Spectral theory and approximation of Koopman operators in chaos

Part 1: smooth functional analysis of dynamical operators

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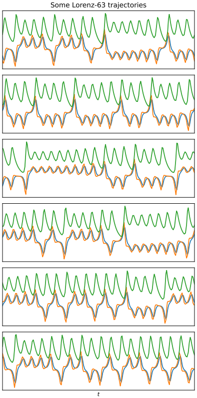
Trying to make sense of dynamical systems

The basic object of study in “dynamics” is a trajectory $\{x_t\}_{t=0,1,2,\ldots}$ in a state space D.

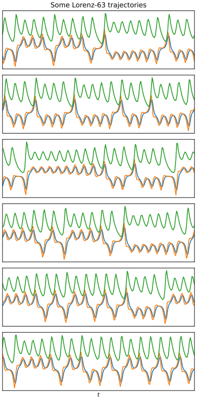
- The set of trajectories of a dynamical system can be very complicated (like people)
- Tendency to study mean behaviour (or expected behaviour with respect to some loss. . .)
- “Mean”,“expectation” implies study with respect to probability. In dynamics, this is *ergodic theory*



Smooth ergodic theory

$$\frac{1}{T}\sum_{t=0}^{T}\psi(x_t) \xrightarrow{T\to\infty} \int_D \psi(x_t)\,\mathrm{d}\mu(x) \text{ for } \mu\text{-almost all } x$$

- Birkhoff ergodic theorem (true in many systems): time averages over orbit = spatial average
  - Means almost every orbit looks like every other orbit at some point in time ("almost all orbits are dense").
- We want to compute so want, e.g. stability to error, ideally with quantitative assurances.
- This suggests studying *smooth* ergodic theory.



Operators

So, we are interested in studying

- The Koopman operator
- The transfer operator ($\approx$ Perron-Frobenius operator)

on function spaces involving differentiability.

It is mostly a separate beast to the $L^p(\mu)$ theory.

Stochastic vs deterministic

Broadly are going to consider three kinds of dynamical system (on compact manifolds):

- **Noisy dynamics** (SDEs. . .): theoretically easy, good starting point for comparison
- **Deterministic contractions**: already known, but explains some of the questions in deterministic dynamics. . .
- **Deterministic chaos**: harder and a bit obscure, main goal of this minicourse

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Then you commit to resolving down to the scale of the noise! How can we believe lower-resolution numerics?

Dual notions

(Markov, time-autonomous) stochastic dynamics $x_t \in D$ are typically studied using two linear operators, acting on functions on the state space $\psi, \varphi : D \to \mathbb{R}$:

- The Chapman-Kolmogorov operator: predicting the expected future value of "observable" functions

$$(\mathcal{K}\psi)(x) = \mathbb{E}[\psi(x_{t+1}) | x_t = x]$$

- The Fokker-Planck operator: evolution of probabilities into future (= push-forward of measure density)

$$(\mathcal{L}\varphi)(x) = \int \varphi(y) \, \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}x}[x_{t+1} = x | x_t = y] \, \mathrm{d}y$$

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Dual notions

Define our transition kernel $k(x,y) = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}y}[x_{t+1} = y | x_t = x]$, then

$$(\mathcal{L}\varphi)(y) = \int \varphi(x)\, k(x,y)\, \mathrm{d}x$$

whereas

$$\begin{aligned}(\mathcal{K}\psi)(x) &= \mathbb{E}[\psi(x_{t+1}) | x_t = x] \\ &= \int \psi(y) \mathrm{d}\mathbb{P}[x_{t+1} = y | x_t = x] \\ &= \int \psi(y)\, k(x,y)\, \mathrm{d}y.\end{aligned}$$

So, these operators are dual:

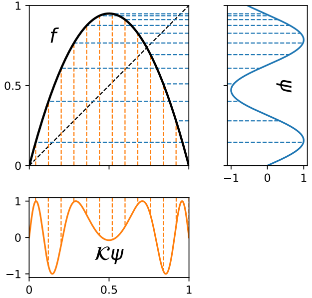
$$\int (\mathcal{L}\varphi)(y)\, \psi(y)\, \mathrm{d}y = \int \varphi(y)\, (\mathcal{K}\psi)(y)\, \mathrm{d}y.$$

Dual notions

These actually work with deterministic maps $f : D \to D$ as well (modulo "some" intricacies. . .)

- The **Koopman operator**: expected future value of functions (aka "observables")

$$(\mathcal{K}\psi)(x) = \psi(f(x))$$

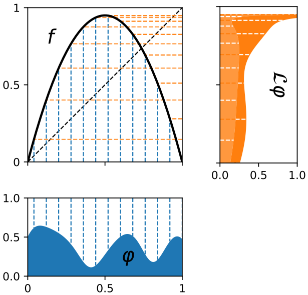


Dual notions

- The **transfer operator**, which gives you the evolution of probabilities into the future (= push-forward of measure density)

$$(\mathcal{L}\varphi)(x) = \int \text{“}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}x}[f(y) = x]\text{”}\, \varphi(y)\,\mathrm{d}y$$

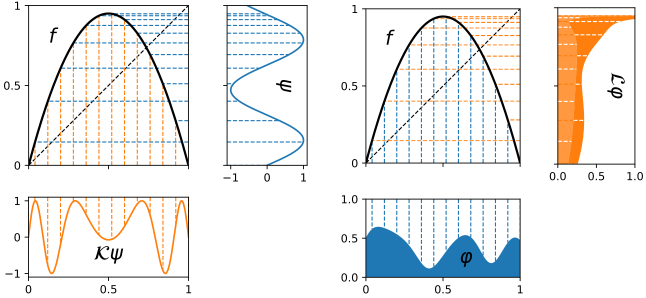
$$= \sum_{f(y)=x} \frac{\varphi(y)}{|\det Df(y)|}$$



