Let us review the situation we have been discussing in the last few lectures. Given a finite group G, we started by choosing irreducible complex representations $R^{(1)}$, $R^{(2)}$, ..., $R^{(s)}$ such that $R^{(i)}$ and $R^{j)}$ are not equivalent if $i \neq j$. We showed that $\sum_k d_k^2 \leq |G|$, where d_k is the degree of $R^{(k)}$. This limits the number of pairwise inequivalent irreducible complex representations G can have. If we now suppose that we have chosen the above sequence of representations to include as many representations as possible then it will still be a finite sequence, and every irreducible complex representation of G will have to be equivalent to exactly one of the $R^{(i)}$. We express this by saying that $R^{(1)}$, $R^{(2)}$, ..., $R^{(s)}$ constitute a *full set of irreducible complex representations of* G.

Given a full set of irreducible representations of G as above, let $\chi^{(k)}$ be the character of $R^{(k)}$. That is, if the coordinate functions of $R^{k)}$ are denoted by $R_{ij}^{(k)}$, then $\chi^{(k)} = \sum_{j=1}^{d_k} R_{jj}^{(k)}$. We saw in an assignment question that equivalent representations have the same character, the reason being that if R, $S\colon G \to \mathrm{GL}(d, \mathbb{C})$ are equivalent then there exists a $T \in \mathrm{GL}(d, \mathbb{C})$ such that $Sg = T^{-1}(Rg)T$ for all $g \in G$, and this gives

$$\mathrm{trace}(Sg) = \mathrm{trace}(T^{-1}(Rg)T) = \mathrm{trace}(Rg) \qquad \text{for all } g \in G.$$

So the character of any irreducible complex representation of G must be equal to one of the characters $\chi^{(1)}$, $\chi^{(2)}$ ..., $\chi^{(s)}$.

We have also proved the orthogonality of coordinate functions:

$$\frac{1}{|G|} \sum_{g \in G} (R_{pm}^{(k)} g)(R_{nq}^{(l)}(g^{-1})) = (1/d_k)\delta_{kl}\delta_{nm}\delta_{pq}$$

for all meaningful values of k, l, p, m, n and q. Putting $p = m$ and $q = n$ and summing over m from 1 to d_k and n from 1 to d_l gives

$$\frac{1}{|G|} \sum_{g \in G} \Big(\sum_{m=1}^{d_k} R_{mm}^{(k)} g \Big) \Big(\sum_{n=1}^{d_l} R_{nn}^{(l)}(g^{-1}) \Big) = (1/d_k)\delta_{kl} \sum_{m=1}^{d_k} \sum_{n=1}^{d_l} \delta_{nm}\delta_{mn}.$$

The quantity on the right hand side is zero unless $l = k$, in which case $\sum_{m=1}^{d_k} \sum_{n=1}^{d_k} \delta_{nm}\delta_{mn}$ evaluates to d_k (since the d_k terms with $n = m$ each contribute 1 and the other terms are 0). Thus

$$\frac{1}{|G|} \sum_{g \in G} \chi^{(k)}(g)\chi^{(l)}(g^{-1}) = \delta_{kl}.$$

Suppose now that χ and ϕ are characters of irreducible complex representations R and S. As explained above, we know that R is equivalent to $R^{(k)}$ and S to $R^{(l)}$ for some k and l. This gives $\chi = \chi^{(k)}$ and $\phi = \chi^{(l)}$. Since $R^{(k)}$ and $R^{(l)}$ are equivalent if and only if $k = l$, it follows that R and S are equivalent if and only if $k = l$. If they are not equivalent then

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \chi^{(k)}(g)\chi^{(l)}(g^{-1}) = 0,$$

and if they are equivalent then

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\phi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \chi^{(k)}(g)\chi^{(k)}(g^{-1}) = 1.$$

Thus we have proved the following theorem.

