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AUTOMORPHISMS OF COXETER GROUPS OF RANK THREE

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ABSTRACT. If W is an infinite rank 3 Coxeter group, whose Coxeter diagram has no infinite bonds, then the automorphism group of W is generated by the inner automorphisms and any automorphisms induced from automorphisms of the Coxeter diagram. Indeed $\mathrm{Aut}(W)$ is the semi-direct product of $\mathrm{Inn}(W)$ and the group of graph automorphisms.

1. PRELIMINARIES

The main aim of this paper is to prove the following result.

Theorem 1. *If W is an infinite, rank 3, Coxeter group, whose Coxeter diagram has no infinite bonds, then the automorphism group of W is the semi-direct product of $\mathrm{Inn}(W)$ and the group of graph automorphisms.*

We recall that a Coxeter group is a group with a presentation of the form

$$W = \mathrm{gp}\langle \{ r_a \mid a \in \Pi \} \mid (r_a r_b)^{m_{ab}} = 1 \text{ for all } a, b \in \Pi \rangle$$

where Π is some indexing set, whose cardinality is called the *rank* of W, and the parameters m_{ab} satisfy the following conditions: $m_{ab} = m_{ba}$, each m_{ab} lies in the set $\{ m \in \mathbb{Z} \mid m \geq 1 \} \cup \{\infty\}$ and $m_{ab} = 1$ if and only if $a = b$. In this paper we shall usually restrict ourselves to finite values of m_{ab}.

If $w \in W$ then we define $l(w)$ to be the length of the shortest expression for w as a product of generators r_a ($a \in \Pi$).

The (*Coxeter*) *diagram* of W is a graph with vertex set Π in which an edge (or *bond*) labelled m_{ab} joins $a, b \in \Pi$ whenever $m_{ab} \geq 3$. We say that the group is *irreducible* if this graph is connected.

Let V be a real vector space with basis Π, and define a bilinear form B on V by

$$B(a, b) = -\cos(\pi/m_{ab})$$

for all $a, b \in \Pi$. For each $a \in V$ such that $B(a, a) = 1$ we define $\sigma_a : V \to V$ by $\sigma_a v = v - 2B(a, v)a$; it is well known (see, for example, Corollary 5.4 of [5]) that W has a faithful representation on V given by $r_a \mapsto \sigma_a$ for all $a \in \Pi$. We shall identify elements of W with their images in this representation; thus $r_a = \sigma_a$ is the reflection in the hyperplane perpendicular to a. The action of W on V preserves the form B.

The elements of the basis Π are called *simple roots*, and the reflections r_a for $a \in \Pi$ are called *simple reflections*. We call $\Phi = \{ wa \mid w \in W,\ a \in \Pi \}$ the *root system* of W.

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The following lemma collects together some facts which will be useful later.

Lemma 2. *Given the above representation of the Coxeter group W, the following are true.*

(1) *If $v \in \Phi$ and $v = \sum_{a\in\Pi} \lambda_a a$ then either $\lambda_a \geq 0$ for all $a \in \Pi$ or $\lambda_a \leq 0$ for all $a \in \Pi$. In the former case we call v a* positive root*, in the latter case a* negative root*, and we define Φ^+ and Φ^- to be the set of all positive roots and the set of all negative roots respectively.*
(2) *If $w \in W$ is a reflection then $w = r_\alpha$ for some $\alpha \in \Phi$. Furthermore, $\alpha = xa$ for some $x \in W$ and $a \in \Pi$, whence $w = xr_a x^{-1}$ is conjugate to a simple reflection.*
(3) *For all $w \in W$ define $N(w) = \{\, v \in \Phi^+ \mid wv \in \Phi^- \,\}$. Then $l(w) = |N(w)|$; in particular, $N(w)$ is a finite set.*
(4) *W is a finite group if and only if the bilinear form B is positive definite.*
(5) *Φ is finite if and only if W is finite.*

Proof. Section 5.4 in [5] contains a proof of 1, while 2 and 3 appear as Proposition 1.14 and Corollary 1.7 respectively. Theorem 4.1 in [4] includes both 4 and 5. □

For each $I \subseteq \Pi$ we define $W_I = \mathrm{gp}\langle \{\, r_a \mid a \in I \,\} \rangle$; these subgroups are called the *standard parabolic subgroups* of W. Clearly W_I preserves the subspace V_I spanned by I; furthermore it acts on this subspace as a Coxeter group with root system $\Phi_I = \Phi \cap V_I$. A *parabolic subgroup* of W is any subgroup of the form $wW_I w^{-1}$ for some $w \in W$ and $I \subseteq \Pi$.

