Let D and D' be diagrams corresponding to partitions π and π' of n. It is clear that the set

$$e(D)\mathbb{C}S_n e(D') = \{\, e(D)ae(D') \mid a \in \mathbb{C}S_n \,\}$$

is a vector subspace of the group algebra $\mathbb{C}S_n$. It will be useful for us to determine the dimension of this space.

Suppose first that $\pi > \pi'$, and let $\tau \in S_n$ be arbitrary. By the proposition from Lecture 20 there exist numbers i and j collinear in D and co-columnar in $\tau D'$. So the transposition (i,j) is in the row group of D and the column group of $\tau D'$. So

$$[R(D)]_1(i,j) = \sum_{\sigma\in R(D)} \sigma(i,j) = \sum_{\tau\in R(D)} \tau = [R(D)]_1$$

since $\tau = \sigma(i,j)$ runs through $R(D)$ as σ does. Similarly,

$$(i,j)[C(\tau D')]_\varepsilon = \sum_{\sigma\in C(\tau D')} \varepsilon(\sigma)(i,j)\sigma = \sum_{\tau\in C(\tau D')} -\varepsilon(\tau)\tau = -[C(\tau D')]_\varepsilon,$$

since if $\tau = (i,j)\sigma$ then $\varepsilon(\tau) = \varepsilon((i,j))\varepsilon(\sigma) = -\varepsilon(\sigma)$. Hence it follows that

$$[R(D)]_1 C[\tau D']_\varepsilon = ([R(D)]_1(i,j))C[\tau D']_\varepsilon = [R(D)]_1((i,j)C[\tau D']_\varepsilon) = -[R(D)]_1 C[\tau D']_\varepsilon,$$

and therefore $[R(D)]_1 C[\tau D']_\varepsilon = 0$. Note also that

$$\tau[C(D')]_\varepsilon\tau^{-1} = \sum_{\sigma\in C(D')} \varepsilon(\sigma)\tau\sigma\tau^{-1} = \sum_{\rho\in C(\tau D')} \varepsilon(\tau^{-1}\rho\tau)\rho = \sum_{\rho\in C(\tau D')} \varepsilon(\rho)\rho = [C(\tau D']_\varepsilon.$$

So for all $\tau \in S_n$ we have $[R(D)]_1\tau[C(D')]_\varepsilon = [R(D)]_1[C(\tau D')]_\varepsilon\tau = 0$, and thus if $a = \sum_{\tau\in S_n} \lambda_\tau\tau$ is any element of the group algebra $\mathbb{C}S_n$ then

$$[R(D)]_1 a[C(D')]_\varepsilon = \sum_{\tau\in S_n} \lambda_\tau[R(D)]_1\tau[C(D')]_\varepsilon = 0.$$

In particular,

$$e(D)be(D') = [R(D)]_1([C(D)]_\varepsilon b[R(D')]_1)[C(D')]_\varepsilon = 0$$

for all $b \in \mathbb{C}S_n$.

Now suppose instead that $\pi' > \pi$, and let $\tau \in S_n$ be arbitrary. The proposition from Lecture 20 now tells us that there is a transposition (i,j) in the row group of $\tau D'$ and the column group of D. So $[C(D)]_\varepsilon(i,j) = -[C(D)]_\varepsilon$ and $(i,j)[R(\tau D']_1 = [R(\tau D']_1$. Hence

$$[C(D)]_\varepsilon[R(\tau D')]_1 = [C(D)]_\varepsilon((i,j)[R(\tau D')]_1) = ([C(D)]_\varepsilon(i,j))[R(\tau D')]_1 = -[C(D)]_\varepsilon[R(\tau D')]_1,$$

and so $[C(D)]_\varepsilon[R(\tau D')]_1 = 0$. If $a = \sum_{\tau\in S_n} \lambda_\tau\tau$ is any element of the group algebra $\mathbb{C}S_n$ then

$$[C(D)]_\varepsilon a[R(D')]_1 = \sum_{\tau\in S_n} \lambda_\tau[C(D)]_\varepsilon\tau[R(D')]_1 = \sum_{\tau\in S_n} \lambda_\tau[C(D)]_\varepsilon[R(\tau D')]_1\tau = 0.$$

In particular,

$$e(D)ae(D') = [R(D)]_1([C(D)]_\varepsilon a[R(D')]_1)[C(D')]_\varepsilon = 0$$

for all $a \in \mathbb{C}S_n$.

We have now proved the following result.