Theorem. *If χ and ϕ are characters of irreducible complex representations of the finite group G then R and S are equivalent if and only if $\chi = \phi$. Furthermore,*

$$\sum_{g\in G} \chi(g)\phi(g^{-1}) = \begin{cases} 1 & \text{if } \chi = \phi, \\ 0 & \text{if } \chi \neq \phi. \end{cases}$$

We have seen that every complex representation of a finite group G is equivalent to a unitary representation; so every character of G is the character of a unitary representation. But if R is unitary then

$$R_{ij}(g^{-1}) = \overline{R_{ji}g} \qquad \text{(for all } g \in G)$$

where the R_{ij} are the coordinate functions of R. Now if χ is the character it follows that

$$\chi(g^{-1}) = \sum_i R_{ii}(g^{-1}) = \sum_i \overline{R_{ii}g} = \overline{\chi(g)},$$

and so we deduce that $\chi(g^{-1}) = \overline{\chi(g)}$ for every complex character of the finite group G. (Another way to see this is to observe that $\chi(g) = \mathrm{trace}(Rg)$ is the sum of the eigenvalues of the matrix Rg, while $\chi(g^{-1})$ is the sum of the eigenvalues of $R(g^{-1}) = (Rg)^{-1}$, which are the inverses of the eigenvalues of Rg. But $g^n = 1$ for some n, so that $(Rg)^n = I$, from which it follows that the eigenvalues of Rg are nth roots of 1. And the inverse of an nth root of 1 coincides with its complex conjugate.)

Let us abandon the $*$ notation introduced in Lecture 12 in favour of something more standard: for functions $f_1,\ f_2\colon G \to \mathbb{C}$ define

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g\in G} \overline{(f_1 g)}(f_2 g).$$

This is an inner product on the space V_G of all such functions. If χ and ϕ are characters of irreducible representations then (χ, ϕ) is 0 if $\chi \neq \phi$ and is 1 if $\chi = \phi$. This is the famous *orthogonality of irreducible characters*. We may choose the representations $R^{(1)}$, $R^{(2)}$, …, $R^{(s)}$ to all be unitary, and then orthogonality of coordinate functions becomes

$$(R^{(k)}_{pm}, R^{(l)}_{qn}) = (1/d_k)\delta_{kl}\delta_{pq}\delta_{mn}.$$

We showed in Lecture 10 that these coordinate functions form a basis for V_G, whence $\sum_{k=1}^{s} d_k^2 = |G|$. We showed in Lecture 12 that the characters $\chi^{(1)}$, $\chi^{(2)}$, …, $\chi^{(s)}$ span the space of class functions on G, and since orthogonality of irreducible characters gives

$$(\chi^{(i)}, \chi^{(j)}) = \delta_{ij}.$$

it follows that $\chi^{(1)}$, $\chi^{(2)}$ …, $\chi^{(s)}$ form an orthonormal basis for the space of class functions. Hence s, the total number of irreducible complex characters of G, equals the number of conjugacy classes of G.

The values of the irreducible characters can be conveniently displayed in an array known as the *character table* of the finite group G. This table has one row for each irreducible character $\chi^{(j)}$ and one column for each conjugacy class, the entry in the jth row and kth column being the value

$\chi^{(j)}(g)$ for elements g in the kth class. For example, here is the character table of a certain group with 168 elements:

	g_1	g_2	g_3	g_4	g_5	g_6
$\chi^{(1)}$	1	1	1	1	1	1
$\chi^{(2)}$	3	-1	1	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
$\chi^{(3)}$	3	-1	1	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$
$\chi^{(4)}$	6	2	0	0	-1	-1
$\chi^{(5)}$	7	-1	-1	1	0	0
$\chi^{(6)}$	8	0	0	-1	1	1

Character tables carry quite a lot of information about the group in question. We shall proceed to derive general facts about character tables, referring to the table above to illustrate them.

Let $\mathcal{C}_1$, $\mathcal{C}_2$, …, $\mathcal{C}_s$ be the conjugacy classes of G, and choose from each class $\mathcal{C}_j$ a representative element g_j. Let h_j be the number of elements in $\mathcal{C}_j$. Since one of the classes consists of the identity element alone, for some j we have $g_j = 1_G$ and $h_j = 1$. The following lemma enables us to see at a glance which column of the character table coresponds to the identity.

