The University of Sydney

Automorphisms of Coxeter groups

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Introduction

A *Coxeter group* is a group W generated by a set S subject to defining relations of the form $s^2 = 1$ for all $s \in S$ and $(st)^{m_{st}} = 1$ for some or all pairs $s, t \in S$.

The pair (W, S) is called a Coxeter system.

Remarkably, every Coxeter group can be faithfully represented as a group of linear transformations on a real vector space.

In this representation, elements of the set S are reflections.

The finite Coxeter groups are exactly the finite groups generated by reflections in Euclidean space. e.g. $W = \mathrm{Sym}(n)$, realized as the group of all $n \times n$ permutation matrices, with $S = \{(1,2),(2,3),(3,4),\ldots,(n-1,n)\}$.

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A noteworthy automorphism

The only finite Coxeter group with an "interesting" automorphism is Sym(6).

It happens that there is an automorphism taking transpositions to permutations of cycle type $(a, b)(c, d)(e, f)$.

$$\begin{array}{ll} (1,2) \mapsto (1,2)(3,4)(5,6) & (2,3) \mapsto (1,4)(2,5)(3,6) \\ (3,4) \mapsto (1,3)(2,4)(5,6) & (4,5) \mapsto (1,2)(4,5)(3,6) \\ (5,6) \mapsto (1,4)(2,3)(5,6) & \end{array}$$

For $n > 6$ this kind of thing can't happen since the conjugacy class consisting of all transpositions is distinguished by its size. And it is easy to show that an automorphism that takes transpositions to transpositions must be inner.

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Diagrams

A Coxeter system is conveniently specified by its *Coxeter graph*.

This is just a diagrammatic notation for the presentation.

Example:

There are 8 generators. Generators s, t corresponding to non-adjacent vertices satisfy $(st)^2 = 1$; for adjacent vertices the relation is $(st)^3 = 1$.

$$\langle s_1, \ldots, s_8 \mid s_i^2 = 1, (s_2s_3)^3 = (s_3s_4)^3 = (s_4s_5)^3 = (s_5s_6)^3 \\ = (s_6s_7)^3 = (s_7s_8)^3 = (s_1s_4)^3 = 1, (s_is_j)^2 = 1 \text{ (otherwise)} \rangle$$

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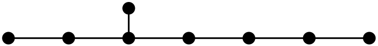
Example: 

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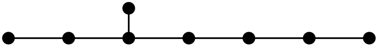
Example: 

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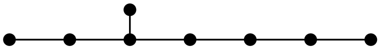
Example: 

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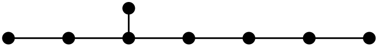

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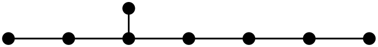

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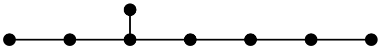
Example: 

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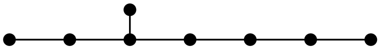
Example: 

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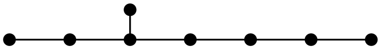

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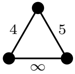

$$\langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^4 = (rt)^5 = 1 \rangle$$

Reducible Coxeter systems

Since the generators are involutions, $(st)^2 = 1 \Leftrightarrow st = ts$.

Recall that $(st)^2 = 1$ means no edge joining s and t.

(W, S) is *reducible* if the diagram is disconnected.

Then W is the direct product of Coxeter groups corresponding to the component diagrams.

W is an elementary abelian 2-group: $\mathrm{Aut}(W)$ is $\mathrm{GL}(n, 2)$.

But today I will focus on irreducible Coxeter systems.

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Geometrical Realization

Given (W, S), let V be a vector space over $\mathbb{R}$ with basis Π in 1-1 correspondence with the vertices of the diagram.

