EFFICIENT COMPUTATION OF STATISTICAL PROPERTIES OF INTERMITTENT DYNAMICS

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ABSTRACT. Intermittent maps of the interval are simple and widely-studied models for chaos with slow mixing rates, but have been notoriously resistant to numerical study. In this paper we present an effective framework to compute many ergodic properties of these systems, in particular invariant measures and mean return times. The framework combines three ingredients that each harness the smooth structure of these systems' induced maps: Abel functions to compute the action of the induced maps, Euler-Maclaurin summation to compute the pointwise action of their transfer operators, and Chebyshev Galerkin discretisations to compute the spectral data of the transfer operators. The combination of these techniques allows one to obtain exponential convergence of estimates for polynomially growing computational outlay, independent of the order of the map's neutral fixed point. This enables effective numerical exploration of intermittent dynamics in all parameter regimes, including in the infinite ergodic regime.

1. INTRODUCTION

Intermittent dynamics, wherein long periods of regular dynamics alternate with bursts of chaotic dynamics, is a feature of many physical systems around a bifurcation between chaotic and regular dynamics, such as in turbulence [16]. The ergodic and statistical behaviour of intermittent dynamics is commonly studied using a prototypical class of so-called Pomeau-Manneville-type maps, which we denote by PM. These maps are defined on the interval $[0, 1]$, with phase spaces that can be divided into a "good" set $[a, 1]$, on which the map is uniformly expanding, and a "bad" set close to an unstable but linearly neutral fixed point at 0:

$$(1) \qquad f(x) = \begin{cases} f_b(x) := xh(x^\alpha)), & x \in [0, a) \\ f_g(x), & x \in [a, 1], \end{cases}$$

where $\alpha > 0$, $f_b' \geq 1$, $h(0) = 1$, $h'(0) > 0$ and $f_g : [a, 1] \to [0, 1]$ is a full-branch expanding Markov map in class U_{NP}, as defined in Appendix A. For simplicity we will assume f_g satisfies an analytic distortion condition (B_δ), also described in Appendix A, and h extends analytically into the complex plane, but our proofs may be appropriately modified to cover the differentiable case.
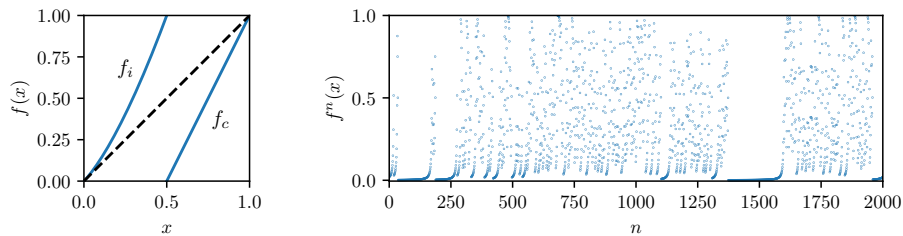

FIGURE 1. Left: Graph of the LSV map (2) for $\alpha = 0.8$. Right: Time series of its dynamics.

A standard example of such a map is the Liverani-Saussol-Vaienti map [13] on the interval $[0,1]$ with $a = \frac{1}{2}$, given by

(2)
$$f(x) = \begin{cases} x(1 + (2x)^\alpha), & x \in [0, \frac{1}{2}) \\ 2x - 1, & x \in [\frac{1}{2}, 1]. \end{cases}$$

This map and its typical dynamics for $\alpha = 0.8$ are shown in Figure 1.

Maps of class PM are endowed with absolutely continuous invariant measures (acims), which are finite for $\alpha \in (0, 1)$, and have summable correlations for $\alpha \in (0, \frac{1}{2})$ [10].

The standard framework for theoretical study of intermittent maps is via the so-called induced map $f^\tau : [a, 1] \to [a, 1]$ (i.e. $f(x)$ iterated $\tau(x)$ times). Here $\tau : [a, 1] \to \mathbb{N}^+$ is the return time to the inducing set, the "good" set $[a, 1]$:

(3)
$$\tau(x) := \inf\{n \in \mathbb{N}^+ : f^n(x) \in [a, 1]\}.$$

The induced map is uniformly expanding (see Figure 2), and it is therefore possible to apply results on uniformly expanding dynamics to it, as well as various numerical methods [18, 5, 4].
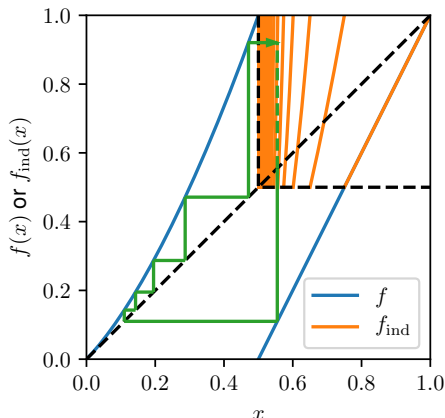

FIGURE 2. The LSV map (2) (blue) for $\alpha = 0.8$ with its induced map on $[\frac{1}{2}, 1]$ (orange).

The non-mixing dynamics near the fixed point poses a problem for obtaining accurate numerical estimates for these maps. This is true even when one naively attempts to estimate expectations of observables by taking Birkhoff sums: expectations converge as a stable law for $\alpha \in [\frac{1}{2}, 1]$ [10], and do not converge for $\alpha \geq 1$ since for these values of α the acim has infinite measure [1].

Nonetheless, a variety of transfer-operator based numerical methods have been considered. Various authors have proposed applying Ulam's method or modifications thereof to the full intermittent system: that is, partitioning the phase space into intervals and calculating statistical properties from an associated discretisation of the transfer operator [14, 8, 9]. With a suitable choice of partitions estimates of the invariant measure were found to converge for $\alpha < 1$ as $\mathcal{O}(N^{-(1-\alpha)})$, where N is the cardinality of the partition: the source of this slow convergence is the map's weak expansion near the neutral fixed point.

By discretising the transfer operator of the induced map, the slow convergence with respect to the partition size was notionally avoided by [4] and the standard convergence rate of Ulam's method, $N^{-1} \log N$ in the partition cardinality N, was obtained. However, to calculate an Ulam-like discretisation of the transfer operator, one must repeatedly evaluate the induced map: this requires iterating the full map past the neutral fixed point, a procedure whose computational expense grows as N^α. Furthermore, the number of evaluations of the induced map required to estimate the transfer operator to a given accuracy is proportional to the norm of the derivative of the induced map, which increases polynomially with the length of the orbit of the full dynamics corresponding to the step of the induced map. The computational time required to estimate an order N Ulam matrix is consequently $\mathcal{O}(N^{\alpha+2})$ as opposed to the $\mathcal{O}(N)$ for uniformly expanding maps, and perhaps due to the inefficiency of this method, this numerical approach has not yet been implemented. Consequently, it is clear that for good numerics it is not enough just compute with the induced map: it is necessary also to avoid iterating through the full dynamics.

In this paper, we will present a numerical method that both solves this problem and harnesses the smooth structure to attain fast convergence rates with low numerical overhead. There are three main ingredients at play. The first ingredient solves the problem of efficiently computing the induced map: this is achieved by employing the Abel function, a concept developed in the area of functional iterative equations [2, 11], with parallels in the theory of complex dynamics [7]. The Abel function conjugates the map f close to the neutral fixed point to a unit shift, thus allowing the induced map to be calculated in closed form. Furthermore, we find that the Abel function possesses an asymptotic expansion that enables efficient computation.

