INDUCING W-GRAPHS II

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Abstract. Let $\mathscr{H}$ be the Hecke algebra associated with a Coxeter group W, and $\mathscr{H}_J$ the Hecke algebra associated with W_J, a parabolic subgroup of W. In [5] an algorithm was described for the construction of a W-graph for an induced module $\mathscr{H} \bigotimes_{\mathscr{H}_J} V$, where V is an $\mathscr{H}_J$-module derived from a W_J-graph. This note is a continuation of [5], and involves the following results:

- inducing ordered and bipartite W-graphs;
- the relationship between the cell decomposition of a W_J-graph and the cell decomposition of the corresponding induced W-graph;
- a Mackey-type formula for the restriction of an induced W-graph;
- a formula relating the polynomials used in the construction of induced W-graphs to Kazhdan-Lusztig polynomials.

The result on cells is a version of a Theorem of M. Geck [4], dealing with cells in W (allowing unequal parameters).

1. Preliminaries

Let W be a Coxeter group with S the set of simple reflections, and let $\mathscr{H}$ be the corresponding Hecke algebra. We use a variation of the definition given in [6], taking $\mathscr{H}$ to be an algebra over $\mathcal{A} = \mathbb{Z}[q^{-1}, q]$, the ring of Laurent polynomials with integer coefficients in the indeterminate q, having an $\mathcal{A}$-basis $\{ T_w \mid w \in W \}$ satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + (q - q^{-1})T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for all $w \in W$ and $s \in S$. We also define $\mathcal{A}^+ = \mathbb{Z}[q]$, the ring of polynomials in q with integer coefficients, and let $a \mapsto \overline{a}$ be the involutory automorphism of $\mathcal{A}$ such that $\overline{q} = q^{-1}$. This involution on $\mathcal{A}$ extends to an involution on $\mathscr{H}$ satisfying $\overline{T_s} = T_s^{-1} = T_s + (q^{-1} - q)$ for all $s \in S$. This gives $\overline{T_w} = T_{w^{-1}}^{-1}$ for all $w \in W$.

For each $J \subseteq S$ define $W_J = \langle J \rangle$, the corresponding parabolic subgroup of W, and let $D_J = \{ w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J \}$, the set of minimal coset representatives of W/W_J. Let $\mathscr{H}_J$ be the Hecke algebra associated with W_J. As is well known, $\mathscr{H}_J$ can be identified with a subalgebra of $\mathscr{H}$.

1.1. **Ordered W-graphs.** Modifying the definitions in [6] to suit our definition of the Hecke algebra, a W-graph is a set Γ (the vertices of the graph) with a set Θ of two-element subsets of Γ (the edges) together with the following additional data: for each vertex γ we are given a subset I_γ of S, and for each ordered pair of vertices δ, γ we are given an integer $\mu(\delta, \gamma)$ which is nonzero if and only if $\{\delta, \gamma\} \in \Theta$.

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These data are subject to the requirement that $\mathcal{A}\Gamma$, the free $\mathcal{A}$-module on Γ, has an $\mathscr{H}$-module structure satisfying

$$T_s\gamma = \begin{cases} -q^{-1}\gamma & \text{if } s \in I_\gamma \\ q\gamma + \sum_{\{\delta\in\Gamma \mid s\in I_\delta\}} \mu(\delta,\gamma)\delta & \text{if } s \notin I_\gamma, \end{cases} \tag{1}$$

for all $s \in S$ and $\gamma \in \Gamma$. If τ_s is the $\mathcal{A}$-endomorphism of $\mathcal{A}\Gamma$ such that $\tau_s(\gamma)$ is the right-hand side of Eq. (1) then this requirement is equivalent to the condition that for all s, $t \in S$ such that st has finite order,

$$\underbrace{\tau_s\tau_t\tau_s\ldots}_{m \text{ factors}} = \underbrace{\tau_t\tau_s\tau_t\ldots}_{m \text{ factors}}$$

where m is the order of st.

To avoid over-proliferation of symbols, we shall use the name of the vertex set of a W-graph to also refer to the W-graph itself. We call I_γ the *descent set* of the vertex $\gamma \in \Gamma$, and we call $\mu(\delta,\gamma)$ and $\mu(\gamma,\delta)$ the *edge weights* associated with the edge $\{\delta,\gamma\}$.

Given a W-graph Γ we define

$$\Gamma_s^- = \{\,\gamma \in \Gamma \mid s \in I_\gamma\,\},$$
$$\Gamma_s^+ = \{\,\gamma \in \Gamma \mid s \notin I_\gamma\,\}.$$

We make the following definition.

Definition 1.1. An *ordered W-graph* is a set Γ with a W-graph structure and a partial order $\leqslant$ satisfying the following conditions:

(i) for all θ, $\gamma \in \Gamma$ such that $\mu(\theta,\gamma) \neq 0$, either $\theta < \gamma$ or $\gamma < \theta$;
(ii) for all $s \in S$ and $\gamma \in \Gamma_s^+$ the set $\{\,\theta \in \Gamma_s^- \mid \gamma < \theta \text{ and } \mu(\theta,\gamma) \neq 0\,\}$ is either empty or consists of a single element $s\gamma$;
(iii) for all $s \in S$ and $\gamma \in \Gamma_s^+$, if $s\gamma$ exists then $\mu(s\gamma,\gamma) = 1$.

The following lemma is well known.

Lemma 1.2 (Deodhar [2, Lemma 3.2])**.** *Let $J \subseteq S$ and $s \in S$, and define*

$$D_{J,s}^- = \{\,d \in D_J \mid \ell(sd) < \ell(d)\,\},$$
$$D_{J,s}^+ = \{\,d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \in D_J\,\},$$
$$D_{J,s}^0 = \{\,d \in D_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin D_J\,\},$$

so that D_J is the disjoint union $D_{J,s}^- \cup D_{J,s}^+ \cup D_{J,s}^0$. Then $sD_{J,s}^+ = D_{J,s}^-$, and if $d \in D_{J,s}^0$ then $sd = dt$ for some $t \in J$.

1.2. **Construction of induced W-graphs.** Following the notation and terminology of [5], we assume that Γ is a W_J-graph and M the corresponding induced $\mathscr{H}$-module.

Theorem 1.3 ([5, Theorem 5.1])**.** *The module M has a unique basis*

$$\{\,C_{w,\gamma} \mid w \in D_J,\ \gamma \in \Gamma\,\}$$

such that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ for all $w \in D_J$ and $\gamma \in \Gamma$, and

$$C_{w,\gamma} = \sum_{y\in D_J,\delta\in\Gamma} P_{y,\delta,w,\gamma} T_y\delta$$

for some elements $P_{y,\delta,w,\gamma} \in \mathcal{A}^+$ with the following properties:

(i) $P_{y,\delta,w,\gamma} = 0$ *if* $y \nleq w$;

(ii) $P_{w,\delta,w,\gamma} = \begin{cases} 1 & \text{if } \delta = \gamma, \\ 0 & \text{if } \delta \neq \gamma; \end{cases}$

(iii) $P_{y,\delta,w,\gamma}$ *has zero constant term if* $(y,\delta) \neq (w,\gamma)$.

The following recursive formula for the polynomials $P_{y,\delta,w,\gamma}$ is proved in [5]: $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$, where

$$(2)\quad P'_{y,\delta,w,\gamma} = \begin{cases} P_{sy,\delta,v,\gamma} - qP_{y,\delta,v,\gamma} & \text{if } y \in D^+_{J,s}, \\ P_{sy,\delta,v,\gamma} - q^{-1}P_{y,\delta,v,\gamma} & \text{if } y \in D^-_{J,s}, \\ (-q-q^{-1})P_{y,\delta,v,\gamma} + \sum_{\theta\in\Gamma_t^+} \mu(\delta,\theta)P_{y,\theta,v,\gamma} & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma_t^-, \\ 0 & \text{if } y \in D^0_{J,s} \text{ and } \delta \in \Gamma_t^+; \end{cases}$$

$$(3)\qquad P''_{y,\delta,w,\gamma} = \sum_{\substack{(z,\theta)\prec(v,\gamma)\\ (z,\theta)\in\Lambda_s^-}} \mu(z,\theta,v,\gamma)P_{y,\delta,z,\theta}.$$

Given $y, w \in D_J$ and $\delta, \gamma \in \Gamma$ with $(y,\delta) \neq (w,\gamma)$, we define an integer $\mu(y,\delta,w,\gamma)$ as follows. If $y < w$ then $\mu(y,\delta,w,\gamma)$ is the coefficient of q in $-P_{y,\delta,w,\gamma}$, and if $w < y$ then it is the coefficient of q in $-P_{w,\gamma,y,\delta}$. If neither $y < w$ nor $w < y$ then

$$\mu(y,\delta,w,\gamma) = \begin{cases} \mu(\delta,\gamma) & \text{if } y = w, \\ 0 & \text{if } y \neq w. \end{cases}$$

We write $(y,\delta) \prec (w,\gamma)$ if $y < w$ and $\mu(y,\delta,w,\gamma) \neq 0$.