Dual notions

These two operators are also still dual!

$$\int \varphi(x)\,(\mathcal{K}\psi)(x)\,\mathrm{d}x = \int (\mathcal{L}\varphi)(x)\,\psi(x)\,\mathrm{d}x$$



Dynamics

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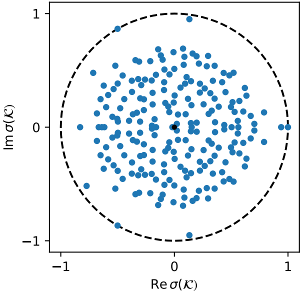
- We can describe dynamics over long times by $\mathcal{K}^n$, $\mathcal{L}^n$, for n large.
- These are best described by the *spectrum* of $\mathcal{K}, \mathcal{L}$.
- Consequently, Koopman/transfer spectra can:
  - Provide reductions for the dynamics
  - Give you statistical information about the system
  - Make sense of the emergent dynamical geometry...

Koopman spectra

What are the spectra of these (infinite-dimensional, weirdly-posed) operators?

Generally computer approximations of the Koopman spectrum look like:



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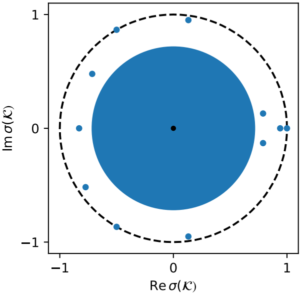

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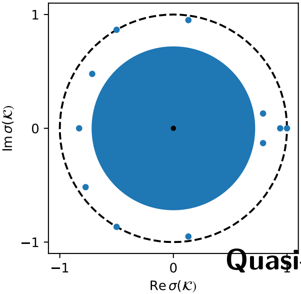
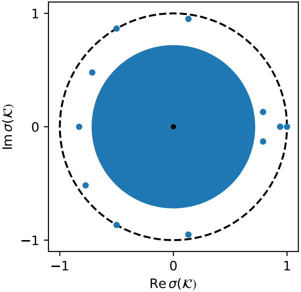


Quasi-compact operator

Structure

A lot of numerics around dynamical systems come down to studying the spectra of these operators numerically.
This lecture series will talk about

1. A mathematical framework that explains quasi-compact operators
2. How different kinds of dynamics fit into the framework
3. How this translates to computation

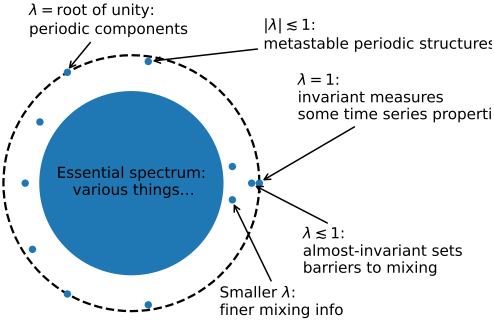


Meaning of the spectrum

Quasicompactness (and $\sigma_{\mathrm{d}}(\mathcal{K})$ with no Jordan blocks) gives:

$$\mathcal{K}^n \psi = \sum_{\substack{\lambda_k \in \sigma_{\mathrm{d}}(\mathcal{K}) \\ \mathcal{K}\psi_k = \lambda_k \psi_k}} c_k(\psi)\, \lambda_k^n\, \psi_k + \mathcal{O}(\rho_{\mathrm{ess}}(\mathcal{K})^n)$$

Different parts of the spectrum have various interpretations:



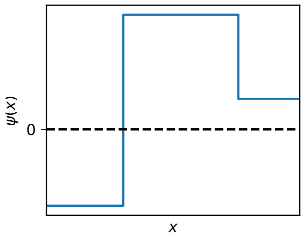
$\lambda = 1$: ergodic components

Proposition

Suppose a **function** $\psi : D \to \mathbb{R}$ *satisfies* $\psi \circ f = \psi$. *Then the level sets of* ψ *are invariant sets.*

Proof.

If $x \in \psi^{-1}(c)$, then $\psi(f(x)) = \psi(x) = c$, so $f(x) \in \psi^{-1}(c)$. □



$|\lambda| = 1$: periodicity

Proposition

Suppose a **function** $\psi : D \to \mathbb{R}$ *satisfies* $\psi \circ f = e^{i\theta\psi}$ *for some* $\theta \in [0, 2\pi]$. *Let* $E_z = \{x \in D : \psi(x) = z\}$.
Then f *maps* E_z *into* $E_{e^{i\theta}z}$.

Proof.

If $x \in E_z$, then $\psi(f(x)) = e^{i\theta}\psi(x) = e^{i\theta}z$, so $f(x) \in E_{e^{i\theta}z}$. □

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$\lambda \lesssim 1$: almost invariant sets

Proposition (à la Froyland and Stancevic '10)

Suppose ψ *satisfies* $\mathcal{K}\psi = \lambda\psi$ *and* $\sup|\psi| \leq 1$ *for some* $\lambda \in (0,1)$. *Let*

$$E_+ = \{x \in D : \psi(x) > 0\}.$$

Then for some $C > 0$,

$$\int_D \mathbb{P}\left(f^t(x) \in E_+ \text{ for } t = 1, \ldots, n\right) \mathrm{d}x \geq C\lambda^{n-1}$$

That is, the Lebesgue measure of the set of points that don't leave E_+ within n steps decays as $\mathcal{O}(\lambda^n)$.
In particular $n \sim 1/(1-\lambda)$, this set is of $\mathcal{O}(1)$ Lebesgue measure.

$\lambda \lesssim 1$: almost invariant sets

Proof.