To save space in our later calculations we shall write $\mathrm{s}(\theta)$ for $\sin\theta$ and $\mathrm{c}(\theta)$ for $\cos\theta$. We also use π_k for π/k (for any positive integer k) and $u \cdot v$ for $B(u, v)$. It is readily checked that if $I = \{a, b\}$ is a two-element subset of Π then Φ_I consists of all vectors v of the form

$$\frac{\mathrm{s}((h-1)\pi_m)}{\mathrm{s}(\pi_m)}a + \frac{\mathrm{s}\left(h\pi_m\right)}{\mathrm{s}(\pi_m)}b, \tag{1}$$

where $h \in \mathbb{Z}$. Observe that $v \cdot a = -\,\mathrm{c}\left(h\pi_m\right)$ and $v \cdot b = \mathrm{c}((h-1)\pi_m)$. Replacing h by $m - h + 1$ gives the equivalent formula

$$\frac{\mathrm{s}\left(h\pi_m\right)}{\mathrm{s}(\pi_m)}a + \frac{\mathrm{s}((h-1)\pi_m)}{\mathrm{s}(\pi_m)}b, \tag{2}$$

where now $v \cdot a = \mathrm{c}((h-1)\pi_m)$ and $v \cdot b = -\,\mathrm{c}\left(h\pi_m\right)$. The positive roots in Φ_I are the vectors of the form (1) or (2) with $1 \leq h \leq m$. We see that positive numbers appearing as coefficients of a or b in roots $v \in \Phi^+_{\{a,b\}}$ are never less than 1. This result in fact extends to the entire root system.

Lemma 3 (Brink [1])**.** *Suppose that $v = \sum_{a\in\Pi} \lambda_a a \in \Phi^+$. For each $a \in \Pi$, if $\lambda_a > 0$ then $\lambda_a \geq 1$.*

(The proof of this proceeds by induction on the length of a shortest element $w \in W$ such that $v = wb$ for some $b \in \Pi$, the point being that if $w = w'r_c$ with $l(w') = l(w) - 1$ then $v = w'(b + \lambda c)$ for some $\lambda \geq 1$, and the inductive hypothesis applies to $w'b$ and $w'c$.)

For convenience we record the following trivial fact.

Lemma 4. *Let u, v be unit vectors in a Euclidean plane with $u \cdot v = \mathrm{c}(\theta) \neq \pm 1$. Then the unique x in the plane such that $x \cdot u = \lambda$ and $x \cdot v = \mu$ is given by*

$$x = \frac{\lambda - \mathrm{c}(\theta)\mu}{\mathrm{s}(\theta)^2} u + \frac{\mu - \mathrm{c}(\theta)\lambda}{\mathrm{s}(\theta)^2} v.$$

2. Finite subgroups

Suppose that W is any Coxeter group with Π as the set of simple roots. We shall make use of the following result, which is due to Tits and appears in [2], Exercise 2d, p. 130.

Lemma 5. *If W is a Coxeter group and $H \leq W$ is finite then H is contained in a finite parabolic subgroup of W.*

One immediate consequence of Lemma 5 is that every maximal finite subgroup of a Coxeter group is parabolic.

The following can be found as Theorem 2.7.4 of [3].

Lemma 6 (Kilmoyer)**.** *Let $I, J \subseteq \Pi$ and suppose that $d \in W$ is the minimal length element of $W_I d W_J$. Then $W_I \cap dW_J d^{-1} = W_K$, where $K = I \cap dJ$.*

Corollary 7. *Let I, $J \subset \Pi$ and $t \in W$. Then for some $\alpha \in W_I$,*

$$W_I \cap tW_J t^{-1} = \alpha W_K \alpha^{-1}$$

where $K = I \cap dJ \subseteq \Pi$, with d the minimal length element of $W_I t W_J$.

Proof. We can write $t = \alpha d \beta$ where $\alpha \in W_I$ and $\beta \in W_J$, and d is the minimal length element in $W_I t W_J$. By Lemma 6

$$W_I \cap tW_J t^{-1} = \alpha(W_I \cap dW_J d^{-1})\alpha^{-1} = \alpha W_K \alpha^{-1}. \quad \square$$

Suppose that $I \subseteq \Pi$ is such that $|W_I|$ is finite and W_J is infinite for all J with $I \subsetneqq J \subseteq \Pi$. Then W_I is a *maximal finite standard parabolic subgroup* of W.

Lemma 8. *If W is any infinite Coxeter group, then all maximal finite standard parabolic subgroups of W are maximal finite subgroups of W.*

Proof. Suppose that W_I is a maximal finite standard parabolic subgroup but is not a maximal finite subgroup. By Lemma 5, we can find $K \subset \Pi$ and $t \in W$ such that $W_I \subsetneqq tW_K t^{-1}$, where W_K is finite. Without loss we may assume that t has minimal length in tW_K. Then, by Corollary 7,