Define a bilinear form on V via $\alpha \cdot \alpha = 1$ for all $\alpha \in \Pi$, and $\alpha \cdot \beta = -\cos(\pi/m)$ if the vertices corresponding to $\alpha, \beta \in \Pi$ are joined by an edge labelled m. (No edge $\Rightarrow \alpha \cdot \beta = 0$.)

For each $\alpha \in \Pi$ let $s_\alpha \colon V \to V$ be the reflection in $\langle \alpha \rangle^\perp$.

That is, $v \mapsto v - 2(v \cdot \alpha)\alpha$ for all $v \in V$.

It is easy to see that $\{ s_\alpha \mid \alpha \in \Pi \}$ generates a subgroup of $\mathrm{O}(V)$ that is a homomorphic image of W.

It is a beautiful fact that the homomorphism is an isomorphism.

And W is finite iff V is Euclidean.

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Type A_n: (n vertices, for any $n \geq 1$)

Type B_n: (n vertices, for any $n \geq 2$)

Type D_n: (n vertices, for any $n \geq 4$)

Type $I_2(m)$: (any $m > 4$)

Type H_3:

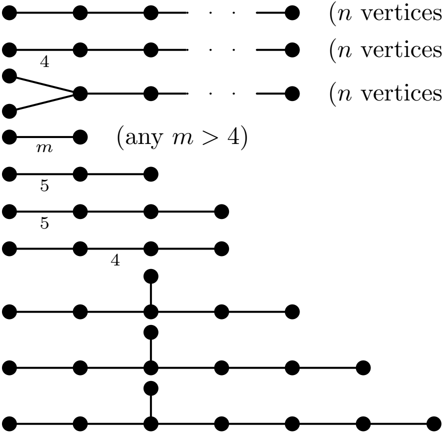
Type H_4:

Type F_4:

Type E_6:

Type E_7:

Type E_8:



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$\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+ \subseteq \{$ positive linear combinations of $\Pi \}$ and $\Phi^- = \{ -\gamma \mid \gamma \in \Phi^+ \}$.

$\mathrm{Ref}(W) =$ set of reflections in $W = \{ s_\gamma \mid \gamma \in \Phi^+ \}$.

Note that $s_\gamma = s_{-\gamma}$ (for every $\gamma \in \Phi$).

$N_{\mathrm{O}(V)}(W) = \{ g \in \mathrm{O}(V) \mid g\Phi = \Phi \}$.

But the only outer automorphisms of W you get like this are *graph automorphisms*, corresponding to symmetries of the Coxeter graph (i.e. symmetries of the presentation).

This is easy if $\#W < \infty$ but decidedly nontrivial otherwise. (RH, Peter Rowley & Don Taylor 1997.)

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Automorphisms of finite Coxeter groups

If $f \in \mathrm{Hom}(W, Z(W))$ then $w \mapsto wf(w)$ is in $\mathrm{End}(W)$.

Let $\mathscr{A}(W)$ be the group of automorphisms of this form.

Since $\#Z(W) \le 2$, it is easy to describe $\mathscr{A}(W)$ in all cases.

Define $R(W) = \{\, \theta \in \mathrm{Aut}(W) \mid \theta(S) \subseteq \mathrm{Ref}(W) \,\}$. (Recall that $\mathrm{Ref}(W)$ = set of reflections in W = set of conjugates of S.)

Theorem: $\mathrm{Aut}(W) = \mathscr{A}(W)R(W)$, *for all irreducible* (W, S) *except* A_5.

Proof (sketch): $\mathrm{Ref}(W)$ generates W and is a union of 1 or 2 classes of involutions. If there is another such set with the same number of elements as $\mathrm{Ref}(W)$, there is a $\theta \in \mathscr{A}(W)$ that maps $\mathrm{Ref}(W)$ to it.

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The reflection preserving automorphisms

Usually, $R(W)$ is generated by the inner automorphisms and graph automorphisms (for finite W).