It is shown in Theorem 5.3 of [5] that the basis elements $C_{w,\gamma}$ can be identified with the vertices of a W-graph for the module M; we shall denote this W-graph by Λ. The descent set of the vertex $C_{w,\gamma}$ of Λ is

$$I(w,\gamma) = \{\, s \in S \mid \ell(sw) < \ell(w) \text{ or } sw = wt \text{ for some } t \in I_\gamma \,\}$$

and the edge weight for $((y,\delta),(w,\gamma))$ is $\mu(y,\delta,w,\gamma)$ (as defined above). Thus $\{C_{y,\delta}, C_{w,\gamma}\}$ is an edge of Λ if and only if $\mu(y,\delta,w,\gamma) \neq 0$, and this occurs if and only if either $(y,\delta) \prec (w,\gamma)$ or $(w,\gamma) \prec (y,\delta)$, or $y = w$ and $\{\delta,\gamma\}$ is an edge of Γ.

We define

$$\begin{aligned} \Lambda_s^- &= \{\, (w,\gamma) \in D_J \times \Gamma \mid s \in I(w,\gamma) \,\} \\ &= \{\, (w,\gamma) \mid w \in D^-_{J,s} \text{ or } w \in D^0_{J,s} \text{ with } t \in I_\gamma \,\}. \end{aligned}$$

Theorem 1.4 ([5, Theorem 5.2])**.** *Let $w \in D_J$ and $\gamma \in \Gamma$. Then for all $s \in S$ such that $\ell(sw) > \ell(w)$ and $sw \in D_J$ we have*

$$(4)\qquad T_sC_{w,\gamma} = qC_{w,\gamma} + C_{sw,\gamma} + \sum \mu(y,\delta,w,\gamma)C_{y,\delta},$$

where the sum is over all $(y,\delta) \in \Lambda_s^-$ such that $(y,\delta) \prec (w,\gamma)$.

It is convenient to distinguish three kinds of edges of the W-graph Λ. Firstly, there is an edge from the vertex $C_{w,\gamma}$ to the vertex $C_{w,\delta}$ whenever there is an edge from γ to δ in Γ. We call these *horizontal* edges. Next, if $s \in S$ and w is in either $D^+_{J,s}$ or $D^-_{J,s}$ then there is an edge joining $C_{w,\gamma}$ and $C_{sw,\gamma}$. We call these *vertical* edges. All other edges are called *transverse.*

2. Inducing ordered W-graphs

Proposition 2.1. *Suppose that vertices $C_{w,\gamma}$ and $C_{z,\theta}$ of Λ are joined by a transverse edge, and suppose that $\ell(w) \leqslant \ell(z)$. Then $I(z,\theta) \subseteq I(w,\gamma)$.*

Proof. Let $s \in I(z,\theta)$, and suppose, for a contradiction, that $s \notin I(w,\gamma)$. Since the edge is not horizontal we have either $(w,\gamma) \prec (z,\theta)$ or $(z,\theta) \prec (w,\gamma)$, and the assumption $\ell(w) \leqslant \ell(z)$ means that the former alternative holds. So we have $(w,\gamma) \prec (z,\theta)$, with $(z,\theta) \in \Lambda_s^-$ and $(w,\gamma) \in \Lambda_s^+$. Since Λ is a W-graph,

$$T_s C_{w,\gamma} = qC_{w,\gamma} + \sum_{(y,\delta)\in\Lambda_s^-} \mu(y,\delta,w,\gamma) C_{y,\delta}$$

and, in particular, one of the terms on the right hand side is $\mu(z,\theta,w,\gamma)C_{z,\theta}$. The coefficient $\mu(z,\theta,w,\gamma)$ is nonzero by the hypothesis that $C_{w,\gamma}$ and $C_{z,\theta}$ are joined by an edge of Λ. But by Theorem 1.4,

$$T_s C_{w,\gamma} = qC_{w,\gamma} + C_{sw,\gamma} + \sum \mu(y,\delta,w,\gamma)C_{y,\delta},$$

with $y \leqslant w$ for all terms in the sum. Since $z \nleqslant w$, it follows that

$$\mu(z,\theta,w,\gamma)C_{z,\theta} = C_{sw,\gamma},$$

which means that the edge $\{C_{w,\gamma}, C_{z,\theta}\}$ is vertical rather than transverse, giving us the desired contradiction. □

Proposition 2.2. *Suppose that the W_J-graph Γ admits a partial order $\leqslant$ satisfying the conditions of Definition 1.1. Then the induced W-graph Λ admits a partial order $\leqslant$ satisfying Definition 1.1 and having the following properties*:

(i) *if $\delta, \gamma \in \Gamma$ and $y, w \in D_J$ are such that $y \leqslant w$ and $\delta \leqslant \gamma$, then $C_{y,\delta} \leqslant C_{w,\gamma}$;*
(ii) *if $\delta, \gamma \in \Gamma$ and $y, w \in D_{J,s}^+$ for some $s \in S$, then $C_{y,\delta} \leqslant C_{w,\gamma}$ implies that $C_{sy,\delta} \leqslant C_{sw,\gamma}$;*
(iii) *if $y \in D_{J,s}^0$ and $w \in D_{J,s}^+$ for some $s \in S$, then $C_{y,\delta} \leqslant C_{w,\gamma}$ implies that $C_{y,t\delta} \leqslant C_{sw,\gamma}$, for all $\gamma \in \Gamma$ and $\delta \in \Gamma_t^+$ such that $t\delta$ exists, where $t = y^{-1}sy$;*
(iv) *if $(y,\delta), (w,\gamma) \in D_J \times \Gamma$ satisfy $P_{y,\delta,w,\gamma} \neq 0$ then $C_{y,\delta} \leqslant C_{w,\gamma}$.*

Proof. We define $\leqslant$ on Λ to be the minimal transitive relation satisfying the requirements (i), (ii) and (iii). It is clear that $C_{y,\delta} \leqslant C_{w,\gamma}$ implies that $y \leqslant w$, with equality only if $\delta \leqslant \gamma$. Hence the fact that the relation $\leqslant$ on Γ is antisymmetric implies the same for the relation $\leqslant$ on Λ.

We prove first that Condition (iv) is satisfied, using induction on $\ell(w)$. In the case $\ell(w) = 0$ the assumption that $P_{y,\delta,w,\gamma} \neq 0$ forces $(y,\delta) = (w,\gamma)$, and so $C_{y,\delta} \leqslant C_{w,\gamma}$. So suppose that $\ell(w) > 0$, and choose $s \in S$ with $\ell(sw) < \ell(w)$. Recall that $P_{y,\delta,w,\gamma} = P'_{y,\delta,w,\gamma} - P''_{y,\delta,w,\gamma}$; hence either $P''_{y,\delta,w,\gamma} \neq 0$ or $P'_{y,\delta,w,\gamma} \neq 0$.

If $P''_{y,\delta,w,\gamma} \neq 0$ then by Eq. (3) there exists a pair (z,θ) with $(z,\theta) \prec (sw,\gamma)$ and $P_{y,\delta,z,\theta} \neq 0$. The inductive hypothesis then yields both $C_{y,\delta} \leqslant C_{z,\theta}$ and $C_{z,\theta} \leqslant C_{sw,\gamma}$, and since also $C_{sw,\gamma} \leqslant C_{w,\gamma}$ it follows that $C_{y,\delta} \leqslant C_{w,\gamma}$, as required. So we may assume that $P'_{y,\delta,w,\gamma} \neq 0$.

Suppose first that $y \in D_{J,s}^+$. By Eq. (2) either $P_{y,\delta,sw,\gamma} \neq 0$ or $P_{sy,\delta,sw,\gamma} \neq 0$, and so the inductive hypothesis yields that either $C_{y,\delta} \leqslant C_{sw,\gamma}$ or $C_{sy,\delta} \leqslant C_{sw,\gamma}$. Since $C_{y,\delta} \leqslant C_{sy,\delta}$ we obtain $C_{y,\delta} \leqslant C_{sw,\gamma}$ in either case, and hence $C_{y,\delta} \leqslant C_{w,\gamma}$.

Now suppose that $y \in D^-_{J,s}$. Again Eq. (2) and the inductive hypothesis combine to yield that either $C_{y,\delta} \leqslant C_{sw,\gamma}$ or $C_{sy,\delta} \leqslant C_{sw,\gamma}$. The former alternative yields $C_{y,\delta} \leqslant C_{w,\gamma}$ as in the previous cases, while the latter alternative yields the same result since (ii) above holds.