Proposition. *Let D, D' be diagrams corresponding to partitions π, π' of n. If $\pi \neq \pi'$ then $e(D)\mathbb{C}S_n e(D') = \{0\}$.*

Consider now the case $\pi' = \pi$ and $D' = D$, and let $\tau \in S_n$. If there are two numbers i and j which are collinear in D and co-columnar in τD then (as above) we see that $[R(D)]_1\tau[C(D)]_\varepsilon = 0$. If there are no such numbers i and j then the proposition from Lecture 20 tells us that $\tau D = \rho\sigma D$ for some $\rho \in R(D)$ and $\sigma \in C(D)$. But $\tau D = \rho\sigma D$ implies $\tau = \rho\sigma$, since, as we noted in Lecture 20, the diagrams for a given partition are in one to one correspondence with the elements of S_n. So it follows that if $[R(D)]_1\tau[C(D)]_\varepsilon \neq 0$ then for some $\rho \in R(D)$ and $\sigma \in C(D)$ we have

$$[R(D)]_1\tau[C(D)]_\varepsilon = ([R(D)]_1\rho)(\sigma[C(D)]_\varepsilon) = \varepsilon(\sigma)[R(D)]_1[C(D)]_\varepsilon = \varepsilon(\sigma)e(D).$$

Thus $[R(D)]_1\tau[C(D)]_\varepsilon$ is a scalar multiple of $e(D)$ for all $\tau \in S_n$, and so if $a = \sum_{\tau\in S_n} \lambda_\tau\tau$ is any element of $\mathbb{C}S_n$ it follows that $[R(D)]_1a[C(D)]_\varepsilon = \sum_{\tau\in S_n} \lambda_\tau[R(D)]_1\tau[C(D)]_\varepsilon$ is a scalar multiple of $e(D)$. So for any $b \in \mathbb{C}S_n$,

$$e(D)be(D) = [R(D)]_1([C(D)]_\varepsilon b[R(D)]_1)[C(D)]_\varepsilon = \lambda e(D) \tag{1}$$

for some scalar λ, and hence $e(D)\mathbb{C}S_n e(D)$ is equal either to $\{0\}$ or to the one dimensional space $\mathbb{C}e(D) = \{\, \lambda e(D) \mid \lambda \in \mathbb{C}\,\}$. The following lemma shows that in fact the latter alternative always holds.

Proposition. *Let D be any diagram corresponding to a partition π of n. Then $e(D)^2$ is a nonzero scalar multiple of $e(D)$.*

Proof. Taking $b = \mathrm{id}$ in Eq. (1) gives $e(D)^2 = \lambda e(D)$ for some scalar $\lambda \in \mathbb{C}$; our aim is to show that $\lambda \neq 0$.

If ρ, $\rho' \in R(D)$ and σ, $\sigma' \in C(D)$ with $\rho\sigma = \rho'\sigma'$ then

$$(\rho')^{-1}\rho = \sigma'\sigma^{-1} \in R(D) \cap C(D) = \{\mathrm{id}\},$$

and so $\rho = \rho'$ and $\sigma = \sigma'$. Hence in the expression

$$e(D) = \sum_{\rho\in R(D)} \sum_{\sigma\in C(D)} \varepsilon(\sigma)\rho\sigma$$

all the terms are distinct; so if we write $e(D) = \sum_{\tau\in S_n} \alpha_\tau\tau$ then $\alpha_{\mathrm{id}} = 1$.

Our strategy is to compute in two different ways the trace of the linear transformation $f\colon \mathbb{C}S_n \to \mathbb{C}S_n$ given by right multiplication by $e(D)$. First we use the obvious basis of $\mathbb{C}S_n$, consisting of all the elements of S_n. If $\sigma \in S_n$ is arbitrary then

$$f(\sigma) = \sigma e(D) = \sum_{\tau\in S_n} \alpha_\tau\sigma\tau = \sum_{\tau\in S_n} \alpha_{sigma^{-1}\tau}\tau,$$

and the coefficient of σ in this is $\alpha_{\text{id}} = 1$. Thus, when we compute the matrix of f relative to this basis of $\mathbb{C}S_n$, all the diagonal entries are 1. So the trace of f is $\dim \mathbb{C}S_n = |S_n| = n!$.

Now choose elements $a_1, a_2, \ldots, a_d$ which form a basis for $\mathbb{C}S_n e(D)$, and extend this to a basis $a_1, \ldots, a_d, a_{d+1}, \ldots, a_{n!}$ of $\mathbb{C}S_n$. For $1 \le i \le d$ we can write $a_i = b_i e(D)$ for some $b_i \in \mathbb{C}S_n$, and since $e(D)^2 = \lambda e(D)$ we deduce that

$$f(a_i) = a_i e(D) = b_i e(D) e(D) = \lambda b_i e(D) = \lambda a_i.$$

So the first d columns of the matrix of f relative to this basis coincide with the first d columns of λI. On the other hand if $i > d$ then $f(a_i) = a_i e(D) \in \mathbb{C}S_n e(D)$, and so $f(a_i)$ is a linear combination of $a_1, a_2, \ldots, a_d$ only. In particular, the coefficient of a_i is zero. The matrix of f relative to this basis has the form

$$\begin{pmatrix} \lambda I & Q \\ 0 & 0 \end{pmatrix}$$

for some $d \times (n! - d)$ matrix Q, and the trace of f is λd. So $\lambda d = n!$, and so $\lambda \neq 0$, as required. □