We are interested in

$$P_n = \int_D \mathbb{P}\left(x_t \in E_+ \text{ for } t = 0\dots, n-1 | x_t = x\right) \mathrm{d}x.$$

The interior probability we can rewrite as

$$\mathbb{E}\left(\prod_{t=0}^{n-1} \mathbb{1}_{E_+}(x_t) | x_0 = x\right)$$

$$\mathbb{1}_{E_+}(x) = \begin{cases} 1 & x \in E_+ \\ 0 & x \notin E_+ \end{cases}$$

- If $n = 1$, this is $\mathbb{E}[\mathbb{1}(x_0 | x_0 = x] = \mathbb{1})(x)$.
- If $n = 2$, this is $\mathbb{E}[\mathbb{1}_{E_+}(x_0)\mathbb{1}_{E_+}(x_1) | x_0 = x] = \mathbb{1}_{E_+}(x)\mathcal{K}[\mathbb{1}_{E_+}](x)$.
- If $n = 3$, this is $\mathbb{E}[\mathbb{1}_{E_+}(x_0)\mathbb{1}_{E_+}(x_1)\mathbb{1}_{E_+}(x_2) | x_0 = x] = \mathbb{1}_{E_+}(x)\mathcal{K}[\mathbb{1}_{E_+}\mathcal{K}[\mathbb{1}_{E_+}]](x)$.
- By induction, we have $(\mathbb{1}_{E_+}\mathcal{K})^{n-1}[\mathbb{1}_{E_+}](x)$.

$\lambda \lesssim 1$: almost invariant sets

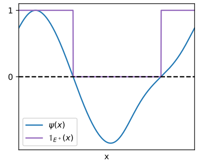
Proof (continued).

Now, $\mathcal{K}$ is a positive operator (i.e. $a \geq b \implies \mathcal{K}a \geq \mathcal{K}b$), and $\mathbb{1}_{E_+} \geq \psi$, so

$$(\mathcal{K}\mathbb{1}_{E_+})^{n-1}[\mathbb{1}_{E_+}](x) \geq (\mathcal{K}\mathbb{1}_{E_+})^{n}[\psi](x)$$

Furthermore, $\mathbb{1}_{E_+}\mathcal{K}\psi = \lambda\mathbb{1}_{E_+}\psi \geq \lambda\psi$.
So, inductively,

$$(\mathbb{1}_{E_+}\mathcal{K})^{n-1}[\psi](x) \geq \lambda^{n-1}\mathbb{1}_{E_+}\psi(x).$$



So for some $C > 0$,

$$P_n \geq \int_D \lambda^{n-1}\mathbb{1}_{E_+}(x)\psi(x)\,\mathrm{d}x = C\lambda^{n-1}.$$

□

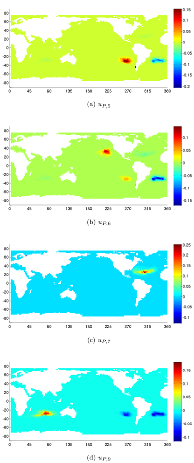
$\lambda \lesssim 1$: almost invariant sets

- Comparable results for complex λ. If

$$E_\theta = \{x \in D : \Re[e^{i\theta}\psi(x)] > 0\},$$

  you have that E_θ mostly maps to $E_{\theta+\arg\lambda}$.
- Same result for transfer operator $\mathcal{L}$ (again if bounded eigenfunctions).

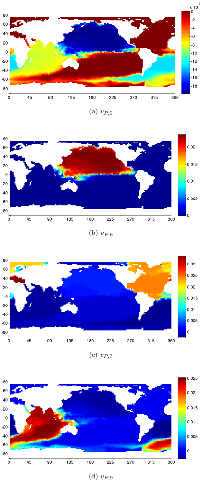
$\lambda \lesssim 1$: garbage patch example



(a) $u_{P,5}$ (b) $u_{P,6}$ (c) $u_{P,7}$ (d) $u_{P,9}$

FIG. 4: Maps of selected left eigenvectors of P showing the locations of the five great ocean garbage patches.

Transfer operator eigenfunctions



(a) $v_{P,5}$ (b) $v_{P,6}$ (c) $v_{P,7}$ (d) $v_{P,9}$

FIG. 5: Maps of right eigenvectors $\{v_{P,r}\}$ of P.

Koopman eigenfunctions

λ	P
λ_1	1.0000
λ_2	1.0000
λ_3	1.0000
λ_4	1.0000
λ_5	0.9999
λ_6	0.9999
λ_7	0.9996
λ_8	0.9991
λ_9	0.9975
λ_{10}	0.9913
λ_{11}	0.9852
λ_{12}	0.9838
λ_{13}	0.9826
λ_{14}	0.9680
λ_{15}	0.9645

Note: in a Lebesgue-orthogonal basis, $\mathcal{L}$ is the transpose of $\mathcal{K}$

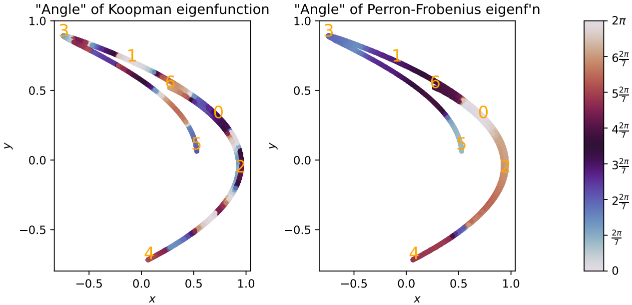
Same spectrum, different eigenfunctions.

Transfer: attractors

Koopman: basins of attraction

Koopman and transfer operator eigenfunctions

Using EDMD I just computed that a Hénon-like map has an eigenvalue $\lambda \approxeq 0.92 e^{-6i\pi/7}$.
This suggests some sort of 7-periodic behaviour persisting over timescale $\sim 1/-\log 0.92 = 13$.