The dihedral groups are obvious exceptions:
$W = \langle s_\alpha, s_\beta \rangle$, where $\alpha \cdot \beta = -\cos(\pi/m)$;
if $\gcd(k, m) = 1$ there exists $\beta' \in \Phi$ with $\alpha \cdot \beta' = \cos(k\pi/m)$;
and there is an automorphism with $s_\alpha \mapsto s_\alpha$ and $s_\beta \mapsto s_{\beta'}$.

When $m = 3$ this doesn't give any outer automorphisms;
when $m = 4$ or 6 you get one, but it is a graph automorphism;
when $m = 5$ you get one non-graph outer automorphism.

It is natural to ask if this automorphism of $I_2(5)$ extends to H_3 and H_4 (the only irred groups of rank > 2 with edge labels > 4).

It does.

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Infinite groups

Assume that W is infinite and the diagram of W has no edges labelled ∞.

Howlett-Rowley-Taylor (1997) proved that the outer automorphism group of W is necessary finite.

Bill Franzsen, RH and Bernhard Mühlherr (2005) improved this, showing that the outer automorphism group of W is isomorphic to the group of graph automorphisms. (That is, all the outer automorphisms come from permutations of S that preserve the defining relations.)

The remaining challenge is to deal with diagrams that have ∞'s.

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Standard parabolic (or visible) subgroups

The geometrical realization of W identifies S (the generating set from the presentation) with $\{ s_\alpha \mid \alpha \in \Pi \}$.

$\dim V = \#\Pi = \#S$ is called the *rank* of W.

Subgroups generated by subsets of S are called *standard parabolic* subgroups.

Let $J \subseteq \Pi$ and $W_J = \langle \{ s_\alpha \mid \alpha \in J \} \rangle$.

W_J preserves the subspace V_J spanned by J; so restriction gives a homomorphism $W_J \to \mathrm{O}(V_J)$.

The image of this is the geometrical realization of a Coxeter group of rank $\#J$.

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The *Tits cone* is $U = \bigcup_{w \in W} wC$. (It is convex – not obviously.)

Stabilizers of points in U are *parabolic subgroups* (= conjugates of W_J's).

Now $C \cap \mathrm{Int}(U) = \bigcup\limits_{W_J \text{ finite}} C_J$, and it follows that if $f \in U$ then $\mathrm{Stab}(f)$ is finite iff $f \in \mathrm{Int}(U)$.

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Now $C \cap \mathrm{Int}(U) = \bigcup_{W_J \text{ finite}} C_J$, and it follows that if $f \in U$ then Stab(f) is finite iff $f \in \mathrm{Int}(U)$.

If $H \leq W$ is finite then H stabilizes some $f \in \mathrm{Int}(U)$
(take the average over an H-orbit),
and so H is contained in a finite parabolic.

Automorphisms of 2-spherical Coxeter groups

The maximal finite parabolic subgroups of W are the maximal finite subgroups of W.

So automorphisms of W preserve this class of subgroups.

This is our main trick for investigating automorphisms of infinite Coxeter groups. (The idea is due to Tits.)

Since parabolic subgroups = pointwise stabilizers of subsets of U, intersections of parabolic subgroups are parabolic.

So if it were true that for each $s \in S$ the group $\langle s \rangle$ is an intersection of maximal finite subgroups then it would be true that every automorphism preserves reflections.

This is actually true for infinite irreducible Coxeter groups such that $m_{st} < \infty$ for all $s, t \in S$. One can also prove (in this case) that reflection preserving automorphisms are orthogonal.

So for these groups all automorphisms are inner by graph.

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The finite continuation of an element

More generally, if $w \in W$ is an involution then we define the *finite continuation* of w to be the intersection of all the maximal finite subgroups containing w.

Note that $\mathrm{FC}(w)$ is always a parabolic subgroup.

For every finitely generated Coxeter group, we are able to describe $\mathrm{FC}(s)$ for all $s \in S$ (by an algorithm that just requires inspecting the Coxeter graph).