With λ as in the above proof, define $e_D = (1/\lambda)e(D)$. Then e_D is an idempotent element, since

$$e_D^2 = \frac{1}{\lambda^2} e(D)^2 = \frac{1}{\lambda^2} \lambda e(D) = e_D.$$

We shall show next time that e_D is a primitive idempotent of $\mathbb{C}S_n$, from which it follows that the left ideal $\mathbb{C}S_n e(D) = \mathbb{C}S_n e_D$ of $\mathbb{C}S_n$ is an indecomposable $\mathbb{C}S_n$ module (by a result from Lecture 18). Incidentally, since λ is the coefficient of id in $e(D)^2$, which is obviously integral since all the coefficients in $e(D)$ are integral, the proof of the last proposition also shows that the dimension d of the left ideal $\mathbb{C}S_n e(D)$ is a divisor of $n!$.

Lecture 22, 22/10/97

We now come to our main theorem on the representation theory of S_n.

Theorem. *If D is any diagram then the left ideal $\mathbb{C}S_n e(D)$ of $\mathbb{C}S_n$ is an irreducible left $\mathbb{C}S_n$-module. Two modules obtained from diagrams in this way are isomorphic if and only if the partitions associated with the diagrams are the same. Furthermore, choosing one diagram for each partition we obtain a full set of irreducible left $\mathbb{C}S_n$ modules.*

Proof. As we saw at the end of last lecture, $e(D)$ is a scalar multiple of an idempotent e_D. Suppose that this idempotent is not primitive, so that $e_D = e + f$ for some idempotents e and f with $ef = fe = 0$. Then

$$e = (e + f)e(e + f) = e_D e e_D \in e(D)\mathbb{C}S_n e(D) = \mathbb{C}e(D)$$

by a proposition from last lecture. So $e = \mu e_D$ for some $\mu \in \mathbb{C}$, and since $e^2 = e$ and $e_D^2 = e_D$ it follows that $\mu^2 = \mu$. Furthermore, idempotents are nonzero by definition; so $\mu = 1$, and $e = e_D$. But this forces $f = 0$, a contradiction. Hence e_D is primitive; so $\mathbb{C}S_n e_D$ is indecomposable, and hence irreducible by Maschke's Theorem.

Let D, D' be diagrams associated with partitions π, π' such that $\pi \neq \pi'$, and suppose that $\mathbb{C}S_n e(D) \cong \mathbb{C}S_n e(D')$ (as left $\mathbb{C}S_n$-modules). Let $\phi\colon \mathbb{C}S_n e(D) \to \mathbb{C}S_n e(D')$ be a $\mathbb{C}S_n$-module isomorphism. Then for some $a \in S_n$ we have $\phi(e_D) = ae(D')$, and since ϕ commutes with the $\mathbb{C}S_n$ action we see that for all $b \in \mathbb{C}S_n$

$$\phi(be_D) = \phi(be_D^2) = be_D\phi(e_D) = be_D ae(D').$$

But $e_D a e(D') \in e(D)\mathbb{C}S_n e(D') = \{0\}$, as we showed in Lecture 21, since $\pi \neq \pi'$. Hence $\phi(be_D) = 0$, and since every element of $\mathbb{C}S_n e(D)$ is expressible in the form be_D it follows that ϕ is the zero map, contradicting the fact that ϕ is an isomorphism. So $\mathbb{C}S_n e(D)$ and $\mathbb{C}S_n e(D')$ are not isomorphic.

The number of conjugacy classes of S_n is the number of different cycle types of permutations of n, which equals the number of partitions of n. Since the number of irreducible characters equals the number of conjugacy classes it follows that the number of partitions of n also gives the number of modules in a full set of irreducible modules for $\mathbb{C}S_n$. Now if we choose a diagram D arbitrarily for each partition π of n then the left ideals $\mathbb{C}S_n e(D)$ are pairwise nonisomorphic and so must comprise a full set of irreducible modules. Furthermore, if we replace one of the diagrams by another diagram corresponding to the same partition then we still have a full set of irreducible modules, and since only one of the modules has been changed the isomorphism type of that module must be unchanged. In other words, distinct diagrams corresponding to the same partition yield isomorphic modules, as required. □

There is much more that can be said about the representation theory of S_n, but we shall take the subject no further. The remaining lectures will be devoted to revision and calculation of various examples.