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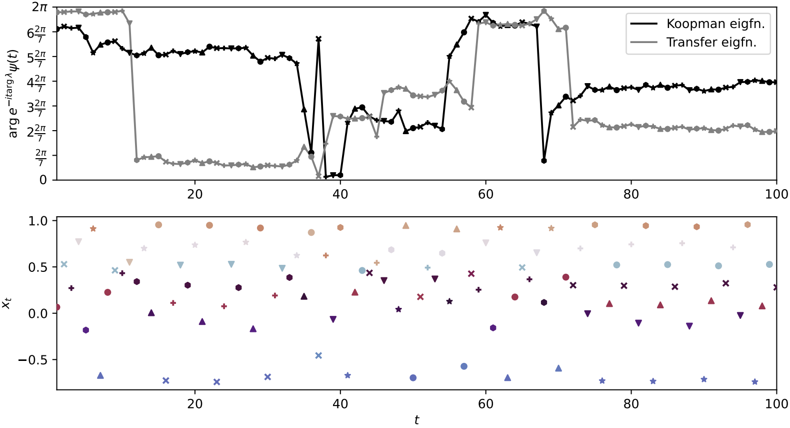


A motivating conundrum

This lecture, let's try and get a theoretical grip on the spectrum of the Koopman operator for a deterministic contraction ($\kappa < 1$):

$$f(x) = \kappa x, \; x \in [-1, 1]$$

So the Koopman operator is

$$(\mathcal{K}\psi)(x) = \psi(\kappa x)$$

What does its spectrum look like?
Let's try and find some eigenfunctions.

A motivating conundrum

Let's sub in a power series $\psi(x) = \sum_{k=0}^{\infty} a_k x^k$:

$$0 = \mathcal{K}\psi - \lambda\psi$$
$$= \sum_{k=0}^{\infty} a_k \kappa^k x^k - \lambda \sum_{k=0}^{\infty} a_k x^k$$

Equating terms we get

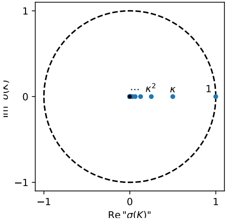
$$(\kappa^k - \lambda) a_k = 0, k \in \mathbb{N}$$

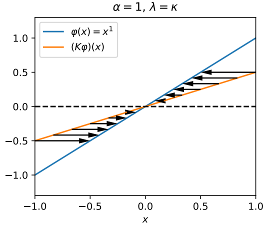
suggesting that our eigenvalues are $\{1, \kappa, \kappa^2, \kappa^3, \ldots\}$ with the respective eigenfunctions $\{1, x, x^2, x^3, \ldots\}$

A motivating conundrum

These eigenfunctions broadly give us what we expect:

- The leading eigenfunction 1 is simple (i.e. no separate basins of attraction*)
- The next eigenfunction has a root at $x = 0$, suggesting a dynamical barrier here (in fact, it is precisely the fixed point—woohoo)
- Some other spectrum accumulating at 0





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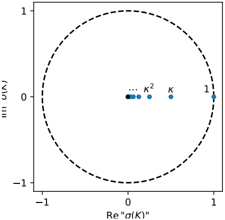
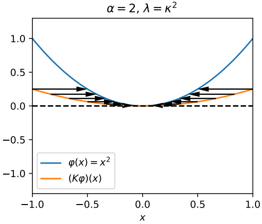
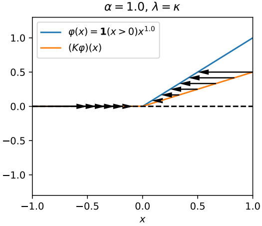




A motivating conundrum

But, by the same token, we could try $\mathcal{K}\mathbb{1}(x > 0)x = \mathbb{1}(x > 0)x$, and so on...



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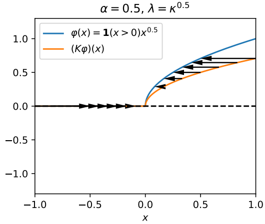
But, by the same token, we could try
$\mathcal{K}\mathbb{1}(x > 0)x^{1/2} = \sqrt{\kappa}\mathbb{1}(x > 0)x^{1/2}$, and so on...



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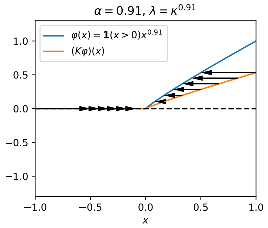

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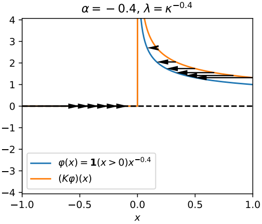

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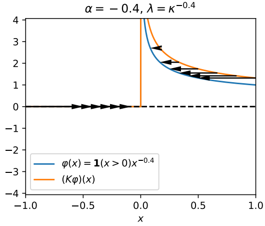


In fact, for any $\alpha \in \mathbb{C}$, we can set $\psi(x) = x^{\alpha} = e^{-\alpha \log x}$ for $x > 0$ and get

$$(\mathcal{K}\phi)(x) = e^{-\alpha(\log x - \log \kappa)} = \kappa^{-\alpha}\psi(x)$$

So, $\mathcal{K}$'s spectrum could cover the complex plane unless we are a bit careful about what functions we allow.

A motivating conundrum

In fact, this comes from a bigger fact which is that Koopman eigenfunctions/eigenvalues are multiplicative in deterministic dynamics.

In general, if

- $\mathcal{K}\psi = \lambda\psi$
- f is deterministic;
- ψ^α is well-defined;

then $\mathcal{K}[\psi^\alpha] = \lambda^\alpha\psi^\alpha$.