$$W_I = W_I \cap tW_K t^{-1} = \alpha W_{I \cap tK} \alpha^{-1},$$

for some $\alpha \in W_I$. Conjugating by α^{-1} shows that $W_I = W_{I \cap tK}$ and therefore $I = I \cap tK \subset tK$, and hence $t^{-1} I \subset K$. By hypothesis W_I is not properly contained in a finite standard parabolic subgroup; so we know that $t \neq 1$, and thus we can find a simple root e such that $t^{-1} e = f$ is negative. As t has minimal length in tW_K it takes positive roots in the root system of W_K to positive roots. But $-f$ is a positive root while $t(-f) = -e$ is negative, and so we can conclude that f is not in the root system of W_K. Thus when $f = t^{-1} e$ is expressed as a linear combination of simple roots, some $g \notin K$ appears with a negative coefficient. It follows that if h is any positive root in the root system of $W_{I \cup \{e\}}$ which is not in the root system of W_I, then $t^{-1} h$ involves g with a negative coefficient. But $W_{I \cup \{e\}}$ is infinite, while W_I is not. So t^{-1} takes an infinite number of positive roots to negative roots, and hence has infinite length, which is a contradiction. $\square$

3. Automorphisms

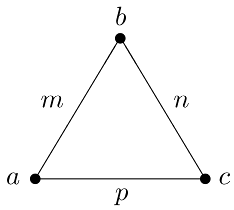
Our objective is to describe the automorphism group of W. Clearly symmetries of the Coxeter diagram give rise to automorphisms which permute the simple reflections, we call these *graph automorphisms*. They clearly form a group. We begin by showing that graph automorphisms are always outer, given only that W is infinite and irreducible. The remainder of the paper is devoted to showing that these give the full outer automorphism group if W is a rank three infinite Coxeter group.

Lemma 9. *If W is any infinite irreducible Coxeter group then the only graph automorphism which is inner is the identity.*

Proof. Suppose that conjugation by $w \in W$ is a graph automorphism. We now let $J = \{\, \alpha \in \Pi \mid w\alpha \in \Phi^- \,\}$. If $J = \emptyset$ then $N(w) = \emptyset$, whence $l(w) = 0$ by Lemma 2, giving $w = \mathrm{id}$. If $J = \Pi$ then $w\Phi^+ = \Phi^-$ and $l(w) = |\Phi^+|$. But this is impossible since Φ is infinite (by Lemma 2). Thus $\emptyset \subsetneqq J \subsetneqq \Pi$; so both J and $\Pi\backslash J$ are non-empty.

As conjugation by w is a graph automorphism, for each $\alpha \in \Pi$ there is an $\alpha' \in \Pi$ such that $wr_\alpha w^{-1} = r_{\alpha'}$. This gives $w\alpha = \pm\alpha'$. It follows that $wJ = -K$ for some $K \subset \Pi$ and $w(\Pi\backslash J) = \Pi\backslash K$. Now let $\alpha \in J$ and $\beta \in \Pi\backslash J$. Then $-w\alpha = \alpha' \in K$ and $w\beta = \beta' \in \Pi\backslash K$, and by the definition of B it follows that $B(\alpha, \beta) \le 0$ and $B(\alpha', \beta') \le 0$. But $B(\alpha, \beta) = B(w\alpha, w\beta) = -B(\alpha', \beta')$, and so we conclude that $B(\alpha, \beta) = 0$. Since this holds for all $\alpha \in J$ and $\beta \in \Pi\backslash J$, and the two sets are non-empty, this contradicts the irreducibility of W. □

For the remainder let W be an infinite Coxeter group, with the following diagram:



where $2 \le m, n, p < \infty$. The group associated with this diagram is finite if (m, n, p) is $(m, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$ or $(5, 3, 2)$ (permuting m, n and p if necessary); so these cases do not arise. Initially we shall ignore the cases $(m, n, p) = (6, 3, 2)$, $(4, 4, 2)$ or $(3, 3, 3)$, where the form B is degenerate. It is readily verified that the discriminant of B is negative in all other cases. When mnp is even we choose the labelling so that m is even and $n \ge p$, while if mnp is odd we choose it so that $m \ge n \ge p$. Then label the simple roots a, b and c as indicated in the diagram.

In view of Lemma 8 we have the following fact.

Lemma 10. *The maximal finite subgroups of W have the form tW_It^{-1} where $I = \{a, b\}$, $\{a, c\}$ or $\{b, c\}$ and $t \in W$.*

Suppose ϕ is an automorphism of the rank three infinite Coxeter group W. The parabolic subgroup $\langle r_a, r_b \rangle$ is a maximal finite subgroup of W, and therefore $\phi\langle r_a, r_b \rangle$ is also a maximal finite subgroup; hence it follows that $\phi\langle r_a, r_b \rangle$ is conjugate to a standard parabolic subgroup. So, modifying ϕ by an inner automorphism if necessary, we may assume that $\phi\langle r_a, r_b \rangle = \langle r_x, r_y \rangle$ for some $x, y \in \Pi$. Furthermore, $\phi\langle r_a, r_c \rangle = t\langle r_\alpha, r_\beta \rangle t^{-1}$ for some $\alpha, \beta \in \Pi$ and $t \in W$, and since $\langle r_a \rangle = \langle r_a, r_b \rangle \cap \langle r_a, r_c \rangle$, it follows that $\phi\langle r_a \rangle = \langle r_x, r_y \rangle \cap t\langle r_\alpha, r_\beta \rangle t^{-1}$ is a parabolic subgroup of W of order 2. Thus $\phi(r_a)$ is a reflection. The same holds for r_b and r_c. This, together with Lemma 2 part 2, proves the following lemma.