Lemma. *The trace of a $d \times d$ unitary matrix has absolute value at most d, with equality occurring if and only if the matrix is a scalar multiple of I.*

Proof. Let M be a $d \times d$ unitary matrix. Then the columns of M form an orthonormal basis of $\mathbb{C}^d$, and in particular $\sum_{i=1}^{d} |M_{ij}|^2 = 1$ for each j. Thus $|M_{jj}| \le 1$, with equality if and only if $M_{ij} = 0$ for all $i \ne j$. So

$$|\mathrm{trace}(M)| = \Big|\sum_{j=1}^{d} M_{jj}\Big| \le \sum_{j=1}^{d} |M_{jj}| \le d \tag{1}$$

with equality only possible if M is diagonal and $|M_{jj}| = 1$ for all j. Indeed, since for any $z_j \in \mathbb{C}$

$$|1 + z_2 + z_3 + \cdots + z_d| < 1 + |z_2| + |z_3| + \cdots + |z_d|$$

unless all the z_j are real and positive‡, and equality in (1) requires that

$$|M_{11}|\,\big|\big(1 + \sum_{j=2}^{d}(M_{jj}/M_{11})\big)\big| = \big|\sum_{j=1}^{d} M_{jj}\big| = \sum_{j=1}^{d}|M_{jj}| = |M_{11}|\big(1 + \sum_{j=2}^{d}|M_{jj}/M_{11}|\big),$$

it follows that $|\mathrm{trace}(M)| < d$ unless M is diagonal, $|M_{jj}| = 1$ for all j, and M_{jj}/M_{11} is real and positive for all j. These latter two conditions clearly force all the diagonal entries to be equal. □

It follows from the lemma that $|\chi^{(k)}(g)| \le d_k$ for all $g \in G$; furthermore, since $R^{(k)}(1_G)$ is the $d_k \times d_k$ identity matrix, which has trace d_k, we see that $\chi^{(k)}(g_j) = d_k$ when $\mathcal{C}_j = \{1_G\}$. The column of the character table corresponding to the identity element thus has positive integer entries which are also the maximum absolute values of the entries in the various rows. In the table above it is the first column.

‡ The student who is unfamiliar with this fact can easily prove it by induction, showing first that $|1 + z| < 1 + |z|$ unless z is real and positive.

The orthogonality relation $(1/|G|)\sum_{g\in G}\chi^{(k)}(g)\overline{\chi^{(l)}(g)} = \delta_{kl}$ can be rewritten as

$$\frac{1}{|G|}\sum_{j=1}^{s} h_j\chi^{(k)}(g_j)\overline{\chi^{(l)}(g_j)} = \delta_{kl} \tag{2}$$

since $\chi^{(k)}(g)\overline{\chi^{(l)}(g)} = \chi^{(k)}(g_j)\overline{\chi^{(l)}(g_j)}$ for each of the h_j elements g in the class $\mathcal{C}_j$. Regarding the character table of G as a matrix T whose (k,j) entry is $\chi^{(k)}(g_j)$, the above equation says that if U is the matrix whose (j,l) entry is $(1/|G|)h_j\overline{\chi^{(l)}(g_j)}$ then $TU = I$. Now this forces $UT = I$ also, and thus

$$\frac{1}{|G|}\sum_{l=1}^{s} h_j\overline{\chi^{(l)}(g_j)}\chi^{(l)}(g_k) = \delta_{jk}$$

for all j and k. Note that h_j is a common factor on the left hand side of this formula. Remember also (see Lecture 4) that h_j is the index in G of the centralizer of g_j. So multiplying through by $|C_G(g_j)|$ gives us the *column orthogonality* relation

$$\sum_{l=1}^{s}\overline{\chi^{(l)}(g_j)}\chi^{(l)}(g_k) = \delta_{jk}|C_G(g_j)|. \tag{3}$$

This is, in a way, simpler than the row orthogonality relation (Eq. (2)) since you do not need to know the h_j's to apply it. Note that putting $k = j$ in Eq. (3) yields that $\sum_{l=1}^{s}|\chi^{(l)}(g_j)|^2$ is the order of the centralizer of g_j. In the case when g_j is the identity element, whose centralizer is of course the whole group, we recover the fact that $\sum_l d_l^2 = |G|$.