The second ingredient is the Chebyshev Galerkin transfer operator method [18, 5] (re-presented here in Appendix A), which enables statistical properties of the induced map to be computed exponentially accurately. The Chebyshev method requires pointwise evaluation of the action of the transfer operator of the induced map, which has an infinite number of branches, each of which contribute to the transfer operator.

To treat this efficiently, the final ingredient is the Euler-Maclaurin formula, which allows us to very efficiently evaluate infinite sums over the branches. The end result is that statistical properties of the induced map can practically be computed to very

high accuracy: because many statistical properties of the full map can be obtained
by summing appropriate statistics of the induced map over backward orbits (see
Proposition 2.3 and Remark 2.1 below), the full map's statistical properties can
then be accurately computed by the same methods.

These ingredients are extremely accurate and work more or less equally well
for large $\alpha > 1$ or near critical thresholds for statistics as for small α. This is
in great contrast to Birkhoff averaging or Ulam-style methods. To illustrate, we
use the methods presented here to obtain highly accurate numerical estimates,
which would be unattainable with previous techniques, and which moreover are
rigorously validated. The methods we present therefore open up fine exploration
of intermittent maps in the infinite ergodic case, as well as in limiting parameter
regimes.

The paper is organised as follows. In Section 2 we state the main theorems
that form the core ideas in our numerical methods, and in Section 3 we give some
numerical results. In Section 4 we prove Theorems 2.1-2.2and in Section 5 we give
explicit bounds on the convergence of Euler-Maclaurin summation over backward
orbits.

2. Main theorems

We first state two theorems which allow us to efficiently compute the induced
map by means of Abel functions, which allows us to express the induced map

$$f^\tau(x) = (\underbrace{f \circ \cdots \circ f}_{\tau(x) \text{ times}})(x)$$

in closed form. A cursory background on Abel functions is given in Section 4.

In Theorem 2.1 we show that the induced map and return times can both be
expressed in terms of an Abel function, if it exists; in Theorem 2.2 we show that
such a function exists and is the principal Abel function: we give an asymptotic
expansion for it around the neutral fixed point.

Theorem 2.1. *For maps in class PM, the return map $f^\tau : [a, 1] \to [a, 1]$ has the
explicit expression*

(4)
$$f^\tau = A^{-1}(\{A(f_g(x))\})$$

where $\{y\}$ denotes the fractional part of y.

*The return time $\tau : [a, 1] \to \mathbb{N}$ given by (3) is also explicitly given in terms of
the Abel function by*

(5)
$$\tau = \begin{cases} \lfloor A(f_g(x)) \rfloor + 1, & x \neq a \\ 1, & x = a, \end{cases}$$

where the bijection $A : [0, 1] \to [0, \infty]$ is an Abel function with

(6)
$$A(f_b(x)) = A(x) - 1$$

for $x \in [0, a]$ and $A(1) = 0$.

Theorem 2.2. *For maps in class PM there exists a (principal) Abel function
$A : [0, 1] \to [0, \infty]$ such that*
(a) A satisfies

(7)
$$A(x) = \lim_{k \to \infty} \frac{1}{\alpha h'(0)} \left(f_b^{-k}(x)^{-\alpha} - f_b^{-k}(1)^{-\alpha} \right).$$

(b) *There is an analytic extension of A into the complex plane having asymptotic expansion uniformly as $z \to 0, \Re z^{-1} \geq R_1 < \infty$*

(8) $$A(z) \sim a_{-1}z^{-\alpha} + a_\ell \log x + a_0 + \sum_{n=1}^{\infty} a_n z^{n\alpha}.$$

Furthermore, this expansion, truncated after the $z^{N\alpha}$ term for $N \geq 1$, has error

$$|A(z) - A_N(z) = \mathcal{O}(N^{N+2}|z|^{-(N+1)\alpha})$$

with explicit constants given in (24).

An example of a principal Abel function is plotted on the map's real domain in Figure 3.
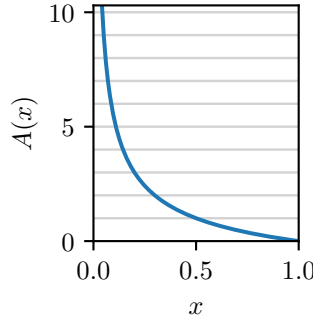

FIGURE 3. Plot of the principal Abel function for the LSV map for $\alpha = 0.8$.

Because the induced map has many desirable properties for the computation of statistical properties, in particular being uniformly expanding, we will compute statistical properties of the full map using those of the induced map. This will at various points require the computation of sums over backward orbits of the intermittent dynamics: for example, using the chain rule the transfer operator of the induced map has the form

(9)
$$\mathcal{L}_{ind}\phi(z) = \sum_{f^\tau(y)=z} |(f^\tau)'(y)|^{-1}\phi(y) = \sum_{n\in\mathbb{N}, f_b^n(f_g(y))=z} (f_b^n)'(f_g(y))^{-1}|f_g'(y)|^{-1}\phi(y).$$

To deal with these sums in a unified way, we will suppose that the summands of these systems can be written as functions Q of the backward orbit $x = f_g(y)$, the derivative $d = (f_b^n(x))^{-1}$, and the orbit index x: for example, from (9) we can see that the induced map's transfer operator has summand function $Q_{\mathcal{L}_{ind},\phi}(x,d,n)$

$$Q_{\mathcal{L}_{ind},\phi}(x,d,n) = d(\mathcal{L}_g\phi)(x),$$

where $\mathcal{L}_g$ is the transfer operator of $f_g : [a,1] \to [0,1]$:

$$(\mathcal{L}_g\phi)(x) = \sum_{f_g(y)=x} |f_g'(y)|^{-1}\phi(y).$$

We introduce the operator $\mathbb{T}[n; \cdot]$ which acts on the summand functions Q : $(0, 1] \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ so as to output the nth summand:

$$\tag{10} \mathbb{T}[Q](n; x) = Q(f_b^{-n}(x), (f_b^{-n})'(x), n)$$

$$\tag{11} = Q(A^{-1}(A(x) + n), (A'(x)(A^{-1})'(A(x) + n), n).$$

We further introduce the operator $\mathbb{S}$ which acts on the summand functions Q to output the sum over all n:

$$\tag{12} \mathbb{S}[Q](x) = \sum_{n=0}^{\infty} \mathbb{T}[Q](n; x)..$$

In Section 5 we will show that, when Q extends to a complex analytic function, these sums $\mathbb{S}[Q](x)$ may be very efficiently estimated using the Euler-Maclaurin formula.

The proposition provides the recipe to compute the transfer operator $\mathcal{L}_{ind}$ and the acim of the induced map, via Abel functions. We consider the so-called solution operator for the induced map: $\mathcal{S}_{ind} = (\mathrm{id} - \mathcal{L}_{ind} + u \int)^{-1}$, where $\int$ is the Lebesgue integral functional on $[a, 1]$ and $u(x) = \frac{1}{1-a}$.