Lemma 11. *All automorphisms of W map reflections to reflections.*

Without loss of generality we may assume that $\phi\langle r_a, r_b\rangle = \langle r_x, r_y\rangle$, where $x, y \in \Pi$. If $\{x, y\} = \{a, c\}$ then $m = p$ and we have a graph automorphism fixing r_a and interchanging r_b and r_c; so, up to inner and graph automorphisms, ϕ fixes $\langle r_a, r_b\rangle$. If $\{x, y\} = \{b, c\}$ we can use a similar argument. Thus we may assume that $\phi\langle r_a, r_b\rangle = \langle r_a, r_b\rangle$.

Now $\phi(r_a)$ is a reflection in $\langle r_a, r_b\rangle$; so $\phi(r_a) = tr_at^{-1}$ or tr_bt^{-1} for some $t \in \langle r_a, r_b\rangle$. As conjugating by t^{-1} fixes $\langle r_a, r_b\rangle$, we may assume that, up to inner and graph automorphisms,

$$\phi\langle r_a, r_b\rangle = \langle r_a, r_b\rangle$$
$$\phi(r_a) = r_a \text{ or } r_b.$$

If m is odd then r_a and r_b are conjugate in $\langle r_a, r_b\rangle$; moreover, if $n = p$ then we have a graph automorphism that interchanges r_a and r_b. So we only need consider the possibility that $\phi(r_a) = r_b$ when m is even and $n > p$, remembering that we have labelled the graph so that $n \geq p$.

Since $\phi\langle r_a, r_b\rangle = \langle r_a, r_b\rangle$, we know that $\phi(r_b)$ is the reflection corresponding to some root in $\Phi_{\{a,b\}}$. We can choose this root to be positive, since a root and its negative correspond to the same reflection. Similarly, $\phi(r_c)$ is the reflection corresponding to some positive root. Thus the two cases we must consider are as follows:

(A) $\phi(r_a) = r_a$, $\phi(r_b) = r_{b'}$ and $\phi(r_c) = r_x$, where $x \in \Phi^+$, and $b' \in \Phi^+_{\{a,b\}}$ has the form (1) for some h such that $1 \leq h \leq m-1$ (since $b' \neq a$);

(B) $\phi(r_a) = r_b$, $\phi(r_b) = r_{a'}$ and $\phi(r_c) = r_{x'}$, where $x' \in \Phi^+$, and $a' \in \Phi^+_{\{a,b\}}$ has the form (2) for some h such that $1 \leq h \leq m-1$.

In Case (A), since r_ar_x and $r_{b'}r_x$ must have the same orders as r_ar_c and r_br_c respectively, we must have $a \cdot x = -\operatorname{c}(i\pi_p)$ and $b' \cdot x = -\operatorname{c}(j\pi_n)$ for some integers i and j, while in Case (B) we similarly obtain $b \cdot x' = -\operatorname{c}(i\pi_p)$ and $a' \cdot x' = -\operatorname{c}(j\pi_n)$. We may choose i and j so that $1 \leq i \leq p-1$ and $1 \leq j \leq n-1$. We also have $a \cdot b' = -\operatorname{c}(h\pi_m)$ (Case (A)) and $b \cdot a' = -\operatorname{c}(h\pi_m)$ (Case (B)).

Noting that the restriction of B to the subspace $V_{\{a,b\}}$ is positive definite, let $\operatorname{proj} : V \to V_{\{a,b\}}$ be the orthogonal projection, and define $x_\perp = c - \operatorname{proj}(c)$, $x_0 = \operatorname{proj}(x)$ and $x'_0 = \operatorname{proj}(x')$. By Lemma 4

$$\operatorname{proj}(c) = -\frac{\operatorname{c}(\pi_m)\operatorname{c}(\pi_n) + \operatorname{c}(\pi_p)}{\operatorname{s}(\pi_m)^2}a - \frac{\operatorname{c}(\pi_m)\operatorname{c}(\pi_p) + \operatorname{c}(\pi_n)}{\operatorname{s}(\pi_m)^2}b$$

and so

$$\begin{aligned} x_\perp &= \frac{\operatorname{c}(\pi_m)\operatorname{c}(\pi_n) + \operatorname{c}(\pi_p)}{\operatorname{s}(\pi_m)^2}a + \frac{\operatorname{c}(\pi_m)\operatorname{c}(\pi_p) + \operatorname{c}(\pi_n)}{\operatorname{s}(\pi_m)^2}b + c \\ &= \delta a + \varepsilon b + c, \end{aligned}$$