A motivating conundrum

How do we allow/banish functions from our linear operator $\mathcal{K}$? We **set a function space as the domain of $\mathcal{K}$.**
Crucial properties of this function space $\mathcal{B}$:

- It is a vector space.
- It has a norm $\|\cdot\|$, with respect to which it is complete (i.e. it's a *Banach space*)
- $\mathcal{K}$ maps $\mathcal{B}$ to itself.
- It doesn't have to contain *only* functions, but should contain all sufficiently nice functions (e.g. C_c^∞)

Note that to do theory it isn't very helpful to have a Hilbert space, except in some cases.

Spectrum of an infinite-dimensional operator

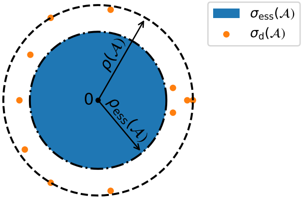
Define the resolvent of an operator $\mathcal{A} : \mathcal{B} \to \mathcal{B}$:

$$R_\lambda(\mathcal{A}) = (\mathcal{A} - \lambda I)^{-1} : \mathcal{B} \to \mathcal{B}$$

The spectrum $\sigma(\mathcal{A})$ is the set of $\lambda \in \mathbb{C}$ where $R_\lambda(\mathcal{A})$ is either not well-defined, or unbounded. It is always closed.
The spectrum includes:

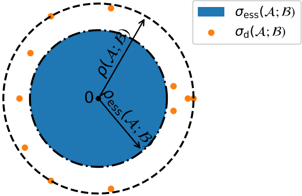
- The discrete spectrum $\sigma_{\mathrm{d}}(\mathcal{A})$, i.e. isolated eigenvalues λ of $\mathcal{A}$ with finite "algebraic multiplicity".
  The nice normal stuff we love from finite-dimensional operators.
- The rest $\sigma_{\mathrm{ess}}(\mathcal{A})$—the "essential spectrum". For Koopman operators in discrete time it is *usually* a ball around 0.

Spectral radii



- Spectral radius $\rho(\mathcal{A}) = \max|\sigma(\mathcal{A})|$
- Essential spectral radius $\rho_{\mathrm{ess}}(\mathcal{A}) = \max|\sigma_{\mathrm{ess}}(\mathcal{A})|$

Spectral radii



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Important to remember these depend on the function space $\mathcal{B}$...

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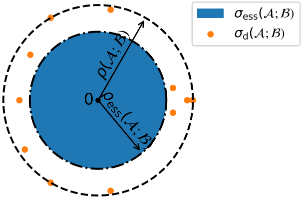


- Spectral radius $\rho(\mathcal{A};\mathcal{B}) = \max|\sigma(\mathcal{A};\mathcal{B})| \leq \|\mathcal{A}\|_{\mathcal{B}})$
- Essential spectral radius $\rho_{\mathrm{ess}}(\mathcal{A};\mathcal{B}) = \max|\sigma_{\mathrm{ess}}(\mathcal{A};\mathcal{B})|$

Important to remember these depend on the function space $\mathcal{B}$...

Compact operators

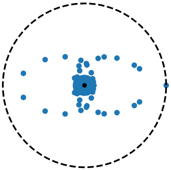
An operator $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ is *compact* if $\overline{\mathcal{A}(B_{\mathcal{B}_1}(0,1))}$ is a compact subset of $\mathcal{B}_2$.

If there exist some operators $\mathcal{A}_N : \mathcal{B}_1 \to \mathcal{B}_2$ such that

$$\|\mathcal{A}_N - \mathcal{A}\|_{\mathcal{B}_1 \to \mathcal{B}_2} \xrightarrow{N \to \infty} 0,$$

then $\mathcal{A}$ is compact. In most reasonable cases (e.g. Hilbert spaces, $\mathcal{B}_1 = \mathcal{B}_2$ has a countable Schauder basis. . .) this is iff.

- Compact operators' only essential spectrum is at $\lambda = 0$. So $\rho_{\text{ess}} = 0$.
- However, there can be countably discrete eigenvalues, which then accumulate at zero.



Stochastic systems

Remember that for most nice stochastic systems (e.g. SDE maps), the Koopman operator is a kernel operator:

$$(\mathcal{K}\psi)(y) = \mathbb{E}[\psi(x_{t+1})|x_t = y] = \int \psi(x)\, k(y,x)\, \mathrm{d}x.$$

Usually k is reasonably regular: for example,
$\int_D |\nabla_y k(y,x)|\, \mathrm{d}x \leq C$ for all x.
In this case, we find that for all $y \in D$,

$$|\nabla_y (\mathcal{K}\psi)(y)| = \int_D |\nabla_y k(y,x)||\psi(x)|\, \mathrm{d}x \leq C \sup_{x\in D} |\psi(x)|.$$

This is a nice bound, which we can translate into functional analysis as follows. . .

An estimate that is always true

Let's take $\mathcal{B}$ to be the space of bounded functions on D with the norm:

$$\|\psi\|_{\mathcal{B}} = \sup_{x \in D} |\psi(x)|.$$

Then, $\mathcal{K}$ always maps bounded functions to bounded functions by virtue of the following:

$$\|\mathcal{K}\psi\|_{\mathcal{B}} = \sup_{x \in D} |\mathbb{E}[\psi(x_{t+1})|x_t = x]| \leq \sup_{y \in D} |\psi(y)| = \|\psi\|_{\mathcal{B}}$$

And you can see it has norm (so spectral radius) bounded by 1!

