functions is nonempty. If e and f are class functions and if x and y are arbitrary conjugate elements of G then $fx = fy$ and $ex = ey$ (since e, f are class functions, and so

$$(e + f)x = ex + fx = ey + fy = (e + f)y$$

by the definition of the sum of two functions. Thus $e + f$ is a class function, and so we have shown that the set of class functions is closed under addition. Similarly, if f is a class function and λ any scalar then for all conjugate elements x, $y \in G$,

$$(\lambda f)x = \lambda(fx) = \lambda(fy) = (\lambda f)y,$$

which shows that λf is a class function. (The student should take care to examine every step in this calculation and make sure that (s)he knows exactly what is being asserted and why it is true. It is very easy to look at equations like the above and believe them because they seem vaguely reasonable, but that is not good enough in pure mathematics.) So the set of all class functions is also closed under scalar multiplication. Hence it is a subspace of V_G.

Let $\mathcal{C}_1$, $\mathcal{C}_2$, …, $\mathcal{C}_t$ be all the conjugacy classes of G, and for each i from 1 to t let F_i be the function $G \to \mathbb{C}$ given by $F_i = \sum_{y\in\mathcal{C}_i} f_{y^{-1}}$, where the functions $f_x \in V_G$ are as defined at the start of this lecture. Then

$$F_i g = \sum_{y\in\mathcal{C}_i} f_{y^{-1}}(g) = \begin{cases} 1 & \text{if } g \in \mathcal{C}_i, \\ 0 & \text{if } g \notin \mathcal{C}_i, \end{cases}$$

since $f_{y^{-1}}(g)$ is 1 if $g = y$ and is zero otherwise. Now every class function on G can be expressed as a linear combination of the F_i; specifically, if $f: G \to \mathbb{C}$ takes the value λ_i on elements in the class $\mathcal{C}_i$ (for i from 1 to t) then $f = \sum_i \lambda_i F_i$. Thus the F_i span the space of class functions. Furthermore, it can be seen that for all choices of the coefficients λ_i the function $\sum_i \lambda_i F_i$ takes the value λ_i on elements of class $\mathcal{C}_i$. Thus if $\sum_i \lambda_i F_i = 0$ then all the coefficients λ_i must be 0, which means that the F_i are linearly independent. So F_1, F_2, …, F_t form a basis for the space of class functions, which therefore has dimension t, as required. □

The functions f_x for $x \in G$ form a basis of V_G. But we also saw in Lecture 10 that if $R^{(1)}$, $R^{(2)}$, …, $R^{(s)}$ are a full set of pairwise inequivalent irreducible matrix representations of G then the collection $\mathcal{S}$ of all coordinate functions of all the $R^{(k)}$ also forms a basis of V_G (whence we deduced that the sum of the squares of the degrees of the $R^{(k)}$ equals $|G|$). We shall show that the characters—see definition below—$\chi^{(1)}$, $\chi^{(2)}$, …, $\chi^{(s)}$ of the representations $R^{(1)}$, $R^{(2)}$, …, $R^{(s)}$ form a basis of the space of class functions on G.

Definition. The *character* of a matrix representation R of G is the function $\chi: G \to \mathbb{C}$ defined by $\chi(g) = \operatorname{trace}(Rg)$. Thus if the degree of R is d and R_{ij} (where $1 \le i, j \le d$) are the coordinate functions of R then $\chi = \sum_{i=1}^{d} R_{ii}$.

We have seen in an assignment question that the character of a representation is always a class function. The point is that similar matrices have the same trace, and so whenever g, $x \in G$ the matrix

$$R(g^{-1}xg) = (Rg)^{-1}(Rx)(Rg)$$

has the same trace as Rx, and this shows that the character χ takes the same value on $g^{-1}xg$ as it does on x. Linear independence of the collection $\mathcal{S}$ of all coordinate functions of the $R^{(k)}$ implies linear independence of the characters $\chi^{(k)}$, for if $\sum_k \lambda_k \chi^{(k)} = 0$ then

$$0 = \sum_k \lambda_k \Bigl(\sum_{i=1}^{d_k} R_{ii}^{(k)}\Bigr) = \sum_{i,k} \lambda_k R_{ii}^{(k)},$$

and this implies that all the coefficients λ_k are zero. So to prove that the characters of the irreducible representations form a basis for the space of all class functions it remains to prove that they span.