Proposition 2.3. *Let $f \in PM$ as in (1). The induced map's invariant probability measure ρ_{ind} is given by $\mathcal{S}_{ind}u$.*

Furthermore, the induced map's transfer operator $\mathcal{L}_{ind} : BV([a, 1]) \to BV([a, 1])$ can be written as $\mathcal{L}_{ind}[\phi](x) = \mathbb{S}[Q_{\mathcal{L}_{ind}, \phi}](x)$, where $Q_{\mathcal{L}_{ind}, \phi}(x, d, n) = d(\mathcal{L}_g \phi)(x)$.

Remark 2.1. *Many statistics of the full dynamics may be efficiently computed through similar formulations. In particular, we have the following formulae for some statistical quantities associated the full dynamics:*

(1) *Expectations of functions ψ of the return time to the inducing set are given by*

$$\tag{13} \rho_{ind}(\psi(\tau)) := \int_a^1 \psi(\tau(x)) \rho_{ind}(x) \mathrm{d}x = \int_a^1 \mathbb{S}[Q_{\tau, \psi}](x) \mathrm{d}x,$$

where

$$Q_{\tau, \psi}(x, d, n) = \psi(n) d(\mathcal{L}_g \phi)(x).$$

(2) *The full invariant measure $\rho \mathrm{d}x$ evaluated pointwise is given by*

$$\tag{14} \rho(x) = \mathbb{S}[Q_{\mathcal{L}_{ind}, \rho_{ind}}](x).$$

This is normalised so its restriction to $[a, 1]$ is a probability measure; for $\alpha < 1$ it may be renormalised to a probability measure on the full set by the constant factor $\frac{1}{\rho_{ind}(\tau)}$.

(3) *The average of an observable $A : [0, 1] \to \infty$ over ρ is given by*

$$\rho(A) := \int_0^1 A(x) \rho(x) \, \mathrm{d}x.$$

Analyticity-preserving properties of $\mathbb{S}$ ensure that $\rho(z)$ extends into the complex plane sufficiently as to allow for accurate quadrature.

(4) *For $\alpha < \frac{1}{2}$, the diffusion coefficient of an observable $A : [0, 1] \to \infty$ is given by*

$$\sigma_f^2(A) = \rho(\phi_A + \mathbb{S}[Q_{\mathcal{L}_{ind}, \mathcal{S}_{ind}(\phi_A|_{[a,1]})}]),$$

where $\phi_A(x) = \mathbb{S}[Q_{\phi, A}(x)]$ and $Q_{\phi, A}(x, d, n) = d \, \phi(x)(A(x) - \rho(A))$.

Theorems 2.1-2.2 and Proposition 2.3 can be exploited to design algorithms to capture statistical properties of intermittent maps with high accuracy. We give some results from such an algorithm in Section 3.

3. Numerical results

We have implemented rigorously validated algorithms suggested by the work in this paper to compute acims and return times.

The first goal is to be able to evaluate the Abel function. One first estimates the coefficients of the Abel function's asymptotic expansion (8) by matching Taylor coefficients at $x = 0$ of the Abel equation (6), with rigorous bounds on the error of this expansion given in Theorem 2.2. This immediately enables accurate evaluation of the Abel function for x near the fixed point: away from the fixed point accurate estimates may be calculated by numerically iterating backwards until for some $k \in \mathbb{N}$ an iterate $f_b^{-k}(x)$ sufficiently close to 0 is reached, and then by using that $A(f_b^{-k}(x)) = A(x) + k$.

To compute statistical quantities, the algorithm computes a Chebyshev Galerkin matrix as in [18, 5]: the action of the transfer operator on Chebyshev basis functions is evaluated pointwise by using Proposition 2.3 and (27). By the Chebyshev method the acim of the induced map can be rigorously estimated as in Algorithm 1 in [18]. Estimates of the return time are obtained by a rigorous computation of the return time formula in Remark 2.1(a) using the Euler-Maclaurin formula (27); pointwise estimates of the full map's acim are obtained similarly using Remark 2.1(b).

We applied these methods to LSV maps (2) for various values α. Plots of the acims obtained using our method are given in Figure 4 with comparisons to estimates obtained using long time series. An example of the rigorous estimate is given in the following theorem:

Theorem 3.1. *For the LSV map with parameter $\alpha = 0.95$, the expected return time to the set $[\frac{1}{2}, 1]$ is*

$$\rho_{ind}(\tau) = 14.073\ 323\ 220\ 001\ 939\ 529\ 241\ 549\ 699$$

$$610\ 756\ 609\ 803\ 3171 \pm 10^{-43}.$$

It is illustrative of the power of the method, particularly of the Abel function numerics, to contrast this with an estimate of the expected return time obtained via iterating the LSV map: the sample of 10^8 iterates used in Figure 4 furnished an estimate $\rho_{ind}(\psi(\tau)) \approx 8.63$, a 40% error. This large error arises because the distribution of the return time $\tau|_{[\frac{1}{2},1]}$, which the iterates sample, is for maps with $\alpha = 0.95$ very heavy-tailed: in fact it becomes non-integrable at the nearby value $\alpha = 1$. The Euler-Maclaurin summation however allows these tails to be summed over very easily, regardless of their decay rates.

The results for different values of α were each obtained in 6 hours over 15 hyper-threaded cores of a research server running 2 E5-2667v3 CPUs with 128GB of memory. The number of basis elements used in the Chebyshev Galerkin method was $N = 512$, and 256-bit extended floating point arithmetic was used using the ValidatedNumerics library in Julia [6].

Let us briefly note that at a preliminary stage in the development of these numerical methods, we implemented adaptive algorithms to compute acims of the
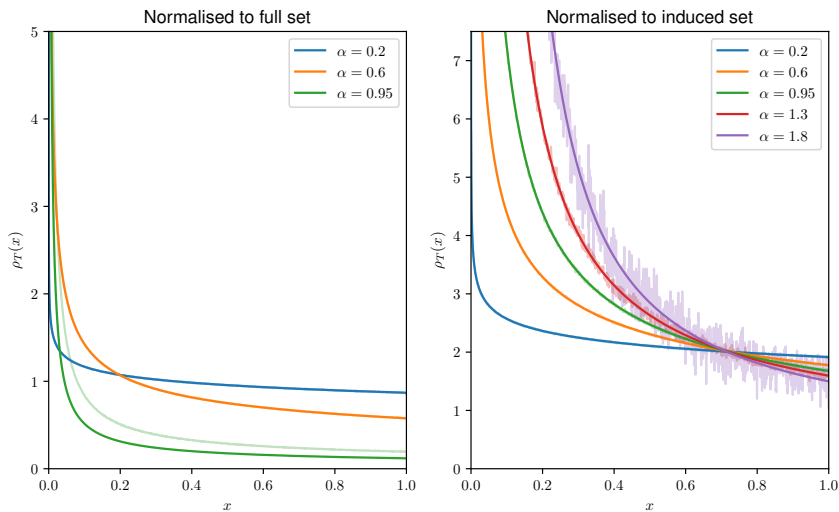
FIGURE 4. Absolutely continuous invariant measures of the LSV map for varying α, normalised to a probability measure: (left) on the full domain when $\alpha < 1$, i.e. for finite absolutely continuous invariant measures, (right) for on the inducing domain. In semi-transparency, histograms of long time series of 10^8 iterates of LSV maps for various α (see discussion of long time series estimates in Section 3).

induced and full systems using floating-point arithmetic. These algorithms were similar in spirit to, and made use of, the Poltergeist package discussed in [18]. In place of the Euler-Maclaurin formula algorithms we propose, a poorly optimised version of the already less numerically efficient Abel-Plana formula [15], the adaptive method could obtain acim estimates accurate to 13 decimal places in around 20 seconds. With good numerical optimisation and using the Euler-Maclaurin formula we believe that it would be possible to obtain these estimates in around 2 seconds, and intend on implementing these methods in future.