Stochastic systems

In our stochastic system, let's also define a "strong" space C^1, of all the continuously differentiable functions on $[-1, 1]$, with the norm

$$\|\psi\|_{C^1} = \sup_{x \in D} |\nabla \psi(x)| + \sup_{x \in D} |\psi(x)| = \|\nabla \psi\|_{\mathcal{B}} + \|\psi\|_{\mathcal{B}}.$$

Then, our bound from before translates to saying

$$\|\mathcal{K}\psi\|_{C^1} \leq C\|\psi\|_{\mathcal{B}}.$$

So $\mathcal{K}$ makes our functions smoother!

Compact embedding

Can we use this to say anything about the compactness of $\mathcal{K}$ in $\mathcal{B}$?

Proposition

The product of a bounded operator and a compact operator (resp. approximable by finite rank) is compact (resp. approximable by finite rank).

Imagine $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ as the following chain:

$$\mathcal{B} \xrightarrow{\mathcal{K}} C^1 \xrightarrow{\text{id}} \mathcal{B}$$

If we can show that $\text{id} : C^1 \to \mathcal{B}$ is compact (aka C^1 embeds compactly into $\mathcal{B}$, which we notate $C^1 \Subset \mathcal{B}$)...

then $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ is compact.

Compact embedding

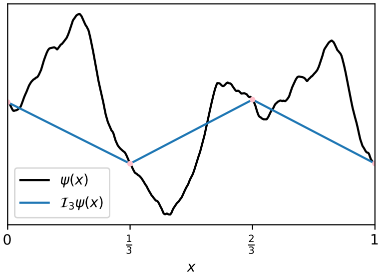
Let's try and construct some finite-dimensional operators that approach $\mathrm{id} : C^1 \to \mathcal{B}$ in norm.

- That is, let's find some finite-dimensional operators that give uniformly good approximations of differentiable functions.

For simplicity, we'll do it on the interval $[0, 1]$.

Compact embedding

For every $\psi \in C^1$, let's define $\mathcal{P}_N \psi$ to linearly interpolate ψ at $S_N = \{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$:



(Exercise: show $\mathcal{P}_N$ is a linear operator.)

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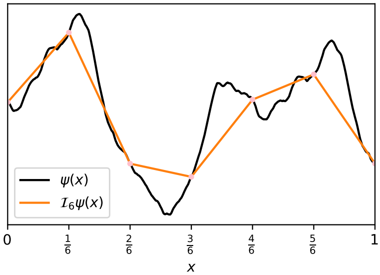

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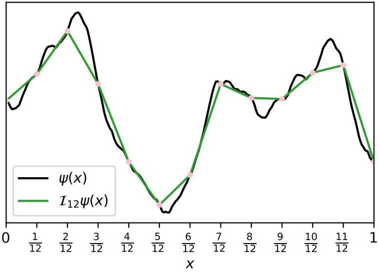

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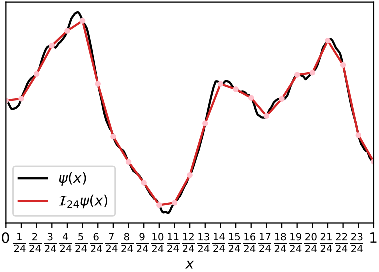

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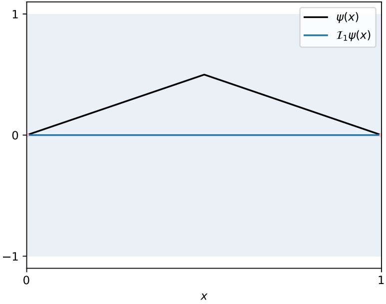
Compact embedding

$\mathcal{P}_N$ is finite rank, and for all $\psi \in C^1$,

$$\|\mathcal{P}_N\psi - \psi\|_{\mathcal{B}} \leq \frac{1}{N}\|\psi\|_{C^1}$$

(Exercise: prove this)



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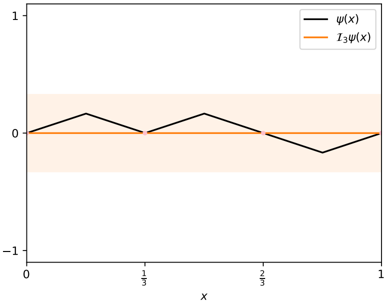

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$\mathcal{P}_N$ is finite rank, and for all $\psi \in C^1$,

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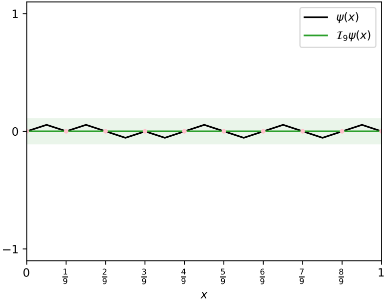


Compact embedding

So $C^1 \Subset \mathcal{B}$.

- The Koopman operator $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ is compact!
- So it only has point spectrum!

Thus, we have proven that all stochastic systems on compact manifolds with differentiable kernels have compact Koopman operators!

Computing with compact operators

- In proving compactness, we came up with a nice approximation scheme (interpolation).
- We could try and approximate our Koopman operator $\mathcal{K}$ by $\mathcal{K}_N := \mathcal{P}_N \mathcal{K}$, perhaps restricting to $\operatorname{im} \mathcal{P}_N$, i.e. piecewise linear functions.
- This approximation $\mathcal{K}_N$ is $\mathcal{O}(1/nN$-close in norm to $\mathcal{K}$, so its simple eigenvalues should be $\mathcal{O}(1/N)$ error...