Looking back at the character table given in Lecture 13, we see that the centralizer of g_2 has order $1^2 + (-1)^2 + (-1)^2 + 2^2 + (-1)^2 = 8$, the centralizer of g_3 has order $1^2 + 1^2 + 1^2 + (-1)^2 = 4$, the centralizer of g_4 has order $1^2 + 1^2 + (-1)^2 = 3$, and g_5 and g_6 each have centralizers of order $1^2 + |(1/2)(-1+i\sqrt{7})|^2 + |(1/2)(-1+i\sqrt{7})|^2 + (-1)^2 + 1^2 = 7$. And summing the squares of the numbers in the first column confirms that the group has order 168. Using the formula $h_j = |G|/|C_G(g_j)|$ we find that $h_2 = 168/8 = 21$, and similarly h_3, h_4, h_5 and h_6 are, respectively, 42, 56, 24 and 24. The total number of elements in the group is the sum of the numbers of elements in all the classes, and this checks since $1 + 21 + 42 + 56 + 24 + 24 = 168$. Column orthogonality is easily checked: for example, multiplying the second last entry of each row by the complex conjugate of the last entry and adding gives $1^2 + (1/4)(-1-i\sqrt{7})^2 + (1/4)(-1-i\sqrt{7})^2 + (-1)^2 + 0^2 + 1^2 = 0$. It is also easy to check row orthogonality: for example, calculating $\sum_j h_j\chi^{(k)}(g_j)\overline{\chi^{(l)}(g_j)}$ with $k = 2$ and $l = 3$ gives $1\times 9 + 21\times(-1)^2 + 42\times 1^2 + 0 + 24\times(1/4)(-1+i\sqrt{7})^2 + 24\times(1/4)(-1-i\sqrt{7})^2 = 0$.

Let us illustrate some of the other deductions that are possible. The centralizer of each element contains as a subgroup the cyclic subgroup generated by the element itself. Thus the order of an element must be a divisor of the order of the centralizer. We deduce immediately that in the example we have been discussing the elements g_5 and g_6 have order 7 and g_4 has order 3. Let C be the cyclic subgroup generated by g_5, and consider the restriction to C of the representation $R^{(2)}$. This gives a representation of C of degree 3. Now the irreducible representations of a cyclic group of order 7 all have degree 1, and are obtained by mapping a generating element to the various complex 7th roots of 1. Writing $\zeta = e^{2\pi i/7}$ we deduce that

$$R^{(2)}(g_5) = X^{-1}\begin{pmatrix}\zeta^k & 0 & 0\\ 0 & \zeta^l & 0\\ 0 & 0 & \zeta^m\end{pmatrix}X$$

for some integers k, l and m. Taking traces we deduce that $\zeta^k + \zeta^l + \zeta^m = (1/2)(-1 + i\sqrt{7})$. Considering all possible values for k, l and m it is not hard to deduce from this that in fact $\{k, l, m\} = \{1, 2, 4\}$. In other words, we have determined the eigenvalues of $R^{(2)}(g_5)$. In a similar fashion one can deduce that the eigenvalues of $R^{(2)}(g_4)$ are 1, ω and ω^2, where $\omega = e^{2\pi i/3}$.