4. RETURN MAPS AND ABEL FUNCTION

Given an iterated function of one dimension $x_{n+1} = g(x_n)$, a function A is considered an Abel function of g if it satisfies the Abel functional equation $A(g(x)) = A(x) + 1$ and is invertible (at least locally). The existence and behaviour of Abel functions around fixed points which are linearly neutral and stable (as opposed to unstable, the case we consider) have been studied in statistics [17, 11]. This corresponds to studying the local inverse f_b^{-1} of our map near the fixed point 0, because the fixed point is linearly neutral and *unstable*. Consequently, our definition assumes iteration of f decrements the Abel function A (as in Theorem 2.1) rather than incrementing it, as is standard in the literature.

Around a fixed point there are an infinite number of continuous or even smooth solutions to the Abel functional equation. It is possible to define a so-called principal Abel function via a certain iterative equation [17], which may be seen to be

equivalent to (7): principal Abel functions have the best regularity properties of all possible solutions to the Abel equation.

We begin by proving Theorem 2.1, which states that the induced map and return time can be appropriately computed using a monotonic function A satisfying the Abel equation (6).

Proof of Theorem 2.1. Suppose A is a bijection $[0, 1] \to [0, \infty]$ satisfying $A(f_b(x)) = A(x) - 1$ and $A(1) = 0$. For $x \in (a, 1]$, the return time, which measures the number of iterates required to return to the inducing set, is given by

$$\tau(x) = \inf\{n \in \mathbb{N}^+ : f^n(x) \in [a, 1]\}.$$

This definition implies that $f^j(x) \in (0, a)$ for $0 < j < \tau(x)$, and consequently $f^j(x) = f_b(f^{j-1}(x)) = A^{-1}(A(f^{j-1}(x) - 1))$ for $0 < j < \tau(x)$. As a result,

$$f^\tau(x) = f^{\tau(x)}(x) = f_b^{\tau(x)-1}(f(x)) = A^{-1}(A(f(x)) - \tau(x) + 1),$$

and since $x \in (a, 1]$, $f(x) = f_g(x)$.

Since $f^\tau(x) \in (a, 1]$ and since $A : [0, 1] \to [0, \infty]$ is a bijection with $A(1) = 0$, we find that A is decreasing and so $0 \leq A(f^\tau(x)) < A(a) = A(f_b(a)) + 1 = A(1) + 1 = 1$, and consequently $A(f(x)) - \tau(x) + 1 \in [0, 1)$. Using that $\tau(x)$ is an integer we obtain (4) and (5). $\qquad\square$

We will now prove the existence of a principal Abel function with nice asymptotic properties (Theorem 2.2). We will do this by showing that an analytic function satisfying part (b) of the theorem must have asymptotic properties as given in part (a). Using results in [17], we then prove the existence of such a function.

Proof of Theorem 2.2. In this setting we find it convenient to transform to coordinates $z = x^\alpha$, considering the conjugated inverse map which we define

$$\hat{T}(z) := z\hat{h}(z) := f_b^{-1}(x)^\alpha.$$

By the implicit function theorem, $\hat{T}$ is uniquely defined for z in a complex neighbourhood of $[0, a^\alpha]$, and in particular for $|z| \leq R$ for some $R > 0$. We consider the principal Abel function for this map, having $\hat{A}(\hat{T}(z)) = \hat{A}(z) + 1$, and set $A(x) = \hat{A}(z)$.

Let the power series at 0 of $\hat{T}(z) = z + \sum_{n=1}^{\infty} \hat{h}_n(-z)^{n+1}$: in particular, $\hat{h}_1 = \alpha h'(0)$. We have that $\hat{T}^{-1}$ is analytic in a neighbourhood of zero with $\hat{T}^{-1}(z) \sim z + \hat{h}_1 z^2 + \ldots$, and consequently that $\hat{T}$ is similarly analytic near zero with $\hat{T}(z) \sim z - \hat{h}_1 z^2 + \ldots$.

For $n \geq 0$ define the following functions, which are holomorphic except at zero,

$$(15) \qquad \hat{A}_n(z) = a_{-1}z^{-1} + a_\ell \alpha^{-1} \log z + C + \sum_{i=1}^{n} a_i z^i,$$

with constant C to be determined later, such that as $z \to 0$,

$$(16) \qquad \hat{A}_n(\hat{T}(z)) - \hat{A}_n(z) + 1 =: D_n(z) = \mathcal{O}(z^{n+2}).$$

Let us define the function g by

$$(17) \qquad \frac{1}{\hat{T}(z)} = \frac{1}{z} + \hat{h}_1 + g(z)z.$$

From the Taylor expansion of $\hat{T}$ at $z = 0$ we can see that the magnitude $|g(z)| \leq G$ for all $|z| \leq R$ and some constant $G < \infty$.

The error D_n is given by the following lemma, whose proof is in Appendix B:

Lemma 4.1. *Let*

$$(18) \qquad r_n = \min\{R, 0.4(\hat{h}_1 + \sqrt{0.4G})^{-1}\}n^{-1},$$

$$(19) \qquad d_2 = 1 + 2.5e^{3/5}(1 + 0.4G\hat{h}_1^{-2}),$$

$$(20) \qquad d_1 = \frac{1 + G\hat{h}_1^{-2}}{d_2^2}.$$

For $|z| \leq r_n$,

$$|D_n(z)| \leq d_1 \left(\frac{d_2|z|}{r_n} \right)^{n+2}.$$

The following lemmas, whose proofs are in Appendix B give bounds on iterates of $\hat{T}$:

Lemma 4.2. *Let $\aleph \in (0, 1)$ and $R_1 = \min\{R, \aleph G^{-1}\hat{h}_1\}$.*
Then for all z with $\Re z^{-1} \geq R_1$, and all $k \in \mathbb{N}$,

$$(21) \qquad \Re z^{-1} + (1 + \aleph)\hat{h}_1 k \geq \Re \hat{T}^k(z)^{-1} \geq \Re z^{-1} + (1 - \aleph)\hat{h}_1 k,$$

and

$$(22) \qquad |z^{-1} + k\hat{h}_1| + \aleph\hat{h}_1 k \geq |\hat{T}^k(z)|^{-1} \geq |z^{-1} + k\hat{h}_1| - \aleph\hat{h}_1 k \geq R_1.$$

Lemma 4.3. *Let $\aleph, R_1$ be as before, and let $\bar{\beta} - 1 \geq \delta \geq 0$. Then for all z with $\Re z^{-1} \geq R_1$,*

$$\sum_{k=0}^{\infty} |\hat{T}^k(z)|^{\bar{\beta}} k^{\delta} \leq \beth_{\bar{\beta}, \delta} |z|^{\bar{\beta} - \delta - 1},$$

where

$$\beth_{\bar{\beta}, \delta} := (1 - \aleph)^{-\bar{\beta}} \hat{h}_1^{-\delta - 1} \left(\frac{1}{\delta + 1} + \frac{1}{\bar{\beta} - \delta - 1} + \hat{h}_1 R_1^{-1} \right).$$

Define the sets

$$S_r = \{z \in \mathbb{C} \mid 0 < |z| < r, \Re z^{-1} \geq R_1\}$$

for $r > 0$, where R_1 is given in Lemma 4.2, and consider a function $\hat{A} : S_r \to \mathbb{C}$ such that for all $z \in S_r$ and $n > 0$,

$$(23) \qquad \lim_{k \to \infty} \hat{A}(\hat{T}^k(z)) - \hat{A}_n(\hat{T}^k(z)) = 0.$$

Later, we will show that $A(z^\alpha)$ where A is given in (7) is such a function. We will prove bounds on $|\hat{T}^n(z)|$ for $z \in S_{R'}$ for some $R' < R$, which will allow us to bound the error between $\hat{A}$ and the $\hat{A}_n$ on this set.