Finally, suppose that $y \in D^0_{J,s}$, and let $t = y^{-1}sy \in J$. By Eq. (2) we see that either $P_{y,\delta,sw,\gamma} \neq 0$, which yields $C_{y,\delta} \leqslant C_{w,\gamma}$ as in the previous cases, or else $\delta \in \Gamma^-_t$ and $\mu(\delta,\theta)P_{y,\theta,sw,\gamma} \neq 0$ for some $\theta \in \Gamma^+_t$. Thus $\{\theta,\delta\}$ is an edge of Γ with $t \in I_\delta$ and $t \notin I_\theta$, and by Conditions (i), (ii) of Definition 1.1 it follows that either $\delta = t\theta$ or $\delta \leqslant \theta$. Moreover, since $P_{y,\theta,sw,\gamma} \neq 0$ the inductive hypothesis yields that $C_{y,\theta} \leqslant C_{sw,\gamma}$. If $\delta \leqslant \theta$ then $C_{y,\delta} \leqslant C_{y,\theta}$, and so $C_{y,\delta} \leqslant C_{sw,\gamma} \leqslant C_{w,\gamma}$. If $\delta = t\theta$ then $C_{y,\delta} \leqslant C_{w,\gamma}$ follows from $C_{y,\theta} \leqslant C_{sw,\gamma}$, in view of (iii) above.

It remains to show that Λ is an ordered W-graph in the sense of Definition 1.1.

Let $C_{y,\delta}$, $C_{w,\gamma} \in \Lambda$ with $\mu(y,\delta,w,\gamma) \neq 0$. If $y = w$ then $\mu(y,\delta,w,\gamma) = \mu(\delta,\gamma)$, and since Γ is an ordered W_J-graph it follows that γ and δ are comparable, whence so are (w,γ) and $(w,\delta) = (y,\delta)$. On the other hand, if $y \neq w$ then $\mu(y,\delta,w,\gamma)$ is a coefficient of one or other of the polynomials $P_{y,\delta,w,\gamma}$ and $P_{w,\gamma,y,\delta}$, and so (iv) above implies that (w,γ) and (y,δ) are comparable. So Condition (i) of Definition 1.1 holds.

Let $s \in S$ and $(w,\gamma) \in \Lambda^+_s$, and suppose that $(y,\delta) \in \Lambda^-_s$ with $C_{w,\gamma} < C_{y,\delta}$ and $\mu(y,\delta,w,\gamma) \neq 0$. We must show that (y,δ) is the unique such element of Λ^-_s.

Suppose first that the edge $\{C_{y,\delta}, C_{w,\gamma}\}$ is transverse. Since $s \in I(y,\delta)$ and $s \notin I(w,\gamma)$, it follows from Proposition 2.1 that $\ell(w) \not\leqslant \ell(y)$, and so $(y,\delta) \prec (w,\gamma)$. But this implies that $P_{y,\delta,w,\gamma} \neq 0$, and in view of (iv) this contradicts the assumption that $C_{w,\gamma} < C_{y,\delta}$. So $\{C_{y,\delta}, C_{w,\gamma}\}$ is either vertical or horizontal.

If the edge $\{C_{y,\delta}, C_{w,\gamma}\}$ is vertical then $\delta = \gamma$ and $y = rw$ for some $r \in S$. Since $C_{w,\gamma} < C_{y,\gamma}$ we have $w \leqslant y$; so $\ell(w) \leqslant \ell(rw)$. Now since $s \in I(rw,\gamma)$ and $s \notin I(w,\gamma)$ it follows readily that $r = s$. So $(y,\delta) = (sw,\gamma)$; moreover, this case can only arise if $w \in D^+_{J,s}$.

Now suppose that $\{C_{y,\delta}, C_{w,\gamma}\}$ is horizontal, so that $y = w$ and $\{\delta,\gamma\}$ is an edge of Γ. Since Γ is an ordered W_J-graph, Condition (i) of Definition 1.1 yields that either $\gamma < \delta$ or $\delta < \gamma$; however, the latter alternative would give $C_{w,\delta} < C_{w,\gamma}$, contradicting our assumption that $C_{w,\gamma} < C_{y,\delta} = C_{w,\delta}$. Now since $s \in I(w,\delta)$ and $s \notin I(w,\gamma)$ we see that $w \in D^0_{J,s}$, and $t = w^{-1}sw$ is in I_δ and not in I_γ. Since Γ satisfies Condition (ii) of Definition 1.1 it follows that $\delta = t\gamma$.

We have shown that

$$(y,\delta) = \begin{cases} (sw,\gamma) & \text{if } w \in D^+_{J,s} \\ (w,t\gamma) & \text{if } w \in D^0_{J,s} \end{cases}$$

where $t = w^{-1}sw$. So (y,δ) is uniquely determined. In accordance with Definition 1.1, we write $C_{y,\delta} = sC_{w,\gamma}$.

It remains to check that Λ satisfies Condition (iii) of Definition 1.1; that is, we must show that if $(w,\gamma) \in \Lambda^+_s$ and $C_{y,\delta} = sC_{w,\gamma}$ then $\mu(y,\delta,s,\gamma) = 1$. If $w \in D^0_{J,s}$ with $w^{-1}sw = t$ then $sC_{w,\gamma}$ is defined if and only if $t\gamma$ is defined, in which case $sC_{w,\gamma} = C_{w,t\gamma}$. Moreover, in this case we have that $\mu(w,t\gamma,w,\gamma) = \mu(t\gamma,\gamma) = 1$, since Γ satisfies Condition (iii) of Definition 1.1. On the other hand, if $w \in D^+_{J,s}$ then $s(w,\gamma) = (sw,\gamma)$, and the desired conclusion that $\mu(sw,\gamma,w,\gamma) = 1$ follows from Theorem 1.4. □

3. Inducing bipartite W-graphs

Definition 3.1. A W-graph is called bipartite if its vertex set Γ is the disjoint union of nonempty sets Γ_1, Γ_2 such that $\mu(\delta, \gamma) = 0$ whenever $\delta, \gamma \in \Gamma_1$ or $\delta, \gamma \in \Gamma_2$.

We assume that a W_J-graph Γ is bipartite and let Γ_1, Γ_2 be the two parts of the vertex set. Then the vertex set of the induced W-graph Λ, namely

$$\{(w, \gamma) \mid \gamma \in \Gamma, w \in D_J\},$$

is the disjoint union of the following two sets:

$$\Lambda_1 = \{(w, \gamma) \mid \ell(w) \text{ is even and } \gamma \in \Gamma_1 \text{ or } \ell(w) \text{ is odd and } \gamma \in \Gamma_2\};$$
$$\Lambda_2 = \{(w, \gamma) \mid \ell(w) \text{ is even and } \gamma \in \Gamma_2 \text{ or } \ell(w) \text{ is odd and } \gamma \in \Gamma_1\}.$$

Proposition 3.2. *Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ is bipartite as above. Then*

(i) *if δ, γ are in the same part Γ_i of Γ and $\ell(w) - \ell(y)$ is even, or δ, γ are in different Γ_i and $\ell(w) - \ell(y)$ is odd, then the polynomial $P_{y,\delta,w,\gamma}$ involves only even powers of q.*

(ii) *if δ, γ are in different parts of Γ and $\ell(w) - \ell(y)$ is even, or δ, γ are in the same part and $\ell(w) - \ell(y)$ is odd, then the polynomial $P_{y,\delta,w,\gamma}$ involves only odd powers of q.*

Proof. Use induction on $\ell(w)$. If $\ell(w) = 0$, it follows from (i) and (ii) of Theorem 1.3. So assume that $\ell(w) > 0$ and let $w = sv$ where $s \in S$ and $\ell(v) = \ell(w) - 1$.

Suppose first that δ, γ are in the same part of Γ and $\ell(w) - \ell(y)$ is even, which is one of the cases in Part (i). The inductive hypothesis immediately implies that the terms on the right hand side of Eq. (2) involve only even powers of q, with the possible exception of the terms $\mu(\delta, \theta)P_{y,\theta,v,\gamma}$ in the sum that appears in the third case (when $y \in D^0_{J,s}$ and $\delta \in \Gamma_t{}^-$). But if $\mu(\delta, \theta) \neq 0$ then θ and δ must be in different parts of Γ, which also implies that θ, γ are in different parts of Γ; so $P_{y,\theta,v,\gamma}$ (where $\ell(v) - \ell(y)$ is odd) involves only even powers of q by the inductive hypothesis. Hence $P'_{y,\delta,w,\gamma}$ involves only even powers of q.