Expressing the regular representation in terms of irreducibles

Let G be a group of order n, with elements $s_1, s_2, \ldots, s_n$. The matrix version of the regular representation we constructed in Lecture 10 maps each $g \in G$ to the matrix whose (i, j)-entry is 1 if $gs_j = s_i$ and 0 if $gs_j \neq s_i$. In particular, for each i the (i, i)-entry is 1 if and only if g is the identity element. So when g is not the identity all the diagonal entries of Rg are zero, and so the trace of Rg is zero. Hence if χ is the character of the regular representation it follows that

$$\chi(g) = \begin{cases} n = |G| & \text{if } g = 1_G, \\ 0 & \text{if } g \neq 1_G. \end{cases}$$

Now for each of the irreducible characters $\chi^{(l)}$ we can easily calculate the inner product $(\chi, \chi^{(l)})$:

$$(\chi, \chi^{(l)}) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi^{(l)}(g)} = \frac{1}{|G|}\chi(1)\overline{\chi^{(l)}(1)} = d_l$$

since $\chi(1) = |G|$ and $\chi^{(k)}(1) = d_l$ (and the terms for $g \neq 1$ are zero). But we also know that χ, like any other character, can be expressed in the form $\sum_{k=1}^{s} m_k \chi^{(k)}$ (see Lecture 10 and the solution to Exercise 2 of Tutorial 5). So

$$(\chi, \chi^{(l)}) = \sum_{k=1}^{s} m_k (\chi^{(k)}, \chi^{(l)}) = \sum_{k=1}^{s} m_k \delta_{kl} = m_l.$$

We conclude that $m_l = d_l$ for all l: each irreducible character occurs as a constituent of χ with multiplicity equal to its degree. Another way to prove this is to calculate the matrix representation of G obtained by using the basis of the G-module V_G consisting of the coordinate functions of the representations $R^{(1)}$, $R^{(2)}$, $\ldots$, $R^{(s)}$. Recall that we showed in Lecture 10 that these coordinate functions do form a basis for V_G, and that V_G has another basis (consisting of the functions f_x defined in Lecture 11) which is in bijective correspondence with G, and which elements of G permute according to the left multiplication action of G on itself. So the representation obtained from the basis of coordinate functions will be equivalent to the matrix version of the regular representation described above.

We need to choose an ordering of the basis, and let us do this as follows. Take the coordinate functions of $R^{(1)}$ first, in the order $R^{(1)}_{11}$, $R^{(1)}_{12}$, $\ldots$, $R^{(1)}_{1d_1}$, followed by $R^{(1)}_{21}$, $R^{(1)}_{22}$, $\ldots$, $R^{(1)}_{2d_1}$, and so on, then move on to the coordinate functions of $R^{(2)}$ ordered similarly, then $R^{(3)}$, and so on. We will show that in fact for each fixed k and p the functions the functions $R^{(k)}_{p1}$, $R^{(k)}_{p2}$, $\ldots$, $R^{(k)}_{pd_k}$ span a submodule V_{kp} of V_G, by showing that $gR^{(k)}_{p1}$, $gR^{(k)}_{p2}$, $\ldots$, $gR^{(k)}_{pd_k}$ (for all $g \in G$) are linear combinations of $R^{(k)}_{p1}$, $R^{(k)}_{p2}$, $\ldots$, $R^{(k)}_{pd_k}$. Indeed, for all k, p and m and all $h \in G$,

$$(gR^{(k)}_{pm})h = R^{(k)}_{pm}(hg) = \sum_{l=1}^{d_k} (R^{(k)}_{pl}h)(R^{(k)}_{lm}g) \qquad (\text{since } R^{(k)}(hg) = (R^{(k)}h)(R^{(k)}g))$$

which shows that $gR^{(k)}_{pm} = \sum_{l=1}^{d_k}(R^{(k)}_{lm}g)R^{(k)}_{pl}$. Furthermore, it shows that the matrix of the action of g on V_{kp} relative to this basis has (l, m)-entry $R^{(k)}_{lm}g$. In other words, for each of the submodules V_{k1}, V_{k2}, …, V_{kd_k} we obtain $R^{(k)}$ as the matrix representation. The representation for the whole of V_G is thus the diagonal sum of d_1 copies of $R^{(1)}$, d_2 copies of $R^{(2)}$, and so on, as required.