Now defining

$$E_n(z) = \hat{A}(z) - \hat{A}_n(z),$$

we have that

$$E_n(z) = E_n(\hat{T}(z)) - (\hat{A}(\hat{T}(z)) - \hat{A}(z)) + (\hat{A}_n(\hat{T}(z)) - \hat{A}_n(z)) = E_n(z) + D_n(z)$$

and thus by (23) and the fact that $f^{-k}(z) \to 0$ as $k \to \infty$, we have that

$$E_n(z) = \sum_{k=1}^{\infty} D_n(\hat{T}^k(z))$$

and thus by Lemma 4.1

$$|\hat{A}(z) - \hat{A}_n(z)| \leq d_1 d_2^{n+2} r_n^{-(n+2)} \sum_{k=1}^{\infty} |\hat{T}^k(z)|^{n+2}.$$

When $\Re z^{-1} \geq R_1$ we can apply Lemma 4.3 to obtain that for $n \geq 1$ and $|z| \leq \min\{R_1, r_n\}$,

$$|\hat{A}(z) - \hat{A}_n(z)| \leq d_1 d_2^{n+2} r_n^{-(n+2)} \beth_{n+2,0} |z|^{n+1}$$

(24)
$$\leq d_3 d_2^{n+2} r_n^{-(n+2)} |z|^{-n+1},$$

where

(25)
$$d_3 = \beth_{n+2,0} d_2$$

Consequently $\hat{A}$ satisfies the appropriate asymptotic expansion generated by the $\hat{A}_n$ with the error bound given by (24).

By change of coordinates $x^\alpha = z$ we have that

$$E_n(z) = d_3 (d_2/r_n)^{n+2} \left(|z|^{-\alpha} - dR \right)^{-(n+1)}.$$

We now show that $\hat{A}$ is in fact given by the principal Abel function that we desired. Let

$$\hat{A}(x) = \lim_{k \to \infty} \frac{\hat{T}^k(1) - \hat{T}^k(x)}{\hat{T}^k(1) - \hat{T}^{k+1}(1)}.$$

Note that $\hat{A}(x^\alpha)$ is just our principal Abel function (7), and $\hat{A}(\hat{T}(z)) = \hat{A}(z) - 1$ where $\hat{A}$ is defined. As a result of Lemma 7 in [17], $\hat{A}$ extends into the complex plane, and for a correct choice of C in (15) for every $\mathfrak{C}$ there exists a $\mathfrak{D}$ so that this map satisfies

$$\lim_{z \to 0} \left(\hat{A}(z) - \hat{A}_0(z) \right) = 0$$

for $|\Re z^{-1}| > \mathfrak{D}$, $|\Im z^{-1}| \leq \mathfrak{C}$. By Lemma 4.2 we have that $\Re \hat{T}^n(z)^{-1}$ is increasing in n, and the following lemma, proved in Appendix B, gives that $\Im \hat{T}^n(z)^{-1}$ is bounded:

Lemma 4.4. *For all z with $\Re z^{-1} \geq R_1$,*

$$\sup_{k \in \mathbb{N}} \left| \Im \hat{T}^k(z)^{-1} \right| < \infty.$$

As a result, $\hat{T}^n(z)$ goes to 0 as $n \to \infty$.

Furthermore, as a result of the monotonicity of f_b and (7), A is clearly monotonically increasing on $[0,1] \to [0,\infty]$; because it is analytic and unbounded it must be a bijection. $\qquad\square$

Remark 4.1. *The Thaler map is an interval map with an explicitly known invariant measure that has a neutral fixed point of order $1 + \alpha$ at zero:*

$$f(z) = z \left[1 + z^{\alpha-1} \left((1+z)^{1-\alpha} - 1 \right) \right]^{1/(\alpha-1)}.$$

This map is not in class PM as the series expansion of $f(z)/z$ at zero contains integer powers of z as well as of z^α. However, one could extend the methods in this paper accordingly.

5. Calculating statistical properties via inducing

Because the infinite sums required to evaluate statistical properties, such as in Proposition 2.3 and Remark 2.1, are summing over smooth functions evaluated on a lattice, we can use the Euler-Maclaurin formula to approximate these sums with exponentially decreasing errors. We state a general theorem that in particular allows us to obtain rigorous bounds on the error of these approximations.

Define the small, bounded sets $\mathcal{R}_s := \{x \in \mathbb{C} : \Re(x^{-\alpha}) \geq s^{-1}\}$, and its transform to z coordinates, $\hat{\mathcal{R}}_s := \{z \in \mathbb{C} : \Re(z^{-1}) \geq s^{-1}\}$.

We will first find it useful to define a constant encoding the regularity of our of our map,

$$
G' = \sup_{z \in \hat{\mathcal{R}}_{R_1}} \left| \frac{d}{dz} z g(z) \right|
$$

and a radius

$$
(26) \qquad Z := \min\{R_1, (2G')^{-1/2}, (2G' \mathrm{J}_{2,0})^{-1}\},
$$

which will be used to specify the region inside which the Euler-Maclaurin formula may be used.

Theorem 5.1. *Suppose $Q(x, d, n)$ is analytic such that for some $\bar{Q}$, some non-negative β, γ, δ with $\bar{\beta} := (\beta + (1 + \alpha)\gamma)/\alpha > 1 + \delta$, and for all $z \in \mathcal{R}_R$, all $d \leq 1$ and all n with $\Re n > 0$,*

$$
|Q(z, d, n)| \leq \bar{Q} |z|^{\beta} |d|^{\gamma} |n|^{\delta}.
$$

Let $\rho > 0$.
Then for all z such that

$$
n_{\rho}^* := \inf\{n : f_b^{-n} z \in \mathcal{R}_{(Z^{-\alpha} + 2\hat{h}_1 + \rho)^{-1/\alpha}}\}
$$

is defined, then

$$
(27)
$$

$$
\mathbb{S}[Q](z) = \sum_0^{n_{\rho}^* - 1} \mathbb{T}[Q](n; z) + \frac{1}{2} \mathbb{T}[Q](n_{\rho}^*; z) - \int_0^{f_b^{-n_{\rho}^*}(z)} A'(\zeta) Q\left(\zeta, \frac{A'(z)}{A'(\zeta)}, A(\zeta) - A(z)\right) \mathrm{d}\zeta
$$

$$
- \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} \frac{\partial^{2k-1}}{\partial n^{2k-1}} \mathbb{T}[Q](n_{\rho}^*; z) + \mathcal{E}_K,
$$

where B_p are the Bernoulli numbers, and $\mathcal{E}_K = O(\rho^{-K + \delta - (\beta + (1+\alpha)\gamma)/\alpha})$, with an explicit bound given in (30-31).