while

$$\begin{aligned} x_0 &= -\frac{\operatorname{c}(j\pi_n)\operatorname{c}(h\pi_m) + \operatorname{c}(i\pi_p)}{\operatorname{s}(h\pi_m)^2}a - \frac{\operatorname{c}(i\pi_p)\operatorname{c}(h\pi_m) + \operatorname{c}(j\pi_n)}{\operatorname{s}(h\pi_m)^2}b' \\ &= -\lambda' a - \mu' b' \end{aligned}$$

and

$$\begin{aligned} x_0' &= -\frac{\mathrm{c}(i\pi_p)\,\mathrm{c}(h\pi_m) + \mathrm{c}(j\pi_n)}{\mathrm{s}(h\pi_m)^2} a' - \frac{\mathrm{c}(j\pi_n)\,\mathrm{c}(h\pi_m) + \mathrm{c}(i\pi_p)}{\mathrm{s}(h\pi_m)^2} b \\ &= -\mu' a' - \lambda' b. \end{aligned}$$

The formulas (1) and (2) yield $x_0 = \lambda a + \mu b$ (in Case (A)) and $x_0' = \mu a + \lambda b$ (in Case (B)) for certain scalars λ and μ that we do not need to evaluate.

Lemma 12. *If $\varepsilon = \delta$ then $n = p$.*

Proof. This is clear, since $\varepsilon - \delta = \frac{(\mathrm{c}(\pi_n) - \mathrm{c}(\pi_p))(1 - \mathrm{c}(\pi_m))}{\mathrm{s}(\pi_m)^2}$. □

As B is non-degenerate and $x - x_0$ is perpendicular to both a and b we can see that $x - x_0$ must be a multiple of $x_\perp$. Thus we may write $x = x_0 + \omega x_\perp$ for some real number ω. If $\omega = 0$ then $x = x_0$ is a linear combination of a and b, and so lies in $\Phi \cap V_{\{a,b\}} = \Phi_{\{a,b\}}$. But since this gives $r_x \in \langle r_a, r_b \rangle$, contradicting the surjectivity of ϕ, we conclude that $\omega \neq 0$. Now since x is a root it has length 1, and so $1 = x \cdot x = (x_0 + \omega x_\perp) \cdot (x_0 + \omega x_\perp) = x_0 \cdot x_0 + \omega^2 x_\perp \cdot c$, which gives

$$\omega^2 = \frac{1 - x_0 \cdot x_0}{x_\perp \cdot c}$$

since $x_\perp \cdot c \neq 0$ by the nondegeneracy of B. A similar argument can be applied to $x' = x_0' + \omega x_\perp$. An inspection of the expressions for x_0 and x_0' in terms of a and b shows that $x_0 \cdot x_0 = x_0' \cdot x_0'$ and so we obtain the same expression for ω^2 in each case. Since $x_0 \cdot a = x \cdot a$ and $x_0 \cdot b' = x \cdot b'$, we find that

$$\begin{aligned} \omega^2 &= \frac{1 - x_0 \cdot x_0}{x_\perp \cdot c} \\ &= \frac{1 + x_0 \cdot (\lambda' a + \mu' b')}{x_\perp \cdot c} \\ &= \frac{\mathrm{s}(\pi_m)^2 \left(\mathrm{c}(i\pi_p)^2 + \mathrm{c}(j\pi_n)^2 + 2\,\mathrm{c}(h\pi_m)\,\mathrm{c}(j\pi_n)\,\mathrm{c}(i\pi_p) - \mathrm{s}(h\pi_m)^2\right)}{\mathrm{s}(h\pi_m)^2 \left(\mathrm{c}(\pi_p)^2 + \mathrm{c}(\pi_n)^2 + 2\,\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\,\mathrm{c}(\pi_p) - \mathrm{s}(\pi_m)^2\right)} \\ &= \frac{\mathrm{s}(\pi_m)^2 N}{\mathrm{s}(h\pi_m)^2 D} \end{aligned}$$

where we have $N = -1 + \mathrm{c}(h\pi_m)^2 + \mathrm{c}(i\pi_p)^2 + \mathrm{c}(j\pi_n)^2 + 2\,\mathrm{c}(h\pi_m)\,\mathrm{c}(j\pi_n)\,\mathrm{c}(i\pi_p)$, and $D = -1 + \mathrm{c}(\pi_m)^2 + \mathrm{c}(\pi_p)^2 + \mathrm{c}(\pi_n)^2 + 2\,\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\,\mathrm{c}(\pi_p)$. It can be checked that $-D$ is the discriminant of B (relative to the basis $\{a, b, c\}$), which is negative, as we have noted. Thus $D > 0$, and hence $N > 0$ (since $\omega \neq 0$).