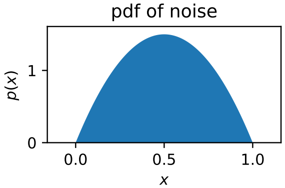
Theorem

Suppose that $\lambda \in \sigma_d(\mathcal{A}; \mathcal{B})$ with algebraic multiplicity L.
Suppose $\|\mathcal{A}_N - \mathcal{A}\|_{\mathcal{B}} \to 0$.
Then each $\mathcal{A}_N$ has L eigenvalues (counting multiplicity) $\lambda_N^1, \dots \lambda_N^L$ such that for large enough N, each

$$|\lambda_N^{(l)} - \lambda_N| \leq C \|\mathcal{A}_N - \mathcal{A}\|_{\mathcal{B}}^{1/L}.$$

Computational example

Let's set $x_{t+1} = 3.54x_t(1 - x_t) + 0.08\Xi$, where Ξ is i.i.d. noise with the following pdf $p(\xi) = \mathbb{1}_{[0,1]}(\xi)6\xi(1 - \xi)$:



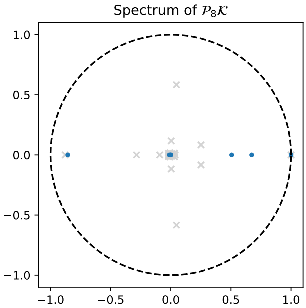
Then the kernel defining the Koopman operator is

$$k(x, y) = 0.08^{-1} p\left(\frac{y - 3.54x(1 - x)}{0.08}\right)$$

and we can try and compute $\mathcal{K}$ on $\operatorname{im} \mathcal{I}_N$.

Computational example

Spectrum converges (as $\mathcal{O}(1/N)$, eventually).



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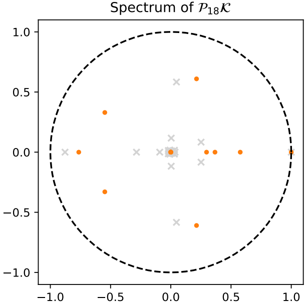

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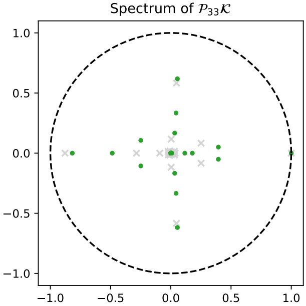

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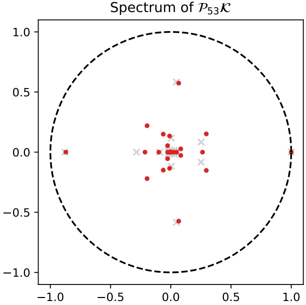

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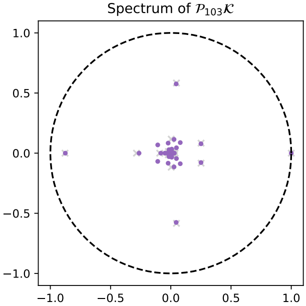

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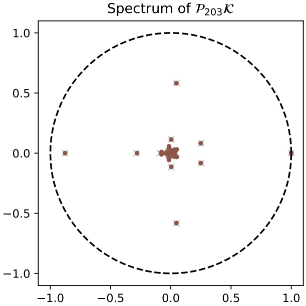

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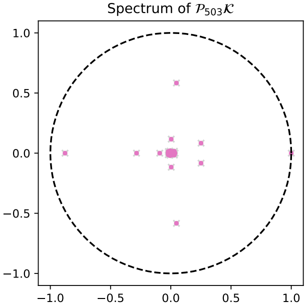
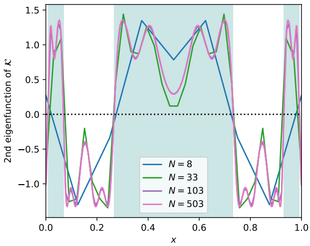
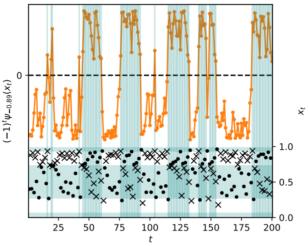


Computational example

Let's look at the Koopman eigenfunction for $\lambda = -0.878$ (so some set for which the period-two map is almost-invariant):





Compactness and Koopmanism

Saddish news: for most deterministic systems, the Koopman operator isn't expected to be compact on any reasonable Banach spaces.
We will see why later.

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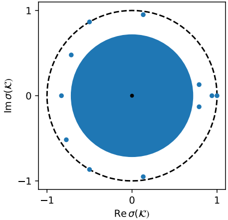
Next best option (should be possible 97%[1] of the time):

A **quasi-compact operator**

[1] 100% minus the probability of having a circle map, essentially.

Quasi-compactness

An operator is quasi-compact if it has this spectral picture:



Suppose we are only thinking about positive operators with spectral radius $= 1$ (e.g. Koopman/transfer). Then

- A quasi-compact operator has $\rho_{\text{ess}}(\mathcal{A}) < 1$.
- A quasi-compact Koopman operator has some discrete spectrum.
- Quasicompact operators are the sum of a compact operator and an operator with an iterate that is a contraction.
  - Why? $\sigma_{\text{ess}}(\mathcal{A} + \mathcal{C}) = \sigma_{\text{ess}}(\mathcal{A})$ when $\mathcal{C}$ is compact.

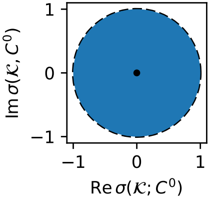
Contraction on C^0

Let's go back to $f(x) = \kappa x, x \in [-1, 1]$.
Let's take $\mathcal{B} = C^0$, the space of bounded, continuous functions on $[-1, 1]$ with the sup-norm. Then $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ since f is continuous, and

$$\|\mathcal{K}\psi\| = \sup_{x\in[-1,1]} |\psi(f(x))| \leq \sup_{x\in[-1,1]} \psi(x) = \|\psi\|$$

so $\rho(\mathcal{K}; C^0) \leq 1$.