Induced representations and induced characters

Linear operators are encountered in many different areas of mathematics, and to deal with them usually requires finding eigenvectors and (if possible) diagonalizing them. If ρ is an invertible linear operator then $n \mapsto \rho^n$ is a representation of the infinite cyclic group $\mathbb{Z}$, and diagonalizing ρ amounts to expressing this representation as a diagonal sum of representations of degree 1. If ρ is not diagonalizable then at least one can put it into Jordan canonical form. Representation theory is concerned with the problem of finding canonical forms not just for single linear transformations but for groups of linear transformations. If the group is finite and we are working over the complex field, the appropriate canonical form for an arbitrary representation should be a diagonal sum of irreducible representations $R^{(k)}$. Thus the two basic problems in the representation theory of finite groups are to find these irreducible representations, and to find methods for expressing an arbitrary representation as a sum of irreducibles.

The orthogonality relations provide a powerful method for dealing with the latter of these two basic problems. As was done above for the regular representation, the orthogonality relations can be used to compute the multiplicities with which the various irreducibles occur as constituents of an arbitrary given representation. Then to explicitly express the representation as a diagonal sum of irreducibles is a matter of solving linear equations. But none of this can be done unless we can find all the irreducible representations in the first place. So the primary fundamental problem of representation theory is to construct irreducible representations.

So far we have not investigated any methods for constructing any representations, let alone irreducible ones. We shall now describe a process for constructing a representation of G given a representation of a subgroup of G. The resulting representation of G need not be irreducible, even if the given representation of the subgroup is irreducible. But if we can construct enough reducible representations of G we will have more chance of finding irreducible ones.

Let L be a subgroup of G and let x_1, x_2, …, x_n be a system of representatives of the left cosets of L in G (see Lecture 3). For each $g \in G$ and each $j \in \{1, 2, \ldots, n\}$ the coset gx_jL must coincide with one of x_1L, x_2L, …, x_nL. Thus G acts on the set $\{x_1L, x_2L, \ldots, x_nL\}$ by left multiplication. This gives us a permutation representation of G which can be converted to a matrix representation by identifying permutations with permutation matrices as described in Lecture 10. To be precise, the matrix associated with $g \in G$ has (i, j)-entry which is 1 if $gx_jL = x_iL$ and 0 otherwise. This is the simplest example of an induced representation: it is the representation induced from the 1-representation of the subgroup L. Note that in the case $L = \{1\}$ this is just the regular representation of G.

Nw let R be a matrix representation of L of degree d. We have seen above that for each $j \in \{1, 2, \ldots, n\}$ there is a unique $i \in \{1, 2, \ldots, n\}$ such that $x_i^{-1}gx_j \in L$ (since this is equivalent to $gx_jL = x_iL$). Let $R^G(g)$ be the $nd \times nd$ matrix which is an $n \times n$ array of $d \times d$ blocks, the (i, j)-block being zero if $x_i^{-1}gx_j \notin L$ and $R(x_i^{-1}gx_j)$ if $x_i^{-1}gx_j \in L$. If we define $\dot{R}g$ to be Rg if $g \in L$ and 0 if $g \notin L$ then we have

$$R^G(g) = \begin{pmatrix} \dot{R}(x_1^{-1}gx_1) & \dot{R}(x_1^{-1}gx_2) & \cdots & \dot{R}(x_1^{-1}gx_n) \\ \dot{R}(x_2^{-1}gx_1) & \dot{R}(x_2^{-1}gx_2) & \cdots & \dot{R}(x_2^{-1}gx_n) \\ \vdots & \vdots & & \vdots \\ \dot{R}(x_n^{-1}gx_1) & \dot{R}(x_n^{-1}gx_2) & \cdots & \dot{R}(x_n^{-1}gx_n) \end{pmatrix}$$

and the (i, j)-block of the product $R^G(g)R^G(h)$ is

$$\sum_{k=1}^{n} \dot{R}(x_i^{-1}gx_k)\dot{R}(x_k^{-1}hx_j) = \dot{R}(x_i^{-1}gx_k)R(x_k^{-1}hx_j)$$

where k is the unique index such that $x_k^{-1}hx_j \in L$. So the (i, j) block of $R^G(g)R^G(h)$ is