Lemma 13. *With m, n, p, i, j and h as above, we have $\omega^2 \leq 1$, with equality if and only if the following conditions all hold: $h = 1$ or $m - 1$; $j = 1$ or $n - 1$; $i = 1$ or $p - 1$; $\mathrm{c}(h\pi_m)\,\mathrm{c}(j\pi_n)\,\mathrm{c}(i\pi_p) = \mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\,\mathrm{c}(\pi_p)$.*

Proof. Since $1 \leq h \leq m - 1$ we see that $|\mathrm{c}(h\pi_m)| \leq \mathrm{c}(\pi_m)$, equality holding if and only if $h = 1$ or $m - 1$. Similar statements hold for j and i, and since

$$\begin{aligned} D - N = {} & \left(\mathrm{c}(\pi_m)^2 - \mathrm{c}(h\pi_m)^2\right) + \left(\mathrm{c}(\pi_n)^2 - \mathrm{c}(j\pi_n)^2\right) + \left(\mathrm{c}(\pi_p)^2 - \mathrm{c}(i\pi_p)^2\right) \\ & + 2\left(\mathrm{c}(\pi_m)\,\mathrm{c}(\pi_n)\,\mathrm{c}(\pi_p) - \mathrm{c}(h\pi_m)\,\mathrm{c}(j\pi_n)\,\mathrm{c}(i\pi_p)\right). \end{aligned}$$

we conclude that $D \geq N$, with equality if and only if the four conditions in the statement of the lemma hold. Furthermore, $\mathrm{s}(\pi_m)^2 \leq \mathrm{s}(h\pi_m)^2$, with equality if and only if $h = 1$ or $m-1$. Since $\omega^2 = \mathrm{s}(\pi_m)^2 N/\,\mathrm{s}(h\pi_m)^2 D$, the result follows. □

Recall that, in Case (A), $\phi(r_c) = r_x$ for some root $x \in \Phi^+$, and

$$x = x_0 + \omega x_\perp = (\lambda + \omega\delta)a + (\mu + \omega\varepsilon)b + \omega c.$$

In Case (B) we have, similarly,

$$x' = x'_0 + \omega x_\perp = (\mu + \omega\delta)a + (\lambda + \omega\varepsilon)b + \omega c.$$

In either case, it follows from Lemma 3 and Lemma 13 that $\omega = 1$, and the four conditions in the statement of Lemma 13 must hold.

If we are in Case (A) then $h = 1$ gives $b' = b$, while $h = m-1$ gives $b' = r_a b$; if we are in Case (B) then $h = 1$ gives $a' = a$, while $h = m-1$ gives $a' = r_b a$. So we have two possibilities in Case (A) and two in Case (B), as follows:

- (A1) $\phi(r_a) = r_a$, $\phi(r_b) = r_b$ and $\phi(r_c) = r_x$, where $x \in \Phi^+$ satisfies $x \cdot a = \eta c \cdot a$ and $x \cdot b = \eta c \cdot b$, where $\eta = \pm 1$;
- (A2) $\phi(r_a) = r_a$, $\phi(r_b) = r_a r_b r_a$ and $\phi(r_c) = r_x$, where $x \in \Phi^+$ satisfies $x \cdot a = \eta c \cdot a$ and $x \cdot b = -\eta c \cdot b$, where $\eta = \pm 1$;
- (B1) $\phi(r_a) = r_b$, $\phi(r_b) = r_a$ and $\phi(r_c) = r_{x'}$, where $x' \in \Phi^+$ satisfies $x' \cdot b = \eta c \cdot a$ and $x \cdot a = \eta c \cdot b$, where $\eta = \pm 1$;
- (B2) $\phi(r_a) = r_b$, $\phi(r_b) = r_b r_a r_b$ and $\phi(r_c) = r_{x'}$, where $x' \in \Phi^+$ satisfies $x' \cdot b = \eta c \cdot a$ and $x' \cdot a = -\eta c \cdot b$, where $\eta = \pm 1$.

Replacing ϕ by itself followed by conjugation by r_a converts Case (A2) into (A1), and similarly replacing ϕ by itself followed by conjugation by r_b converts Case (B2) into (B1).

Since the form is nondegenerate and a, b linearly independent, the following result is clear.

Lemma 14. *Let α and β be any scalars. There are at most two vectors $v \in V$ such that $v \cdot a = \alpha$, $v \cdot b = \beta$ and $v \cdot v = 1$; furthermore, if $v \in V$ satisfies these equations then so does σv, where $\sigma \colon V \to V$ is the reflection in the plane $V_{\{a,b\}}$.*

Recall that Case (B) arises only if m is even and $n > p$. So assume first of all that m is odd. Obviously $x = \eta c$ satisfies $x \cdot a = \eta c \cdot a$ and $x \cdot b = \eta c \cdot b$. By Lemma 14 it follows that the only values of x that can satisfy the requirements of Case (A1) above are $x = \eta c$ and $x = \sigma(\eta c)$, where $\eta = \pm 1$. Clearly $x = c$ corresponds to the identity automorphism, while $x = -c$ is not a solution since $-c \notin \Phi^+$.