Let us consider the powers of q in $P''_{y,\delta,w,\gamma}$. The nonzero terms in Eq. (3) correspond to quadruples (z, θ, v, γ) such that $P_{z,\theta,v,\gamma}$ has a nonzero coefficient of q (since this coefficient is $-\mu(z, \theta, v, \gamma)$). Hence, by the inductive hypothesis, $P_{z,\theta,v,\gamma}$ involves only odd powers of q. There are now two possible cases.

(1) If $\ell(v) - \ell(z)$ is even, then θ, γ must in different parts of Γ; so θ, δ are in different parts of Γ and

$$\ell(z) - \ell(y) = (\ell(w) - \ell(y)) - (\ell(v) - \ell(z)) - 1$$

is odd. So $P_{y,\delta,z,\theta}$ involves only even powers of q, by the inductive hypothesis.

(2) If $\ell(v) - \ell(z)$ is odd, then θ, γ must be in the same part of Γ; so θ, δ are in the same part of Γ and $\ell(z) - \ell(y)$ is even. So again $P_{y,\delta,z,\theta}$ involves only even powers of q, by the inductive hypothesis.

Hence $P''_{y,\delta,w,\gamma}$, like $P'_{y,\delta,w,\gamma}$, involves only even powers of q.

The other three cases are all very similar to the first case; we omit the details. □

As an immediate consequence of Proposition 3.2 we have the following result.

Theorem 3.3. *Assume that W_J-graph Γ is bipartite. Then the induced W-graph Λ is bipartite.*

4. Inducing cells

Let $(w,\gamma)\in D_J\times\Gamma$, and let $s\in S$. If $(w,\gamma)\in\Lambda_s^-$ then $T_sC_{w,\gamma}=-q^{-1}C_{w,\gamma}$, and so

$$-q^{-1}\sum_{\substack{y\in D_J\\ \delta\in\Gamma}}P_{y,\delta,w,\gamma}T_y\delta=\sum_{\substack{y\in D_J\\ \delta\in\Gamma}}P_{y,\delta,w,\gamma}T_sT_y\delta. \tag{5}$$

We also have

$$T_sT_y\delta=\begin{cases}T_{sy}\delta & \text{if } y\in D_{J,s}^+\\ T_{sy}\delta+(q-q^{-1})T_y\delta & \text{if } y\in D_{J,s}^-\\ -q^{-1}T_y\delta & \text{if } y\in D_{J,s}^0 \text{ and } \delta\in\Gamma_t^-\\ qT_y\delta+\sum_{\theta\in\Gamma_t^-}\mu(\theta,\delta)T_y\theta & \text{if } y\in D_{J,s}^0 \text{ and } \delta\in\Gamma_t^+\end{cases}$$

where $t=y^{-1}sy$. Substituting this into Eq. (5) and equating coefficients yields a proof of the following result.

Proposition 4.1. *Let $s\in S$ and $(w,\gamma)\in\Lambda_s^-$. If $y\in D_{J,s}^0$ and $\delta\in\Gamma_t^+$, where $t=y^{-1}sy$, then $P_{y,\delta,w,\gamma}=0$. If $y\in D_{J,s}^+$ then $P_{y,\delta,w,\gamma}=-qP_{sy,\delta,w,\gamma}$ for all $\delta\in\Gamma$.*

Note that this simplifies our original inductive formulas for the polynomials $P_{y,\delta,w,\gamma}$. In particular, in the situation of Eq. (3) we have that $P''(y,\delta,w,\gamma)=0$ when $y\in D_{J,s}^0$ and $\delta\in\Gamma_t^+$.

Let $\leq_\Gamma$ be the preorder on Γ defined in [6] by the rule that $\delta\leq_\Gamma\gamma$ if and only if there exists a finite sequence $\delta=\gamma_0,\gamma_1,\ldots,\gamma_k=\gamma$ of elements of Γ with $\mu(\gamma_{i-1},\gamma_i)\neq 0$ and $I(\gamma_{i-1})\nsubseteq I(\gamma_i)$ for all $i\in\{1,2,\ldots,k\}$.

Proposition 4.2. *Let y, $w\in D_J$ and δ, $\gamma\in\Gamma$ with $\delta\not\leq_\Gamma\gamma$. Then $P_{y,\delta,w,\gamma}=0$.*

Proof. Use induction on $\ell(w)$. Since $\delta\neq\gamma$ the case $\ell(w)=0$ follows from (i) and (ii) of Theorem 1.3. So assume that $\ell(w)>0$, and let $w=sv$ where $s\in S$ and $\ell(v)=\ell(w)-1$.

The inductive hypothesis immediately implies that the terms on the right hand side of Eq. (2) are zero, with the possible exception of the terms $\mu(\delta,\theta)P_{y,\theta,v,\gamma}$ in the sum that appears in the third case (when $y\in D_{J,s}^0$ and $\delta\in\Gamma_t^-$). In all of these terms we have that $I_\delta\nsubseteq I_\theta$, since $t\in I_\delta$ and $t\notin I_\theta$. So either $\delta\leq_\Gamma\theta$ or else $\mu(\delta,\theta)=0$. By the inductive hypothesis, either $\theta\leq_\Gamma\gamma$ or else $P_{y,\theta,v,\gamma}=0$. But since $\delta\not\leq_\Gamma\gamma$ we cannot have both $\delta\leq_\Gamma\theta$ and $\theta\leq_\Gamma\gamma$; so either $\mu(\delta,\theta)=0$ or $P_{y,\theta,v,\gamma}=0$. So all the terms $\mu(\delta,\theta)P_{y,\theta,v,\gamma}$ are zero, and so $P'_{y,\delta,w,\gamma}=0$.

All the elements z appearing on the right hand side of Eq. (3) satisfy $\ell(z)\leqslant\ell(v)$, and so the inductive hypothesis tells us that if $\delta\not\leq_\Gamma\theta$ then $P_{y,\delta,z,\theta}=0$. Furthermore, if $\theta\not\leq_\Gamma\gamma$ then $P_{z,\theta,v,\gamma}=0$, and so $\mu(z,\theta,v,\gamma)=0$. Since $\delta\not\leq_\Gamma\gamma$ we must have either $\theta\not\leq_\Gamma\gamma$ or $\delta\not\leq_\Gamma\theta$, and so all the terms $\mu(z,\theta,v,\gamma)P_{y,\delta,z,\theta}$ are zero. So $P''_{y,\delta,w,\gamma}=0$, and hence $P_{y,\delta,w,\gamma}=0$, as required. □

Suppose now that $C_{z,\theta}$ and $C_{w,\gamma}$ vertices of Λ that are adjacent and satisfy $I(z,\theta)\nsubseteq I(w,\gamma)$. If $w=z$ then $s\in I(w,\theta)$ and $s\notin I(w,\gamma)$ forces $sw=wt$ for some $t\in I_\theta$ with $t\notin I_\gamma$. So in this case θ and γ are adjacent vertices of Γ with $I_\theta\nsubseteq I_\gamma$. In particular, $\theta\leq_\Gamma\gamma$. The same conclusion holds trivially if the edge $\{C_{z,\theta},C_{w,\gamma}\}$ is vertical, since in this case $\theta=\gamma$. If the edge is transverse then by Proposition 2.1 we deduce that $\ell(z)<\ell(w)$, and so we must have $(z,\theta)\prec(w,\gamma)$. Thus $P_{z,\theta,w,\gamma}\neq 0$, and so $\theta\leq_\Gamma\gamma$ by Proposition 4.2.

Let $\leq_\Lambda$ be the preorder relation on the W-graph Λ generated by the requirement that $C_{z,\theta} \leq_\Lambda C_{w,\gamma}$ whenever $C_{z,\theta}$ and $C_{w,\gamma}$ are adjacent and $I(z,\theta) \nsubseteq I(w,\gamma)$. The above calculations have proved the following theorem.

Theorem 4.3. *If $C_{z,\theta}$ and $C_{w,\gamma}$ are vertices of Λ with $C_{z,\theta} \leq_\Lambda C_{w,\gamma}$ then $\theta \leq_\Gamma \gamma$.*

Recall from [6] that vertices $\theta, \gamma \in \Gamma$ lie in the same cell of Γ if and only if $\theta \leq_\Gamma \gamma$ and $\gamma \leq_\Gamma \theta$. Similarly, $C_{z,\theta}$ and $C_{w,\gamma}$ are in the same cell of Λ if and only if $C_{z,\theta} \leq_\Lambda C_{w,\gamma}$ and $C_{w,\gamma} \leq_\Lambda C_{z,\theta}$. Theorem 4.3 shows that if Δ is a cell in Γ then the set $\{\, C_{w,\gamma} \mid w \in D_J \text{ and } \gamma \in \Delta \,\}$ is a union of cells in Λ. In the case that Γ is the Kazhdan-Lusztig W_J-graph for the regular representation, this result (and Theorem 4.3) have been proved by Meinolf Geck [4].