We will find the following proposition useful in proving this theorem:

Proposition 5.2. *Let $m \in \mathbb{C}$. If we restrict $\hat{A}$ to act on $\hat{\mathcal{R}}_Z$, then for any $z_0 \in \hat{\mathcal{R}}_{(Z^{-1} + 2\hat{h}_1 |m|)^{-1}}$,*

$$
|A^{-1}(A(z_0) + m)^{-\alpha} - z_0^{-\alpha}| \leq 2\hat{h}_1 |m|.
$$

Proof of Theorem 5.1. A simple application of the Euler-Maclaurin formula [3] gives most of the terms in (27); we convert the integral expression

$$\int_{n_\rho^*}^\infty \mathbb{T}[Q](n,z)dn = \int_{n_\rho^*}^\infty Q\left(A^{-1}(n+A(z)), \frac{A'(z)}{A'(A^{-1}(n+A(z)))}, n\right) dn$$

$$= \int_{n_\rho^*+A(z)}^\infty Q\left(A^{-1}(n), \frac{A'(z)}{A'(A^{-1}(n))}, n-A(z)\right) dn$$

$$= \int_0^{T^{n_\rho^*}(z)} -A'(\zeta)Q\left(\xi, \frac{A'(z)}{A(\xi)}, n-A(z)\right) d\zeta.$$

The remainder term $\mathcal{E}_K$ can be bounded [12, 3] by

$$(28) \qquad |\mathcal{E}_K| \leq \frac{2}{(2\pi)^{2K+1}} \int_{n_\rho^*}^\infty \left|\frac{\partial^{2K+1}}{\partial n^{2K+1}}\mathbb{T}[Q](n;z)\right| dn.$$

From Lemma 4.2 we have that if $s^{-1} = \Re z^{-\alpha}$ with $s^{-1} > Z^{-1}+2\hat{h}_1$, then $z \in \mathcal{R}_s$ with

$$A^{-1}(n+A(z)) = f_b^{-n}(z) \in \mathcal{R}_{(s^{-1}+n(1-\aleph)\hat{h}_1)^{-1}}$$

for integer n, and from Proposition 5.2 that for $n \in (0,1)$ that

$$A^{-1}(A(z) - n) \in \mathcal{R}_{(s^{-1}-2\hat{h}_1)^{-1}};$$

consequently for all $n > 0$,

$$(29) \qquad A^{-1}(A(z) + n) \in \mathcal{R}_{(s^{-1}+\hat{h}_1((1-\aleph)n-2))^{-1}}.$$

Thus, for any z, $n \geq n_\rho^*$, and $m \in B(0, \rho + (1-\aleph)(n-n_\rho^*)/2)$, $\zeta := A^{-1}(A(z)+n+m) \in \mathcal{R}_Z$ and so

$$|\mathbb{T}[Q](n+m;z)| = \left|Q\left(\zeta, \frac{A'(z)}{A'(\zeta)}, n+m\right)\right|$$

$$\leq \bar{Q}|\zeta|^\beta|\frac{|A'(z)|^\gamma}{|A'(\zeta)|^\gamma}|n+m|^\delta$$

$$= \bar{Q}|A'(z)|^\gamma|\zeta|^\beta\left|\frac{\zeta^{1+\alpha}}{\zeta^{1+\alpha}A'(\zeta)}\right|^\gamma |n+m|^\delta$$

$$\leq \bar{Q}|A'(z)|^\gamma|\zeta|^{\beta+(1+\alpha)\gamma}2^{|\gamma|}h_1^\gamma|n+m|^\delta,$$

by Lemma 5.3.

We know from (29) and Lemma 5.3 that $\zeta \in \mathcal{R}_Z$, and thus if $\bar{\beta} = (\beta + (1+\alpha)\gamma)/\alpha \geq 0$,

$$|\mathbb{T}[Q](n+m;z)| \leq \bar{Q}|A'(z)|^\gamma Z^{-\bar{\beta}}2^{|\gamma|}h_1^\gamma(n+\rho+(1-\aleph)(n-n_\rho^*)/2)^\delta.$$

By applying Proposition 5.2 we have that

$$|\zeta|^{-\alpha} \leq |T^{n_\rho^*}(z)|^{-\alpha} + 2h_1(n+|m|-n_\rho^*) = |T^{n_\rho^*}(z)|^{-\alpha} + 2h_1(3-\aleph)(n-n_\rho^*)/2$$

and thus if $\bar{\beta} \leq 0$,

$$|\mathbb{T}[Q](n+m;z)| \leq \bar{Q}|A'(z)|^\gamma 2^{|\gamma|}h_1^\gamma\left(|T^{n_\rho^*}(z)|^{-\alpha} + 2h_1\frac{3-\aleph}{2}(n-n_\rho^*)\right)^{-\bar{\beta}}\left(n+\rho+\frac{1-\aleph}{2}(n-n_\rho^*)\right)^\delta.$$

We then have by Cauchy's formula that

$$\left| \frac{\partial^{2K+1}}{\partial n^{2K+1}} \mathbb{T}[Q](n;z) \right| \leq (2K+1)!(\rho + (1 - \aleph)(n - n_\rho^*)/2)^{-(2K+2)}.$$

$$\sup \left\{ |\mathbb{T}[Q](n+m;z)| \mid m \in B(0, \rho + (1-\aleph)(n - n_\rho^*)/2) \right\};$$

thus, from (28) and with some simplification,

$$(30) \qquad\qquad |\mathcal{E}_K| \leq \frac{(2K+1)!}{(2\pi\rho)^{2K+1}} \frac{4\bar{Q}|A'(z)|^\gamma}{1 - \aleph} \left(1 + \rho^{-1} n_\rho^*\right)^\delta W,$$

where
$$(31)$$

$$W = \begin{cases} \frac{\rho^\delta}{2K+1-\delta} 2^{|\gamma|} h_1^\gamma Z^{-\bar{\beta}}, & \bar{\beta} \geq 0, \\ \frac{\rho^{\delta - \bar{\beta}}}{2K+1+\bar{\beta}-\delta} 2^{|\gamma|} h_1^\gamma \max\left\{ \rho^{-1} |T^{n_\rho^*}(z)|, 2h_1(1 + 2(1-\aleph)^{-1}) \right\}^{-\bar{\beta}}, & \bar{\beta} < 0, \end{cases}$$

recalling that we defined $\bar{\beta} := (\beta + (1 + \alpha)\gamma)/\alpha$.