Let $w_0 = (r_a r_b)^{m-1/2} r_a$, the maximal length element of $\langle r_a, r_b \rangle$. Then $\sigma(\eta c)$ is a root if and only if $w_0(\sigma(\eta c))$ is a root. Furthermore, since w_0 fixes $x_\perp$, and satisfies $w_0 a = -b$ and $w_0 b = -a$, while σ fixes a and b and satisfies $\sigma x_\perp = -x_\perp$, we find that

$$\begin{aligned} w_0(\sigma c) = w_0(\sigma(\mathrm{proj}(c) + x_\perp)) &= w_0(\mathrm{proj}(c) - x_\perp) \\ &= w_0(\mathrm{proj}(c)) - x_\perp = \delta b + \varepsilon a - (c + \delta a + \varepsilon b). \end{aligned}$$

This cannot be a root if $\delta \neq \varepsilon$, since the coefficients of a and b have opposite signs in this case. If $\delta = \varepsilon$ then $\sigma c = w_0(-c)$, which is a negative root, while $x = -\sigma c = w_0 c$ satisfies all the requirements of Case (A1). In this case we see that ϕ followed by conjugation by w_0 interchanges r_a and r_b and fixes r_c. Since we have seen in Lemma 12 that $\delta = \varepsilon$ gives $n = p$, this is a graph automorphism of W.

Suppose now that m is even. The reflection σ appearing in Lemma 14 satisfies $\sigma v = -(r_a r_b)^{m/2} v$ for all $v \in V$, since $(r_a r_b)^{m/2}$ acts as a half-turn on $V_{\{a,b\}}$ and fixes $x_\perp$. Thus σ preserves the root system Φ. In Case (A) we again see that $x = c$ yields the identity automorphism, and $x = -c$ is inadmissible since $-c \notin \Phi^+$. Similarly, $x = \sigma c$ is inadmissible, since $\sigma c = -(r_a r_b)^{m/2} c$ is a negative root. The other possibility is $x = -\sigma c$; in this case it is readily verified that ϕ is inner, given by conjugation by $(r_a r_b)^{m/2}$.

Turning finally to Case (B), suppose that m is even and $n > p$. Let τ be the reflection that interchanges a and b and fixes $x_\perp$. By Lemma 14 the only vectors x' satisfying $x' \cdot b = \eta c \cdot a$, $x' \cdot a = \eta c \cdot b$ and $x' \cdot x' = 1$ are $\tau(\eta c)$ and $\sigma(\tau(\eta c))$. Since σ preserves Φ, if we can prove that $\tau c \notin \Phi$ then it will follow that there are no vectors x' satisfying the requirements of Case (B1). But $\tau c = c + \alpha(a - b)$ for some scalar α, with $\alpha = 0$ if and only if $c \cdot (a - b) = 0$. This last condition is not satisfied, since $n > p$. Hence the coefficients of a and b in τc have opposite signs, showing that $\tau c \notin \Phi$, as required. So Case (B) never arises.

Thus we have shown that in each case ϕ is either inner or a product of a graph automorphism and an inner automorphism.

Theorem 15. *If W is an infinite rank 3 Coxeter group with B non-degenerate and no infinite bonds then the automorphism group of W is generated by the inner and graph automorphisms.*

4. The Degenerate Case

Now suppose that W is a rank 3 Coxeter group with B degenerate, so that $W \cong \widetilde{A}_2$, $\widetilde{C}_2$ or $\widetilde{G}_2$, the triples (m, n, p) being, respectively, $(3, 3, 3)$, $(4, 4, 2)$ or $(6, 3, 2)$. For consistency we shall continue to assume that elements of Φ are normalized to have length 1, rather than using the standard root systems in types $\widetilde{C}_2$ and $\widetilde{G}_2$. If ϕ is an arbitrary automorphism of W then we have $\phi(r_a) = r_{a'}$, $\phi(r_b) = r_{b'}$ and $\phi(r_c) = r_{c'}$ for some $a', b', c' \in \Phi$.

Since $r_{a'} r_{b'}$ has the same order as $r_a r_b$ we must have $a' \cdot b' = \mathrm{c}(i\pi_m)$ for some i coprime to m, and since m is 3, 4 or 6 this in fact yields $a' \cdot b' = \varepsilon a \cdot b$ for some $\varepsilon = \pm 1$. Replacing a' by $-a'$ if necessary, we may assume that $a' \cdot b' = a \cdot b$. Modifying ϕ by a graph automorphism if necessary, we may assume that the parabolic subgroup $\langle r_{a'}, r_{b'} \rangle$ is in the same W-conjugacy class as $\langle r_a, r_b \rangle$, and then modifying ϕ by an inner automorphism if necessary, we may assume that $\langle r_{a'}, r_{b'} \rangle = \langle r_a, r_b \rangle$. Now $\{a', b'\}$ and $\{a, b\}$ are two simple systems in $\Phi_{\{a,b\}}$, and hence lie in the same $\langle r_a, r_b \rangle$-orbit. So modifying ϕ by a further inner automorphism we may assume that $\{r_{a'}, r_{b'}\} = \{r_a, r_b\}$.