5. W_K-cells in induced W-graphs

Let J, $K \subseteq S$, and let ρ be a representation of W_J. Inducing to W and then restricting to W_K yields a representation $\operatorname{Res}^W_{W_K}(\operatorname{Ind}^W_{W_J}(\rho))$, and by Mackey's formula (see [8, 44.2]) we have

$$\operatorname{Res}^W_{W_K}(\operatorname{Ind}^W_{W_J}(\rho)) \cong \sum_d \operatorname{Ind}^{W_K}_{W_K \cap dW_Jd^{-1}}(\operatorname{Res}^{dW_Jd^{-1}}_{W_K \cap dW_Jd^{-1}}(d\rho)) \tag{6}$$

where d runs through a set of representatives of the $W_K \backslash W / W_J$ double cosets, and $d\rho$ is the representation of dW_Jd^{-1} defined by

$$(d\rho)x = \rho(d^{-1}xd)$$

for all $x \in dW_Jd^{-1}$. Our aim is to describe a W-graph version of Eq. (6).

If J, $K \subseteq S$ we define $D_K^{-1} = \{\, x^{-1} \mid x \in D_K \,\}$ and $D_{KJ} = D_K^{-1} \cap D_J$. It is well known that every $W_K \backslash W / W_J$ double coset contains a unique element $d \in D_{KJ}$, and every $w \in W_K d W_J$ can be expressed in the form $w = udt$ with $u \in W_K$ and $t \in W_J$, and $\ell(w) = \ell(u) + \ell(d) + \ell(t)$.

The following result is proved in [7, Theorem 2.7.4].

Proposition 5.1 (Kilmoyer)**.** *Let K and J be subsets of S. Then each $W_K \backslash W / W_J$ double coset contains a unique element of D_{KJ}. Moreover, whenever $d \in D_{KJ}$ we have $W_K \cap dW_Jd^{-1} = W_L$, where $L = K \cap dJd^{-1}$.*

Note that, as a consequence of Proposition 5.1, if $d \in D_{KJ}$ then the isomorphism $z \mapsto d^{-1}zd$ from $W_K \cap dW_Jd^{-1}$ to $d^{-1}W_Kd \cap W_J$ preserves lengths of elements.

Definition 5.2. Whenever $L \subseteq K \subseteq S$ we define $D_L^K = W_K \cap D_L$, the set of minimal length coset representatives for W_K/W_L.

Let J, $K \subseteq S$ and $w \in W$, and let $d \in W_K w W_J \cap D_{KJ}$. Suppose that $u \in W_K$ is such that $ud \in D_J$. Writing $L = K \cap dJd^{-1}$, we can express u in the form $u'v$ with $u' \in D_L^K$ and $v \in W_L$ and then we have

$$ud = u'vd = u'dv'$$

where $v' = d^{-1}vd \in d^{-1}W_Kd \cap W_J$ and

$$\ell(ud) = \ell(u) + \ell(d) = \ell(u') + \ell(v) + \ell(d) = \ell(u') + \ell(d) + \ell(v').$$

Since $ud \in D_J$ and $v' \in W_J$ this forces $\ell(v') = 0$. We conclude that

$$D_J = \{\, ud \mid d \in D_{KJ} \text{ and } u \in D^K_{K \cap dJd^{-1}} \,\}. \tag{7}$$

Returning now to W-graphs, we start with a trivial observation.

Proposition 5.3. *Any W-graph becomes a W_L-graph if the elements of $S\backslash L$ are ignored.*

We write $\mathrm{Res}^S_L(\Gamma)$ for the W_L-graph obtained in this way. Of course, the W_L-module obtained from $\mathrm{Res}^S_L(\Gamma)$ is simply the restriction of the W-module obtained from Γ.

Now let Γ be a W_J-graph and $\Lambda = \mathrm{Ind}^S_J(\Gamma)$ the induced W-graph, constructed as in Section 1. The vertex set of Λ can be identified with the set

$$D_J \times \Gamma = \{ (x, \gamma) \mid x \in D_J,\ \gamma \in \Gamma \},$$

which is in one to one correspondence with

$$\{ (u, d, \gamma) \mid u \in D^K_{K\cap dJd^{-1}},\ d \in D_{KJ},\ \gamma \in \Gamma \}.$$

Consider a fixed d, and put $L = K \cap dJd^{-1}$. The vertices (ud, γ) of $\mathrm{Res}^S_K(\Lambda)$, as $u \in D^K_L$ and $\gamma \in \Gamma$ vary, span a subgraph of $\mathrm{Res}^S_K(\Lambda)$, which we refer to as the *d-subgraph* of $\mathrm{Res}^S_K(\Lambda)$. We shall show that the d-subgraph of $\mathrm{Res}^S_K(\Lambda)$ is a W_K-graph.

Because $d^{-1}Ld \subseteq J$ and the the isomorphism $z \mapsto dzd^{-1}$ from $W_{d^{-1}Ld}$ to W_L is length preserving, the $W_{d^{-1}Ld}$-graph $\mathrm{Res}^J_{d^{-1}Ld}$ immediately gives rise to a W_L-graph, which, for brevity, we refer to as $d\Gamma$. We write the vertices of $d\Gamma$ as pairs $d\gamma$, where γ varies over vertices of Γ. The descent set of $d\gamma \in d\Gamma$ is

$$I_{d\gamma} = \{ dsd^{-1} \mid s \in I_\gamma \subseteq J \text{ and } dsd^{-1} \in K \} \subseteq K \cap dJd^{-1},$$

and the edges and edge weights of $d\Gamma$ correspond exactly to those of Γ:

$$\mu(d\gamma, d\gamma') = \mu(\gamma, \gamma')$$

for all γ, $\gamma' \in \Gamma$. The vertex set of the induced W_K-graph $\mathrm{Ind}^K_L(d\Gamma)$ is

$$\{ (u, d\gamma) \mid u \in D^K_L \text{ and } \gamma \in \Gamma \},$$

which is in obvious one to one correspondence with the vertex set of the d-subgraph of $\mathrm{Res}^S_K(\Lambda)$. We shall show that these graphs are actually isomorphic.

Lemma 5.4. *The descent set of the vertex $(u, d\gamma)$ of $\mathrm{Ind}^K_L(d\Gamma)$ equals the descent set of the vertex (ud, γ) of $\mathrm{Res}^S_K(\Lambda)$.*

Proof. The descent set of $(u, d\gamma)$ consists of the $s \in K$ such that either $\ell(su) < \ell(u)$ or $u^{-1}su \in I_{d\gamma}$, and the descent set of (ud, γ) consists of the $s \in K$ such that either $\ell(sud) < \ell(ud)$ or $(ud)^{-1}s(ud) \in I_\gamma$. It is clear from the fact that $ud \in D_J$ that $\ell(su) < \ell(u)$ if and only if $\ell(sud) < \ell(ud)$. Moreover, the definition of $d\Gamma$ gives $I_{d\gamma} = d(J \cap I_\gamma)d^{-1}$; so $u^{-1}su \in I_{d\gamma}$ immediately implies that $(ud)^{-1}s(ud) \in I_\gamma$. On the other hand, since $u \in W_K$ and $I_\gamma \subseteq J$, if $(ud)^{-1}s(ud) = s' \in I_\gamma$ then $ds'd^{-1} = u^{-1}su \in W_K \cap dW_Jd^{-1} = W_L$, whence $\ell(ds'd^{-1}) = \ell(s') = 1$, giving $u^{-1}su \in L \cap dI_\gamma d^{-1} = d(J \cap I_\gamma)d^{-1}$. □

The following result shows that the edges and edge weights of $\mathrm{Ind}^K_L(d\Gamma)$ and the d-subgraph of $\mathrm{Res}^S_K(\Lambda)$ also agree.

Lemma 5.5. *Let J, K, d, L, $\Gamma, d\Gamma$ be as in the discussion above. Let $P_{y,\delta,w,\gamma}$ (for y, $w \in D_J$ and $\delta, \gamma \in \Gamma$) be the polynomials appearing in the construction of $\mathrm{Ind}^S_J(\Gamma)$, and let $P^K_{y,d\delta,w,d\gamma}$ (for y, $w \in D^K_L$ and $d\delta$, $d\gamma \in d\Gamma$) be the corresponding polynomials in the construction of $\mathrm{Ind}^K_L(d\Gamma)$. Then $P^K_{y,d\delta,w,d\gamma} = P_{yd,\delta,wd,\gamma}$, for all y, $w \in D^K_L$ and γ, $\delta \in \Gamma$.*

Note that Eq. (7) above shows that yd, $wd \in D_J$, as necessary for the statement to make sense.