It is then possible to classify the involutions that have the same finite continuations as the reflections, and reduce the problem of classifying automorphisms to the problem of classifying reflection-preserving automorphisms.

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Twists

For some cases in which there are ∞'s in the diagram, one can find reflection-preserving automorphisms of W that are “partial conjugations”.

The construction is due to Brady, Mccammond, Mühlherr and Neumann.

Example:

Changing the 8 to a 7 the isomorphism becomes an automorphism.

Let $W = \langle S \rangle$ correspond to the diagram on the left, and let $S = \{r, s, t, u\}$ (top to bottom, left to right).
Then $\{r, s, t, ststsuststs\}$ is a second Coxeter generating set for W, corresponding to the second diagram.

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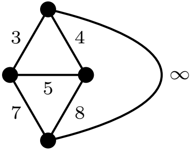
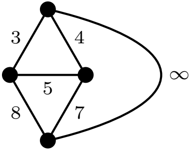
Example: 

is isomorphic to 

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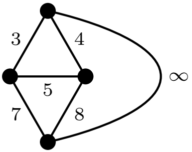
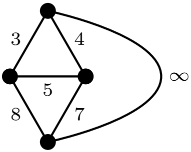
Example: 

is isomorphic to 

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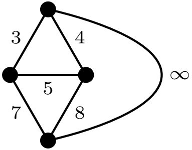
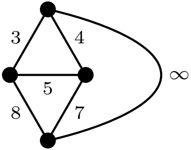
Example: 

is isomorphic to 

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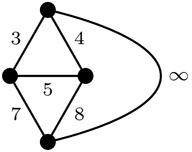
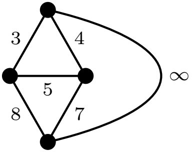
Example: 

is isomorphic to 

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A generalization of twisting

Suppose that S is the disjoint union of P, Q and R, and that $m_{ab} = \infty$ whenever $a \in Q$ and $b \in R$. Then W is a "free product with amalgamated subgroup": $W = A *_C B$, where $A = W_{P \cup Q}$, $B = W_{P \cup R}$ and $C = W_P$.

Conversely, suppose that $G = H *_L K$, where H, K are subgroups of G and $L = H \cap K$. If (H, A) and (K, B) are Coxeter systems and $C = A \cap B$ generates L, then $(G, A \cup B)$ is a Coxeter system.

If (K, B') is another Coxeter system, and $A \cap B' = A \cap B$, then $A \cup B'$ is another Coxeter generating set for G, not necessarily conjugate to $A \cup B$, even if B and B' are conjugate.

Twisting corresponds to the special case where L is finite and $B' = w^{-1} B w$, where w is the longest element of L. (Observation due to Mauro Grassi.)

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The conjecture

Finding a second Coxeter system in a given Coxeter group is obviously the same thing as finding an isomorphism from one Coxeter group to another.

Using inner automorphisms, graph automorphisms and the reflection-preserving automorphisms of dihedral groups and the Coxeter groups of types H_3 and H_4, one can build up more reflection preserving isomorphisms using Mauro's generalization of twisting.

We conjecture that every reflection preserving isomorphism from one Coxeter group to another is obtainable in this way.

For Coxeter diagrams where all edge labels are ∞ you need only twists, not generalized twists.

The conjecture is true in this case. (Proved for tree diagrams by L. James, diagrams with no triangles by Tits, completed by Mühlherr.)

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Finding a second Coxeter system in a given Coxeter group is obviously the same thing as finding an isomorphism from one Coxeter group to another.

Using inner automorphisms, graph automorphisms and the reflection-preserving automorphisms of dihedral groups and the Coxeter groups of types H_3 and H_4, one can build up more reflection preserving isomorphisms using Mauro's generalization of twisting.

We conjecture that every reflection preserving isomorphism from one Coxeter group to another is obtainable in this way.

For Coxeter diagrams where all edge labels are ∞ you need only twists, not generalized twists.

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