Finally, since the integrand in the integral in (27) is $\mathcal{O}(z^{-(\alpha+1)(1-\gamma)+\beta+\alpha\delta})$, we know that it will converge if $\bar{\beta} + \delta < 1$. $\qquad\square$

The following lemma, used in the proof of Theorem 5.1 bounds the derivative of the Abel function:

Lemma 5.3. *For any $z \in \mathcal{R}_Z$,*

$$(2h_1)^{-1} \leq |z^{\alpha+1} A'(z)| \leq 2h_1^{-1}.$$

Proof. It is possible to show from the definition of A in (7) that

$$z^{-2} \hat{A}'(z) = \lim_{k \to \infty} \frac{z^2 (\hat{T}^k)'(z)}{\hat{h}_1 (\hat{T}^k(z))^2} = \hat{h}_1^{-1} \prod_{k=0}^\infty \xi(\hat{T}^k(z)),$$

where

$$\xi(z) := \frac{z^2}{\hat{T}(z)^2} \hat{T}'(z).$$

From the definition of $g(z)$ we have that

$$\xi(z) = 1 + z^2 (zg)'$$

and so for $z \in \hat{\mathcal{R}}_R$ such that $|z| \leq (2G')^{-1/2}$,

$$|\log|\xi(z)|| \leq \log 2 \, 2G'|z|^2.$$

By Lemma 4.2 then, for $z \in \hat{\mathcal{R}}_{\min\{R,(2G')^{-1/2}\}}$,

$$\left| \log \hat{h}_1 |z^{-2} \hat{A}'(z)| \right| l \leq \log 2 \, 2G' \sum_{k=0}^\infty |\hat{T}^k(z)|^2$$

$$\leq \log 2 \, 2G' \beth_{2,0} |z|$$

$$\leq \log 2,$$

for $z \in \hat{\mathcal{R}}_Z$ as required. $\qquad\square$

Remark 5.1. *The choice of K which minimises the bound on $\mathcal{E}_K$ for given ρ is asymptotically $K \approx \pi\rho - \frac{1}{2}$, which gives an error $\log|\mathcal{E}_K| \lesssim -2\pi\rho$.*

APPENDIX A. NUMERICAL CALCULATION OF STATISTICAL PROPERTIES OF
INDUCED MAP

We summarise here briefly the Chebyshev Galerkin spectral method proposed in
[18], which we use to numerically approximate the action of the induced map's so-
called solution operator. This method discretises the transfer operator in Chebyshev
   We define maps in U_{NP} as those self-maps f of an interval $[p, q]$ that satisfy:

(1) There are open disjoint intervals $\mathcal{O}_\iota, \iota \in I$ whose union is of full measure
    in $[p, q]$ such that the inverse of $f|_{\mathcal{O}_\iota}$ extends to a bijection $v_\iota : [p, q] \to \overline{\mathcal{O}_\iota}$.
    (*Full-branch condition*)

(2) The map has *bounded distortion*:

$$(D_1) \qquad\qquad \sup_{x \in \Lambda, \iota \in I} \left| \frac{v_\iota''(x)}{v_\iota'(x)} \right| < \infty.$$

(3) It is *C-uniformly expanding*, that is, that

$$(CE) \qquad\qquad \check{\lambda} = \inf_{x \in \cup_{\iota \in I} \mathcal{O}_\iota} \frac{\sqrt{(q-x)(x-p)}}{\sqrt{(q-f(x))(f(x)-p)}} |f'(x)| > 1.$$

(4) The interval $\mathcal{O}_\iota$ satisfy the *partition spacing condition*:

$$(P) \qquad\qquad \sup \left\{ \frac{|\mathcal{O}_\iota|}{d(\mathcal{O}_\iota, \partial\Lambda)} : d(\mathcal{O}_\iota, \partial\Lambda) > 0 \right\} = \Xi < \infty.$$

Furthermore, we will assume the following analytic distortion bound holds:

$$(B_\delta) \qquad\qquad \sup_{\iota \in I, z \in B_\delta} \left| \frac{v_\iota''(z)}{v_\iota'(z)} \right| = \frac{q+p}{2} C_{1,\delta} < \infty$$

where the Bernstein ellipse B_δ has centre in the complex plane at $\frac{q+p}{2}$ with major
semiaxis $\frac{q-p}{2} \cosh\delta$ and minor semiaxis $\frac{q-p}{2} i \sinh\delta$.

Remark A.1. *If $f_g \in U_{NP}$ considered as a map $[a, 1] \to [0, 1]$, and if $f \in PM$,
then $f^\tau \in U_{NP}$. Furthermore, if f_g also satisfies (B_δ) in the same sense, then f^τ
satisfies (B_δ).*

Define (shifted) Chebyshev polynomials

$$\tilde{T}_k(x) = \cos\left(k \cos^{-1} \frac{2x - p - q}{q - p} \right), \, k \in \mathbb{N}.$$

which are orthogonal on $[p, q]$ with respect to the weight $1/\sqrt{(q-x)(x-p)}$. Let $\mathcal{P}_N$
be the operator projecting a function onto the first $N + 1$ Chebyshev polynomials.

Theorem A.1 (Wormell, 2019 [18])**.** *Let the operator*

$$\mathcal{K}_N = \mathcal{P}_N(I - \mathcal{L}_N + 1 \smallint)|_{\mathcal{P}_N(BV)},$$

where $\smallint$ denotes the Lebesgue total integral functional, and let $\mathcal{S}_N = \mathcal{K}_N^{-1}$.
   *Then for all maps in U_{NP} satisfying B_δ, there exist constants p, K (see Remark
5.4 and the proof of Theorem 2.3 in [18]) such that for all functions $\phi \in \mathcal{P}_N(BV)$,*

$$\|\mathcal{S}_N\phi - \mathcal{S}\phi\|_{BV} \le K e^{-(\delta-p)N}.$$

Furthermore, $\mathcal{S}1$ is the acim of f.

The finite-dimensional operator K_N may be closely approximated in the Chebyshev polynomial basis by Lagrange interpolation of its action on basis functions $\{T_k\}_{k=0,\ldots,N}$ at Chebyshev points $x_{k,N} = \cos(2k+1)\pi/2(N+1)$, $k = 0, \ldots, N$ [18, 5].

APPENDIX B. PROOF OF LEMMAS 4.1-4.4

Proof of Lemma 4.1. Matching power series coefficients at $z = 0$, we have that

$$(32) \qquad a_{-1} = \hat{h}_1^{-1}$$

$$(33) \qquad a_\ell = \hat{h}_2 \hat{h}_1^{-2} - 1$$

$$(34) \qquad a_n = \frac{1}{n\hat{h}_1} \frac{D_{n-1}^{(n+1)}(0)}{(n+1)!}.$$

Suppose

$$(35) \qquad r_n = \min\{R, cn^{-1}(\hat{h}_1 + \sqrt{Gc})^{-1}\}$$

for some $c \in (0,1)$ and let

$$M_{n,r} = \sup_{|z| \leq r} |D_n(z)|.$$

We have as a result of (34) that for any $r \leq r_n$

$$|a_n| \leq \frac{1}{n\hat{h}_1} r^{-n-1} M_{n-1,r}.$$

Consequently, for $n \geq 1$ and $r \leq r_n$ we have that

$$M_{n,r} \leq M_{n-1,r} + |a_n| \sup_{|z| \leq r} |\hat{T}(z)^n - z^n|$$

$$\leq M_{n-1,r} + \frac{r^{-n-1} M_{n-1,r}}{n\hat{h}_1} \sup_{|z| \leq r} r^n |(z^{-1}\hat{T}(z))^n - 1|$$

$$\leq M_{n-1,r}\left(1 + \frac{e^{n(z^{-1}\hat{T}(z)-1)} - 1}{n\hat{h}_1 r}\right).$$