The proof of Lemma 5.4 is a straightforward induction on $\ell(w)$. Since $yd \geqslant d$, Theorem 1.3 gives

$$P^K_{y,d\delta,1,d\gamma} = P_{yd,\delta,d,\gamma} = \begin{cases} 1 & \text{if } (y, d\delta) = (1, d\gamma) \\ 0 & \text{otherwise,} \end{cases}$$

which starts the induction. Turning to the inductive step, let $w \in D^K_L$ with $\ell(w) \geqslant 1$, and write $w = sv$ with $\ell(v) < \ell(w)$. Note that $s \in K$, since $w \in W_K$. Now for all $y \in D^K_L$ we see that $y^{-1}sy \in L$ if and only if $(yd)^{-1}s(yd) \in J$, and it follows readily that the three cases $y \in D^+_{L,s}$, $y \in D^-_{L,s}$, $y \in D^0_{L,s}$ correspond to the three cases $yd \in D^+_{J,s}$, $yd \in D^-_{J,s}$, $yd \in D^0_{J,s}$. When $y \in D^0_{L,s}$ we write $t = y^{-1}sy$; note that (for any $\delta \in \Gamma$) we have $d\delta \in (d\Gamma)^+_t$ if and only if $\delta \in \Gamma^+_{d^{-1}td}$. Following the terminology of Eq. (2), we call the cases $y \in D^+_{L,s}$, $y \in D^-_{L,s}$, $y \in D^0_{J,s}$ with $d\delta \in (d\Gamma)^-_t$ and $y \in D^0_{J,s}$ with $d\delta \in (d\Gamma)^+_t$ respectively cases (a), (b), (c) and (d). Then, as in Eq. (9),

$$P^{K\prime}_{y,d\delta,w,d\gamma} = \begin{cases} P^{K\prime}_{sy,d\delta,v,d\gamma} - qP^{K\prime}_{y,d\delta,v,d\gamma} & \text{(case (a)),} \\ P^{K\prime}_{sy,d\delta,v,d\gamma} - q^{-1}P^{K\prime}_{y,d\delta,v,d\gamma} & \text{(case (b)),} \\ (-q-q^{-1})P^{K\prime}_{y,d\delta,v,d\gamma} + \sum\limits_{d\theta \in d\Gamma^+_t} \mu(d\delta, d\theta)P^{K\prime}_{y,d\theta,v,d\gamma} & \text{(case (c)),} \\ 0 & \text{(case (d)).} \end{cases}$$

Since for all the terms in the sum in case (c) we have $\mu(d\delta, d\theta) = \mu(\delta, \theta)$, with θ running through all elements of $\Gamma^+_{d^{-1}td}$ as $d\theta$ runs through all elements of $(d\Gamma)^+_t$, it follows from the inductive hypothesis that the right hand side above equals the corresponding formula for $P'_{yd,\delta,wd,\gamma}$ obtained from Eq. (2). Thus $P^{K\prime}_{y,d\delta,w,d\gamma} = P'_{yd,\delta,wd,\gamma}$.

In a similar fashion, it follows from Eq. (3) that $P^{K\prime\prime}_{y,d\delta,w,d\gamma} = P''_{yd,\delta,wd,\gamma}$. The point is that $(zd, \theta) \in \Lambda^-_s$ if and only if $zd \in D^-_{J,s}$ or $zd \in D^0_{J,s}$ with $(zd)^{-1}s(zd) \in I_\theta$, and this corresponds to $(z, d\theta) \in (\mathrm{Ind}^K_L)^-_s$; moreover, the inductive hypothesis gives $\nu(zd, \theta, vd, \gamma) = \nu^K(z, d\theta, v, d\gamma)$ and $(zd, \theta) \prec (vd, \gamma)$ if and only if $(z, d\theta) \prec (v, d\gamma)$. So it follows that $P^K_{y,d\delta,w,d\gamma} = P_{yd,\delta,wd,\gamma}$, as required.

Since the coefficients of the polynomials $P^K_{y,d\delta,w,d\gamma}$ and $P_{yd,\delta,wd,\gamma}$ determine the edges and edge weights of $\mathrm{Ind}^K_L(d\Gamma)$ and the d-subgraph of $\mathrm{Res}^S_K(\Lambda)$, Lemmas 5.4 and 5.5 combine to show that these graphs are isomorphic. In particular, the d-subgraph of $\mathrm{Res}^S_K(\Lambda)$ is a W_K-graph.

In view of Eq. (7) we see that the vertex set of $\mathrm{Res}_K(\Lambda)$ is the disjoint union of the vertex sets of its d-subgraphs, as d runs through all elements of D_{KJ}. It seems reasonable to expect, therefore, that each d-subgraph is a union of W_K-cells of $\mathrm{Res}_K(\Lambda)$. To prove this, we make use of the following result.

Lemma 5.6 (Deodhar[2, Lemma 3.5])**.** *Let $d \in D^{-1}_K$ and $w \in W$, and write $w = ue$ with $e \in D^{-1}_K$ and $u \in W_K$. Then $d \leqslant w$ if and only if $d \leqslant e$.*

As above, let Γ be a W_J-graph and $\Lambda = \mathrm{Ind}^S_J(\Gamma)$, and consider the W_K-graph $\mathrm{Res}^S_K(\Lambda)$. Let d, e be distinct elements of D_{KJ} with $\ell(d) \geqslant \ell(e)$. We show that if there is an edge of $\mathrm{Res}^S_K(\Lambda)$ joining a vertex α of the d-subgraph and a vertex β of the e-subgraph then $e \leqslant d$, and the descent set of α is a subset of the descent set of β.

We write $\alpha = (ud, \gamma)$ and $\beta = (ve, \delta)$, where $u \in D^K_{K\cap dJd^{-1}}$ and $v \in D^K_{K\cap eJe^{-1}}$, and γ, $\delta \in \Gamma$. Note that since $d \neq e$ the edge joining α and β is not horizontal. Suppose first that it is transverse. Then either $ud \leqslant ve$ or $ve \leqslant ud$. But the former alternative would give $d \leqslant ve$ and hence $d \leqslant e$ by Lemma 5.6, contradicting our assumptions that $d \neq e$ and $\ell(e) \leqslant \ell(d)$. So we must have $ve \leqslant ud$, and, by the same argument, $e \leqslant d$. Moreover, $I(ud, \gamma) \subseteq I(ve, \delta)$, by Proposition 2.1, and so $I(ud, \gamma) \cap K$, which is the descent set of α in $\mathrm{Res}^S_K(\Lambda)$, is a subset of $I(ve, \delta) \cap K$, the descent set of β.

We now consider the case that $\{\alpha, \beta\}$ is vertical, which means that $\delta = \gamma$ and $ud = sve$ for some $s \in S$. We either have $ud \leqslant ve$ or $ve \leqslant ud$, depending on whether $\ell(sve) = \ell(ve) - 1$ or $\ell(sve) = \ell(ve) + 1$. As in the last paragraph, the former alternative gives $d \leqslant e$, contradicting our hypotheses. So $ve \leqslant ud$, and $e \leqslant d$.

Suppose, for a contradiction, that $I(ud, \gamma) \cap K \nsubseteq I(ve, \delta) \cap K$, so that there exists an $r \in K$ with $r \in I(ud, \gamma)$ and $r \notin I(ve, \gamma)$. Observe first that $r \neq s$, since otherwise we would have

$$W_K dW_J = W_K udW_J = W_K rudW_J = W_K veW_J = W_K eW_J,$$

contradicting the assumption that d and e are distinct elements of D_{KJ}. Now $\ell(rve) > \ell(ve)$, since $r \notin I(ve, \delta)$. Since also $\ell(sve) > \ell(ve)$, it follows that $\ell(rsve) = \ell(ve) + 2$; that is, $\ell(rud) = \ell(ud) + 1$. Since $r \in I(ud, \gamma)$ this forces $rud = udt$ for some $t \in I_\gamma \subseteq J$. Now udt must be the longest element in $W_{\{r,s\}}udt$, since $\ell(rudt) = \ell(du) < \ell(dut)$ and

$$\ell(sdut) = \ell(vet) = \ell(ve) + 1 = \ell(du) < \ell(dut).$$

Moreover, $ve = (sr)(udt)$ is the minimal length element in $W_{\{r,s\}}udt$ since, as noted above, $\ell(rve) > \ell(ve)$ and $\ell(sve) > \ell(ve)$. Thus sr is the longest element of $W_{\{r,s\}}$, and it follows that $rs = sr$. Thus $rve = rsud = srud = sudt = vet$, and since $t \in I_\gamma$ this shows that $r \in I(ve, \gamma)$, contradicting our assumptions.