Then, eigenfunctions $\psi_\alpha(x) := \mathbb{1}(x > 0)e^{\alpha \log x}$ are in C^0 for $\Re\alpha \leq 0$.
Corresponding eigenvalues are κ^α, so $\sigma(\mathcal{A}; C^0([-1, 1]))$ fills the whole (closed) unit ball.



Just cts. spectrum!

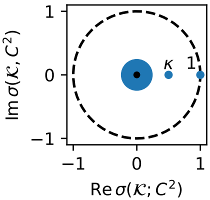
Contraction on C^r spaces

What about some spaces that remove more of the ψ_α? Let's try C^r, the space of r-times continuously differentiable functions on $[-1,1]$. The following norm on C^r works:

$$\|\psi\|_{C^r} = \|\psi^{(r)}\|_{C^0} + \psi(0) + \psi'(0) + \ldots + \psi^{(r-1)}(0).$$

We have that $\psi_\alpha^{(r)}(x) = \alpha(\alpha-1)\cdots(\alpha-r+1)\psi_{\alpha-r}(x)$, so ψ_α is in C^r if either:

- $\psi_{\alpha-r}$ is in C^0, i.e. $\Re\alpha > r$. So $B(0,\kappa^r)$ is in the spectrum.
- α is one of $0,1,2,\ldots,r-1$, i.e. $\psi_\alpha = 1, x, x^2, \ldots, x^{r-1}$. So $1, \kappa, \kappa^2, \ldots, \kappa^{r-1}$ are in the spectrum.



Contraction on C^r spaces

Is there anything else?
Well, let's try and do an eigendecomposition. Recalling that every function in C^r can be written as

$$\psi(x) = \psi(0) + \psi'(0)x + \ldots + \frac{\psi^{(r-1)}(0)}{(r-1)!}x^{r-1} + \mathcal{O}(x^r),$$

we can decompose

$$C^r = \langle 1 \rangle \oplus \langle x \rangle \oplus \cdots \oplus \langle x^{r-1} \rangle \oplus \underbrace{\left\{\psi \in C^r : \psi^{(l)}(0) = 0 \text{ for } l < r\right\}}_{=:V}.$$

All these subspaces are $\mathcal{K}$-invariant, and $\sigma(\mathcal{K})$ is the union of the spectrum of $\mathcal{K}$ restricted to these subspaces.
Only what happens on V we are uncertain of.

Contraction on C^r spaces

$$\|\psi\|_{C^r} = \|\psi^{(r)}\|_{C^0} + \psi(0) + \psi'(0) + \ldots + \psi^{(r-1)}(0).$$

$$\left\{\psi \in C^r : \psi^{(l)}(0) = 0 \text{ for } l < r\right\}$$

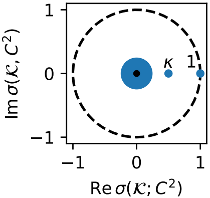
For $\psi \in V$ we have

$$\begin{aligned} \|\mathcal{K}\psi\|_{C^r} = \|(\mathcal{K}\psi)^{(r)}\|_{C^0} &= \sup_{x\in[-1,1]} \|\kappa^r \psi^{(r)}(\kappa x)\|_{C^0} \\ &= \kappa^r \sup_{y\in[-\kappa,\kappa]} |\psi^{(r)}(y)| \leq \kappa^r \|\psi\|_{C^r} \end{aligned}$$

so $\sigma(\mathcal{K}|_V)$ is a subset of $B(0,\kappa^r)$.

This means the spectrum of $\mathcal{K}$ on C^r is

$$\sigma(\mathcal{K}, C^r) = \underbrace{\overline{B(0,\kappa^r)}}_{\text{essential}} \cup \underbrace{\{\kappa^{r-1}, \kappa^{r-2}, \ldots \kappa, 1\}}_{\text{discrete}}.$$

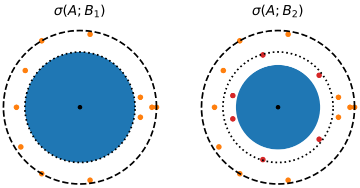


Spectrum vs function space

In general, essential spectrum will vary by function space, but the discrete eigenvalues are more canonical:

Lemma (simplified from Baladi and Tsujii, '08)

Suppose Banach space $\mathcal{B}_2$ is a dense *subset of Banach space $\mathcal{B}_1$, and $\mathcal{A}$ is bounded on both $\mathcal{B}_2$ and $\mathcal{B}_1$.*
Then, the discrete spectrum of $\mathcal{A}$ with absolute value greater than $\max\{\rho_{ess}(\mathcal{A};\mathcal{B}_1), \rho_{ess}(\mathcal{A};\mathcal{B}_2)\}$ *matches in $\mathcal{B}_1$ and $\mathcal{B}_2$ (ditto multiplicity, eigenfunctions).*



Sidenote: spaces of fractional differentiability

We can continuously interpolate between C^r spaces by looking, e.g., at Hölder continuity. The β-Hölder constant of a function is given by

$$H_\beta(\psi) = \sup_{x,y\in[-1,1]} \frac{|\psi(x) - \psi(y)|}{|x-y|^\beta}, \beta \in (-1, 1]$$

Then the $C^{r+\beta}$ norm of ψ is given by

$$\|\psi\|_{C^{r+\beta}} = \|\psi\|_{C^r} + H_\beta(\psi^{(r)}).$$

i.e. $C^{r+\beta}$ consists of functions whose the rth derivative is β-Hölder.

The essential spectral radius is $\rho_{\mathrm{ess}}(\mathcal{K}, C^{r+\beta}) = \kappa^{r+\beta}$, with discrete eigenvalues $\{1, \kappa, \ldots, \kappa^r\}$.