Now, by our stipulation on r_n we have that

$$(36) \qquad |\hat{h}_1 z + g(z)z^2| \leq \hat{h}_1 r_n + G r_n^2 \leq cn^{-1},$$

and so

$$|z^{-1}\hat{T}(z) - 1| = \left| -\frac{\hat{h}_1 + g(z)z}{z^{-1} + \hat{h}_1 + g(z)} \right| \leq \frac{cn^{-1}}{1-c}.$$

Consequently,

$$M_{n,r_n} \leq \left(1 + \frac{e^{c/(1-c)}}{n\hat{h}_1 r_n}\right) M_{n-1,r_n}$$

$$\leq \left(1 + c^{-1} e^{c/(1-c)}\left(1 + \sqrt{G\hat{h}_1^{-2}c}\right)\right) M_{n-1,r_n}$$

$$=: d_2 M_{n-1,r_n} \leq d_2 M_{n-1,r_{n-1}}$$

and thus

$$(37) \qquad M_{n,r} \leq d_2^n M_{0,r_0}.$$

We now aim to bound

$$M_{0,r} = \sup_{|z| \leq r} \left| a_{-1}((\hat{T}(z))^{-1} - z^{-1}) + a_\ell \log(z^{-1}\hat{T}(z)) \right|.$$

From (33) it can be shown that $a_\ell = \hat{h}_1^{-2}g(0)$, and thus $|a_\ell| \leq \hat{h}_1^{-1}G$. Furthermore, (36) gives that

$$|z^{-1}\hat{T}(z)| \geq 1 - n^{-1}c \geq 1 - c,$$

giving that

$$M_{0,r_0} \leq \hat{h}_1^{-1}(\hat{h}_1 + Gr_0) + \hat{h}_1^{-2}G\log((1-c)^{-1})$$
$$\leq 1 + \hat{h}_1^{-2}G(c - \log(1-c)) := d_1 d_2^2$$

where in the last line we used (35).

Thus, since $D_n(z) = \mathcal{O}(z^{n+2})$ as $z \to 0$, for all $|z|$ smaller than r, where r is as in (35),

$$|D_n(z)| \leq (|z|/r_n)^{n+2} \sup_{|w|=r_n} |D_n(w)| \leq d_1 d_2^2 \, d_2^n |z|^{n+2} r_n^{-(n+2)}.$$

Choosing $c = 0.4$ we finally obtain the required bounds. $\qquad\square$

Proof of Lemma 4.2. We proceed by induction on (21) and (22). The base case clearly holds as $\hat{T}^0(z) = z$. Suppose that (21) and (22)hold for some $k \in \mathbb{N}$. Then $|\hat{T}^k(z)| \leq (\mathbb{R}\hat{T}^k(z)^{-1})^{-1} \leq R_1$, where $R_1 := \min\{R, \aleph G^{-1}\hat{h}_1\}$. Because $R_1 \leq R$ we can apply (17), giving

$$(38) \qquad |\hat{T}^{k+1}(z)^{-1} - \hat{T}^k(z)^{-1} - \hat{h}_1| \leq G|\hat{T}^k(z)| \leq \aleph G^{-1}\hat{h}_1.$$

Since

$$\left| \mathbb{R}\hat{T}^{k+1}(z)^{-1} - \mathbb{R}\hat{T}^k(z)^{-1} - \hat{h}_1 \right| \leq \left| \hat{T}^{k+1}(z)^{-1} - \hat{T}^k(z)^{-1} - \hat{h}_1 \right|,$$

we obtain from (38) that (21) must also hold for $k + 1$. Furthermore, since

$$|\hat{T}^{k+1}(z)^{-1} - \hat{T}^k(z)^{-1} - \hat{h}_1| = (\hat{T}^{k+1}(z)^{-1} - (k+1)\hat{h}_1) - (\hat{T}^k(z)^{-1} - k\hat{h}_1),$$

the inequality (38) implies (22) for $k + 1$. $\qquad\square$

Proof of Lemma 4.3. From Lemma 4.2 we have

$$\sum_{k=0}^{\infty} |\hat{T}^k(z)|^{\bar{\beta}} k^\delta \leq \sum_{k=0}^{\infty} (|z^{-1} + \hat{h}_1 k|^{-1} - \aleph\hat{h}_1 k)^{-\bar{\beta}} k^\delta$$
$$\leq \sum_{k=0}^{\infty} \left(\max\{z^{-1}, \hat{h}_1 k\} - \aleph\hat{h}_1 k \right)^{-\bar{\beta}} k^\delta.$$

The summand is increasing for $k \leq \hat{h}_1^{-1}z^{-1}$ and decreasing for larger k. Thus we can use an integral bound:

$$\sum_{k=0}^{\infty} |\hat{T}^k(z)|^{\bar{\beta}} k^\delta \leq \int_0^\infty \left(\max\{z^{-1}, \hat{h}_1 k\} - \aleph\hat{h}_1 k \right)^{-\bar{\beta}} k^\delta \, dk + (1-\aleph)^{-\bar{\beta}}|z|^{\bar{\beta}-\delta}\hat{h}_1^{-\delta}$$
$$= |z|^{-\bar{\beta}-\delta-1}\hat{h}_1^{-\delta-1}(1-\aleph)^{-\bar{\beta}} \left(\frac{2F_1(\bar{\beta}, \delta+1, \delta+2, \aleph)}{(\delta-1)(1-\aleph)^{-\bar{\beta}}} \right.$$
$$\left. + \frac{1}{\bar{\beta}-\delta-1} + \hat{h}_1|z| \right),$$

which using that $_2F_1(\bar{\beta}, \delta+1, \delta+2, \aleph) \leq (1-\aleph)^{-\bar{\beta}}$ and $|z| \leq R_1^{-1}$ gives the desired bound. $\qquad\square$

Proof of Lemma 4.4. We know that $g(z)$ is analytic for complex $|z| \leq R$; as a result, if we define $g_1 := g'(0)$ and $g_2(z) = z^{-1}(g(z) - g_1)$ we have that g_2 is bounded for $|z| \leq R$ by some constant $G_2 < \infty$. Since g maps real inputs to real inputs, we also know that g_1 is real. Combining this with (17), we have for $|z| \leq R_1 \leq R$ that

$$\hat{T}(z)^{-1} - z^{-1} = \hat{h}_1 + g_1 z + g_2(z)z^2,$$

and so taking imaginary parts,

$$\Im\hat{T}(z)^{-1} - \Im z^{-1} = g_1\Im z + \Im(g_2(z)z^2) = -g_1|z|^2\Im z^{-1} + \Im(g_2(z)z^2).$$

We can then bound the growth in the imaginary part of z^{-1} under iteration by $\hat{T}$:

$$|\Im\hat{T}(z)^{-1}| \leq (1 + g_1|z|^2)|\Im z^{-1}| + G_2|z|^2.$$

Since for $\Re z \leq R_1$ we have from Lemma 4.2 that $|\hat{T}^k(z)| \leq R$ for all $k \in \mathbb{N}$, we obtain the linear recurrence relation

$$|\Im\hat{T}^{k+1}(z)^{-1}| \leq (1 + g_1|z|^2)|\hat{T}^k(z)^{-1}| + G_2|z|^2.$$

Since by Lemma 4.2, $|z|^2 = \mathcal{O}(k^{-2})$ for all $\Re z \leq R_1$, iterates of this equation are bounded, as required. $\qquad\square$

Acknowledgements

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