Proposition 5.7. *Let J, $K \subseteq S$ and let Γ be a W_J-graph. For each $d \in D_{KJ}$, the d-subgraph of $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$ is a union of cells.*

Proof. Let α be a vertex in the d-subgraph. We must prove that any vertex β that is in the same cell of $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$ as α is also in the d-subgraph. Recall that the vertex set of $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$ is the disjoint union of the vertex sets of its e-subgraphs, as e runs through D_{KJ}; so β must lie in the e-subgraph for some $e \in D_{KJ}$.

Since α and β are in the same cell we have that $\alpha \leq \beta$ and $\beta \leq \alpha$, where $\leq$ is the Kazhdan-Lusztig preorder on $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$. So there exists a sequence of vertices $\alpha = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = \beta$ with α_{i-1} and α_i adjacent and $I(\alpha_{i-1}) \nsubseteq I(\alpha_i)$ for $1 \leqslant i \leqslant n$, and another such sequence $\beta = \beta_0, \beta_1, \beta_2, \ldots, \beta_m = \alpha$ with β_{j-1} and β_j adjacent and $I(\beta_{j-1}) \nsubseteq I(\beta_j)$ for $1 \leqslant j \leqslant m$.

Let α_i lie in the d_i-subgraph and β_i in the e_j-subgraph, where d_i, $e_j \in D_{KJ}$ (for all $i \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, m\}$). Since α_{i-1} and α_i are adjacent and $I(\alpha_{i-1}) \nsubseteq I(\alpha_i)$ the argument preceding this proposition shows that either $d_{i-1} = d_i$ or $\ell(d_{i-1}) < \ell(d_i)$. So $\ell(d_i) \geqslant \ell(d_{i-1})$, and $d_{i-1} \leqslant d_i$ in the Bruhat order. Thus it follows that $d = d_0 \leqslant e = d_n$. But the same reasoning applied to the sequence of β_j's gives $e \leqslant d$. Hence $e = d$, as required. □

We give an example to illustrate the distribution of W_K-cells in $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$. Let W be the Weyl group of type D_4, with generators r, s, t and u, where r, s, u correspond to the end nodes of the Coxeter graph. Let $J = \{r, s, t\}$ (of type A_3) and Γ the W_J-graph consisting of two vertices γ, δ such that $I_\gamma = \{r, s\}$, $I_\delta = \{t\}$ and $\mu(\delta, \gamma) = \mu(\gamma, \delta) = 1$. Then $D_J = \{1, u, tu, rtu, stu, rstu, trstu, utrstu\}$. Let $K = \{r, t, u\}$. Then there are two $W_K\backslash W/W_J$ double cosets, with shortest elements $d_1 = 1$ and $d_2 = stu$. We find that $K \cap d_1 J {d_1}^{-1} = \{r, t\}$ and $K \cap d_2 J {d_2}^{-1} = \{u, t\}$; so we have $D^K_{K\cap d_1 J d_1} = \{1, u, tu, rtu\}$ and $D^K_{K\cap d_2 J {d_2}^{-1}} = \{1, r, tr, utr\}$. The vertex set of the d_1-subgraph of $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$ is

$$\{(1,\gamma), (u,\gamma), (tu,\gamma), (rtu,\gamma), (1,\delta), (u,\delta), (tu,\delta), (rtu,\delta)\}$$

and the vertex set of the d_2-subgraph is

$$\{(stu,\gamma), (rstu,\gamma), (trstu,\gamma), (utrstu,\gamma), \\ (stu,\delta), (rstu,\delta), (trstu,\delta), (utrstu,\delta)\}.$$

The diagram below shows $\mathrm{Ind}^S_J(\Gamma)$ (on the left) and $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$ (obtained by removing s from all the descent sets of $\mathrm{Ind}^S_J(\Gamma)$). The circles denote vertices of the graphs, and the generators written inside a circle comprise the descent set of the vertex. All edge weights are 1.

The W-graph $\mathrm{Ind}^S_J(\Gamma)$ has two cells of size 3, namely

$$\{(1,\gamma), (1,\delta), (u,\delta)\}$$

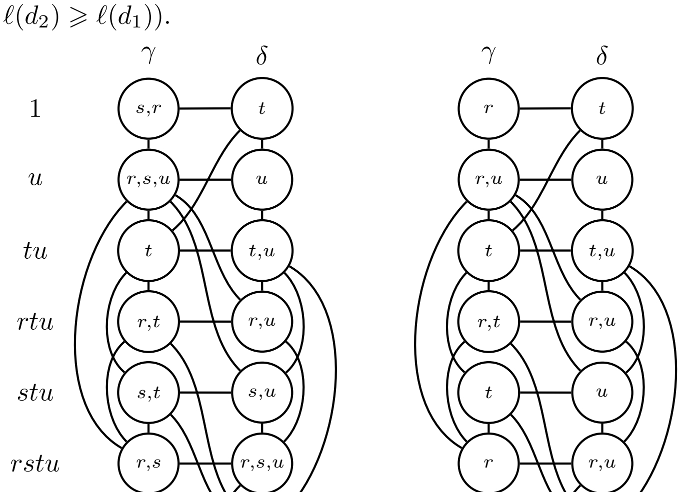
and

$$\{(trstu,\gamma), (utrstu,\gamma), (utrstu,\delta)\},$$

with the remaining 10 vertices constituting a third cell. There are six cells in $\mathrm{Res}^S_K(\mathrm{Ind}^S_J(\Gamma))$, as follows:

$$\{(1,\gamma), (1,\delta), (u,\delta)\},$$
$$\{(u,\gamma), (tu,\gamma)\},$$
$$\{(tu,\delta), (rtu,\gamma), (rtu,\delta)\},$$
$$\{(stu,\gamma), (stu,\delta), (rstu,\gamma)\},$$
$$\{(rstu,\delta), (trstu,\delta)\},$$
$$\{(trstu,\gamma), (utrstu,\gamma), (utrstu,\delta)\}.$$

The first three of these are in the d_1-subgraph, the other three in the d_2-subgraph. Observe that for every edge joining a vertex α of the d_1-subgraph and a vertex β of the d_2-subgraph we have $I(\beta) \subseteq I(\alpha)$, in accordance with the results proved above (since $\ell(d_2) \geqslant \ell(d_1)$).



6. Connection with Kazhdan-Lusztig polynomials

The results of the preceding sections can be applied with $J = \phi$ (so that $W_J = \{1\}$, the trivial subgroup of W) and Γ the trivial W_J-graph consisting of a single vertex (and no edges). In this case $\mathscr{H}_J \simeq \mathcal{A}$ and the $\mathscr{H}_J$-module $\mathcal{A}\Gamma$ is simply a 1-dimensional $\mathcal{A}$-module. Note also that $D_J = W$.

Theorem 6.1. *The algebra $\mathscr{H}$ has a unique basis $\{\, C_w \mid w \in W \,\}$ such that $\overline{C_w} = C_w$ for all w and $C_w = \sum_{y\in W} p_{y,w} T_y$ for some elements $p_{y,w} \in \mathcal{A}^+$ with the following properties::*

(i) $p_{y,w} = 0$ *if* $y \nleqslant w$;
(ii) $p_{w,w} = 1$;
(iii) $p_{y,w}$ *has zero constant term if* $y \neq w$.

The polynomials $p_{y,w}$ are related to the polynomials $P_{y,w}$ of [6] (the genuine Kazhdan-Lusztig polynomials) by $p_{y,w}(q) = (-q)^{\ell(w)-\ell(y)}\overline{P_{y,w}(q^2)}$. That is, to get $p_{y,w}$ from $P_{y,w}$ replace q by q^2, apply the bar involution, and then multiply by $(-q)^{\ell(w)-\ell(y)}$. The quantity $\mu(y,w)$, which is the coefficient of $q^{\frac{1}{2}(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$, is the coefficient of q in $(-1)^{\ell(w)-\ell(y)}p_{y,w}$. However, since Kazhdan and Lusztig show that $\mu_{y,w}$ is nonzero only when $\ell(w) - \ell(y)$ is odd, $\mu(y,w)$ is the coefficient of q in $-p_{y,w}$.

The elements C_w form a W-graph basis for $\mathscr{H}$, and Eq. (2.3a) of [6] (or Theorem 1.4 above) shows the W-graph is ordered, in the sense of Definition 1.1, relative to the Bruhat order on W.