Proposition. *If $\chi^{(1)}$, $\chi^{(2)}$, …, $\chi^{(s)}$ are the characters of a full set of irreducible representations of G then every class function on G can be expressed as a linear combination $\sum_k \lambda_k \chi^{(k)}$.*

Proof. Let f be a class function on G, and for each of the irreducible representations $R^{(h)}$ (notation as above) consider the matrix

$$M_h = \sum_{g\in G} \overline{(fg)}(R^{(h)}g), \tag{1}$$

(where the overline indicate complex conjugation). We show that M_h commutes with $R^{(h)}x$ for all $x \in G$. Indeed, since f is a class function we have that $\overline{f(x^{-1}gx)} = \overline{fg}$ for all $g \in G$, and so

$$\begin{aligned}(R^{(h)}x)^{-1}M_h(R^{(h)}x) &= (R^{(h)}x)^{-1}\Big(\sum_{g\in G}\overline{f(x^{-1}gx)}(R^{(h)}g)\Big)(R^{(h)}x)\\ &= \sum_{g\in G}\overline{f(x^{-1}gx)}((R^{(h)}x)^{-1}(R^{(h)}g)(R^{(h)}x))\\ &= \sum_{g\in G}\overline{f(x^{-1}gx)}R^{(h)}(x^{-1}gx) = \sum_{g\in G}\overline{(fg)}(R^{(h)}g)\end{aligned}$$

since, with x fixed, $x^{-1}gx$ runs through all elements of G as g does. So $(R^{(h)}x)^{-1}M_h(R^{(h)}x) = M_h$. Now because $R^{(h)}$ is irreducible, Schur's Lemma tells us that $M_h = \lambda_h I$ for some scalar λ_h So now looking at the (i,j)-entry in Eq. (1) tells us that

$$\sum_{g\in G}\overline{(fg)}(R^{(h)}_{ij}g) = \lambda_h\delta_{ij}. \tag{2}$$

For all $f_1,\ f_2 \in V_G$ define $f_1 * f_2 \in \mathbb{C}$ by

$$f_1 * f_2 = \frac{1}{|G|}\sum_{g\in G}(f_1g)\overline{(f_2g)}.$$

We saw in Lecture 9 (see Eq. (3) of that lecture) that $R^{(k)}_{pm}*R^{(l)}_{qn}$ is zero unless $k = l$, $p = q$ and $m = n$, in which case it is $1/d_k$. Since the functions $R^{(k)}_{pm}$ span V_G we can write $f = \sum_{k,p,m}\mu_{k\,p\,m}R^{(k)}_{pm}$ for some coefficients $\mu_{k\,p\,m}$, and this gives

$$R^{(h)}_{ij} * f = \sum_{k,p,m}\overline{\mu_{k\,p\,m}}(R^{(h)}_{ij} * R^{(k)}_{pm}) = \sum_{k,p,m}\overline{\mu_{k\,p\,m}}(1/d_k)\delta_{hk}\delta_{ip}\delta_{jm} = (1/d_h)\overline{\mu_{h\,i\,j}}.$$

But Eq. (2) says that $R^{(h)}_{ij} * f = \lambda_h\delta_{ij}/|G|$. Thus we have shown that

$$\mu_{h\,i\,j} = \frac{d_h\overline{\lambda_h}\delta_{ij}}{|G|},$$

and it follows that

$$f = \sum_{h,i,j}\mu_{h\,i\,j}R^{(h)}_{ij} = \frac{1}{|G|}\sum_{h,i,j}d_h\overline{\lambda_h}\delta_{ij}R^{(h)}_{ij} = \frac{1}{|G|}\sum_h d_h\overline{\lambda_h}\Big(\sum_i R^{(h)}_{ii}\Big) = \frac{1}{|G|}\sum_h d_h\overline{\lambda_h}\chi^{(h)}.$$

This is a linear combination of the $\chi^{(h)}$'s, as required. □