$$R(x_i^{-1}gx_k)R(x_k^{-1}hx_j) = R(x_i^{-1}ghx_j)$$

if $x_i^{-1}gx_k$ is also in L and is zero if $x_i^{-1}gx_k \notin L$. Given that $x_k^{-1}hx_j \in L$ we see that the product $(x_i^{-1}gx_k)(x_k^{-1}hx_j) = x_i^{-1}ghx_j$ is in L if and only if $x_i^{-1}gx_k \in L$. So the (i, j) block of $R^G(g)R^G(h)$ is $R(x_i^{-1}ghx_j)$ if $x_i^{-1}ghx_j$ is in L and is 0 otherwise. That is, it equals $\dot{R}(x_i^{-1}ghx_j)$, the (i, j) block of $R^G(gh)$. So $R^G(gh) = R^G(g)R^G(h)$.

To complete the proof that R^G is a matrix representation of G it remains to prove that $R^G(g)$ is invertible for each g. Since $R^G(g)R^G(g^{-1}) = R^G(1)$ it suffices to show that $R^G(1)$ is the identity. But this is clear since the (i, j) block of $R^G(1)$ is $\dot{R}(x_i^{-1}x_j)$, which is $R(1) = I$ if $i = j$ and 0 if $i \neq j$ (since $x_i^{-1}x_j \notin L$ if $i \neq j$).

Our next task is to calculate the character of the representation R^G of G in terms of the character of the representation R of L. There are various forms of this formula, and which is best to use depends on the context. Writing χ for the character of R and χ^G for the character of R^G, and defining $\dot{\chi}(g)$ to be equal to $\chi(g)$ for $g \in L$ and zero for $g \notin L$, we see immediately from the formula for $R^G(g)$ that

$$\chi^G(g) = \sum_{i=1}^{s} \dot{\chi}(x_i^{-1}gx_i) = \sum_{i} \chi(x_i^{-1}gx_i)$$

where this last sum is over those x_i such that $x_i^{-1}gx_i \in L$. For each $l \in L$ the element $l^{-1}x_i^{-1}gx_il$ is in L if and only if $x_i^{-1}gx_i \in L$, and when it is in L then $\chi(l^{-1}x_i^{-1}gx_il) = \chi(x_i^{-1}gx_i)$, since χ is a class function on L. Thus $\chi^G(g) = (1/|L|) \sum_{l\in L} \sum_i \chi(l^{-1}x_i^{-1}gx_il)$ (sum over those i and l such that $l^{-1}x_i^{-1}gx_il \in L$). As each element of G is uniquely expressible in the form x_il (with $i \in \{1, 2, \ldots, n\}$ and $l \in L$) we conclude that

$$\chi^G(g) = \frac{1}{|L|} \sum_{x} \chi(x^{-1}gx)$$

where the sum is over those $x \in G$ such that $x^{-1}gx \in L$.

As x runs through all elements of G every conjugate of g occurs $|C_G(g)|$ times as a value of $x^{-1}gx$. For example, $x^{-1}gx = g$ for each of the $|C_G(g)|$ elements x in $C_G(g)$, and any other given conjugate of g will arise from the elements x in some other coset $C_G(g)h$. So we can write

$$\chi^G(g) = \frac{|C_G(g)|}{|L|} \sum_{l\in\mathcal{L}} \chi(l)$$

where the $\mathcal{L}$ is set of all conjugates of g that are in L. Now $\mathcal{L}$ is a union of conjugacy classes of L, and if l_1, l_2, $\ldots$, l_m are representatives of these classes then we can write

$$\chi^G(g) = \frac{|C_G(g)|}{|L|} \sum_{i=1}^{m} q_i\chi(l_i)$$

where q_i is the number of elements in the L-conjugacy class of l_i. Finally, since $q_i = |L|/|C_L(l_i)|$ we can also write

$$\chi^G(g) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_L(l_i)|}\chi(l_i) = \frac{|G|}{|L|} \sum_{i=1}^{m} \frac{q_i}{h}\chi(l_i),$$

where h is the number of elements in the G-conjugacy class of g.