Applying Theorem 6.1 with W replaced by W_J yields a W_J-graph basis for the regular representation of $\mathscr{H}_J$. The representation of $\mathscr{H}$ obtained by inducing the regular representation of $\mathscr{H}_J$ is, of course, the regular representation of $\mathscr{H}$. Applying our procedure for inducing W-graphs yields a W-graph basis for $\mathscr{H}$ consisting of elements $C_{w,\gamma}$ (for $w \in D_J$ and $\gamma \in W_J$) such that $\overline{C_{w,\gamma}} = C_{w,\gamma}$ and

$$C_{w,\gamma} = \sum_{y\in D_J}\sum_{\delta\in W_J} P_{y,\delta,w,\gamma} T_y C_\delta, \tag{8}$$

where the polynomials $P_{y,\delta,w,\gamma}$ satisfy the conditions given in Theorem (1.3). By Proposition 2.2 there is a partial order on the set $\Lambda = \{\, C_{w,\gamma} \mid w \in D_J,\ \gamma \in W_J \,\}$ such that for all y, $w \in D_J$ and δ, $\gamma \in W_J$,

(i) if $y \leqslant w$ and $\delta \leqslant \gamma$ then $C_{y,\delta} \leqslant C_{w,\gamma}$,
(ii) if $C_{y,\delta} \leqslant C_{w,\gamma}$ and if y, $w \in D^+_{J,s}$ for some $s \in S$, then $C_{sy,\delta} \leqslant C_{sw,\gamma}$,
(iii) if $C_{y,\delta} \leqslant C_{w,\gamma}$ with $w \in D^+_{J,s}$ and $y \in D^0_{J,s}$ for some $s \in S$, and if also $t\delta > \delta$ where $t = y^{-1}sy$, then $C_{y,t\delta} \leqslant C_{sw,\gamma}$.

Furthermore, the partial order on Λ is defined to be the minimal partial order satisfying these three properties.

Note that Λ is in bijective correspondence with W via $C_{w,\gamma} \leftrightarrow w\gamma$.

Proposition 6.2. *The above partial order on Λ corresponds exactly the Bruhat order on W, in the sense that $C_{y,\delta} \leqslant C_{w,\gamma}$ if and only if $y\delta \leqslant w\gamma$ in W.*

Proof. Let us check first that the Bruhat order on W does satisfy the properties (i), (ii) and (iii) above. With regard to (i), it is certainly true that $y \leqslant w$ and $\delta \leqslant \gamma$ implies that $y\delta \leqslant w\gamma$. Turning to (ii), suppose that y, $w \in D^+_{J,s}$ and δ, $\gamma \in W_J$

with $y\delta \leqslant w\gamma$. Since $w < sw \in D_J$ we see that

$$\ell(sw\gamma) = \ell(sw) + \ell(\gamma) = 1 + \ell(w) + \ell(\gamma) = 1 + \ell(w\gamma),$$

and $\ell(sy\delta) = 1 + \ell(y\delta)$ similarly. So $sy\delta \leqslant sw\gamma$, by Deodhar [2, Theorem 1.1]. For (iii), suppose that $w \in D^+_{J,s}$ and $y \in D^0_{J,s}$, and let $\delta, \gamma \in W_J$ with $y\delta \leqslant w\gamma$. Suppose also that $t\delta > \delta$, where $t = y^{-1}sy \in J$. Then

$$\ell(sy\delta) = \ell(yt\delta) = \ell(y) + \ell(t\delta) = 1 + \ell(y) + \ell(\delta) = 1 + \ell(y\delta),$$

and since also $\ell(sw\gamma) = 1 + \ell(w\gamma)$ as above, Deodhar [2, Theorem 1.1] again gives the desired conclusion that $yt\delta = sy\delta \leqslant sw\gamma$.

Since the partial order on Λ is generated by the properties (i), (ii) and (iii), and since also the Bruhat order on W satisfies the same properties, it follows that $C_{y,\delta} \leqslant C_{w,\gamma}$ implies that $y\delta \leqslant w\gamma$ for all $y, w \in D_J$ and $\delta, \gamma \in W_J$.

We must show, conversely, that $y\delta \leqslant w\gamma$ implies that $C_{y,\delta} \leqslant C_{w,\gamma}$. In view of statement IV in [2, Theorem 1.1] it is sufficient to do this when $\ell(w\gamma) = \ell(y\delta) + 1$. Making this assumption, we argue by induction on $\ell(w)$. Observe that if $\ell(w) = 0$ then $w\gamma = \gamma \in W_J$, and since $y\delta \leqslant w\gamma$ it follows that $y\delta \in W_J$. Hence $y = 1$, and $C_{y,\delta} \leqslant C_{w,\gamma}$ by Property (i). So suppose that $\ell(w) > 0$, and choose $s \in S$ with $sw < w$.

Consider first the possibility that $sy\delta > y\delta$. Then we must in fact have $sy\delta = w\gamma$, since, using the terminology of [2, Theorem 1.1], Property $Z(s, sy\delta, w\gamma)$ implies that $sy\delta \leqslant w\gamma$. So either $sy = w$ and $\delta = \gamma$, in which case $C_{y,\delta} \leqslant C_{w,\gamma}$ by Property (i), or else $y = w$ and $\gamma = t\delta$, where $t = y^{-1}sy \in J$, and again Property (i) gives $C_{y,\delta} \leqslant C_{w,\gamma}$.

The only alternative is that $sy\delta < y\delta$, and in this case we have that $sy\delta \leqslant sw\gamma$ (by $Z(s, y\delta, w\gamma)$, in Deodhar's terminology). If $y \in D^-_{J,s}$ then the inductive hypothesis yields that $C_{sy,\delta} \leqslant C_{sw,\gamma}$, and Property (ii) gives $C_{y,\delta} \leqslant C_{w,\gamma}$. Since $y \in D^+_{J,s}$ is not possible given $sy\delta < y\delta$, it remains to deal with the case $y \in D^0_{J,s}$. Writing $t = y^{-1}sy$ we have $sy\delta = yt\delta \leqslant sw\gamma$, and the inductive hypothesis gives $C_{y,t\delta} \leqslant C_{sw,\gamma}$. Note that here $t\delta < \delta$ and $sw \in D^+_{J,s}$; so applying Property (iii) we obtain the desired conclusion that $C_{y,\delta} \leqslant C_{w,\gamma}$. □

Equation (8) and Theorem 6.1 give $C_\delta = \sum_{\theta \in W_J} p_{\theta,\delta} T_\theta$, and we deduce that

$$C_{w,\gamma} = \sum_{y \in D_J} \sum_{\delta,\theta \in W_J} P_{y,\delta,w,\gamma} p_{\theta,\delta} T_{y\theta},$$

since $T_y T_\theta = T_{y\theta}$ for all $y \in D_J$ and $\theta \in W_J$. The coefficient of $T_{y\theta}$ in this expression is $\sum_{\delta \in W_J} P_{y,\delta,w,\gamma} p_{\theta,\delta}$, and for this to be nonzero there must exist a $\delta \in W_J$ such that $P_{y,\delta,w,\gamma}$ and $p_{\theta,\delta}$ are both nonzero. Now $p_{\theta,\delta} \neq 0$ implies that $\theta \leqslant \delta$ by Theorem 6.1, and $P_{y,\delta,w,\gamma} \neq 0$ gives $y\delta \leqslant w\gamma$, by Propositions 2.2 and 6.2. These combine to give $y\theta \leqslant y\delta \leqslant w\gamma$. So if the coefficient of $T_{y\theta}$ in $C_{w,\gamma}$ is nonzero then $y\theta \leqslant w\gamma$. Furthermore, the coefficient is a polynomial in q whose constant term is nonzero only if there exists a $\delta \in W_J$ such that $P_{y,\delta,w,\gamma}$ and $p_{\theta,\delta}$ both have nonzero constant terms. This only occurs when $(y, \delta) = (w, \gamma)$ and $\theta = \delta$; that is, the constant term is nonzero only if $y\theta = w\gamma$. Hence by the uniqueness assertion in Theorem 1.3 we deduce that $C_{w,\gamma} = C_{w\gamma}$, and

$$p_{y\theta,w\gamma} = \sum_{\delta \in W_J} P_{y,\delta,w,\gamma} p_{\theta,\delta} \tag{9}$$

for all y, $w \in D_J$ and θ, $\gamma \in W_J$.

Since the elements $C_{w,\gamma}$ produced by our construction coincide with the elements $C_{w\gamma}$ of the Kazhdan-Lusztig construction, the W-graph data of our construction must also agree with Kazhdan-Lusztig. So if $y\theta \leqslant w\gamma$ then $\mu(y\theta, w\gamma)$, the coefficient of q in $-p_{y\theta,w\gamma}$, must equal the element $\mu(y,\theta,w,\gamma)$ of our construction. That is, if $y < w$ then $\mu(y\theta, w\gamma)$ equals the coefficient of q in $-P_{y,\theta,w,\gamma}$, while if $y = w$ then it equals $\mu(\theta,\gamma)$, which is the coefficient of q in $-p_{\theta,\gamma}$. Eq. (9) above confirms this.

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