In type $\widetilde{A}_2$ conjugating by the element $r_a r_b r_a$ interchanges r_a and r_b; so in this case we may assume that $a' = a$ and $b' = b$. In the other two cases r_a and r_b are not conjugate in W, since there are homomorphisms $W \to \{1, -1\}$ with $a \mapsto -1$ and b, $c \mapsto 1$; furthermore, the conjugacy classes containing r_a and r_b are distinguished by group-theoretic criteria: in type $\widetilde{C}_2$ the reflections conjugate to r_a lie in maximal finite subgroups of order 4, whereas those conjugate to r_b do not, and in type $\widetilde{G}_2$ the reflections conjugate to r_b lie in maximal finite subgroups of order 6, whereas those conjugate to r_a do not. So in either case $\phi(r_a) = r_b$ is not possible, and hence we may assume that $a' = a$ and $b' = b$.

Since $r_b r_{c'}$ has the same order as $r_b r_c$, it follows (by the same reasoning used above) that $b \cdot c' = \pm b \cdot c$, and replacing c' by $-c'$ if necessary, we may assume that

$b \cdot c' = b \cdot c$. In type $\widetilde{A}_2$ we similarly deduce that $a \cdot c' = \varepsilon a \cdot c$ for some $\varepsilon = \pm 1$; however, since B is degenerate the matrix

$$\begin{pmatrix} a \cdot a & a \cdot b & a \cdot c' \\ b \cdot a & b \cdot b & b \cdot c' \\ c' \cdot a & c' \cdot b & c' \cdot c' \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{\varepsilon}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{\varepsilon}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

must be singular, and so it follows that $\varepsilon = 1$. In the other two cases we have $a \cdot c' = 0$, since r_a and $r_{c'}$ commute; so we conclude that $a \cdot c' = a \cdot c$ in all cases.

We conclude from the above discussion that $c' - c$ is in the one-dimensional space which is the orthogonal complement of $V_{\{a,b\}}$. Thus $c' = c + \alpha u$ for some scalar α, where the vector u is given by

$$u = \begin{cases} a + b + c & \text{in type } \widetilde{A}_2, \\ a + \sqrt{2}\, b + c & \text{in type } \widetilde{C}_2, \\ \sqrt{3}\, a + 2\, b + c & \text{in type } \widetilde{G}_2. \end{cases}$$

If the scalar α above is zero, then of course ϕ is the identity automorphism. There is only one other possibility for α.

Lemma 16. *In the above notation, if $\alpha \neq 0$ then $\alpha = -2$.*

Proof. Since $\{r_a, r_b, r_{c'}\}$ generates W, it follows that $\{a, b, c'\}$ spans V. So $1 + \alpha$, the coefficient of c in c', is nonzero. Thus, since $c' \in \Phi$, it follows from Lemma 3 that $|1 + \alpha| \geq 1$. Now let $f \colon V \to V$ be the linear transformation defined by $a \mapsto a$, $b \mapsto b$ and $c \mapsto c'$. Then f preserves the bilinear form B, and so $f r_c f^{-1} = r_{fc} = r_{c'} = \phi(r_c)$, and similarly $f r_a f^{-1} = \phi(r_a)$ and $f r_b f^{-1} = \phi(r_b)$. An obvious induction on length yields $f w f^{-1} = \phi(w)$ for all $w \in W$, and so for all $v \in \Phi$ the element $r_{fv} = f r_v f^{-1} = \phi(r_v)$ is a reflection in W. Moreover, since ϕ is bijective, every reflection in W is obtained in this fashion, and it follows that $v \mapsto fv$ maps Φ to Φ bijectively.

The above discussion shows that there is some $v \in \Phi$ such that $fv = c$. Writing $v = \lambda a + \mu b + \nu c$ gives $c = \lambda a + \mu b + \nu(c + \alpha u)$, and equating the coefficients of c gives $\nu(1 + \alpha) = 1$. But $|\nu| \geq 1$ by Lemma 3, and since we established above that $|1 + \alpha| \geq 1$, this shows that $\nu = 1 + \alpha = \pm 1$. If $\alpha \neq 0$, this gives $\alpha = -2$, as claimed. □

It remains to observe that when $\alpha = -2$ the automorphism ϕ is the composite of an inner automorphism and a graph automorphism. In type $\widetilde{A}_2$ we find that if $w = r_a r_b r_a$ then $wa = -b$, $wb = -a$ and $wc = c + 2a + 2b = -(c - 2u)$. Thus conjugation by w followed by the graph automorphism that interchanges r_a and r_b coincides with ϕ. In the other two cases it is equally easy to check that ϕ is given by conjugation by the longest element of $\langle r_a, r_b \rangle$. We have thus established that for all three of these groups the full automorphism group is generated by the graph automorphisms and the inner automorphisms. Combined with Theorem 15, this immediately yields Theorem 1.

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