Linear response for macroscopic observables in high-dimensional systems

Caroline Wormell

The University of Sydney

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Joint work with Georg Gottwald

Setting

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The physical measure encodes long-term ergodic behaviour of x_n.
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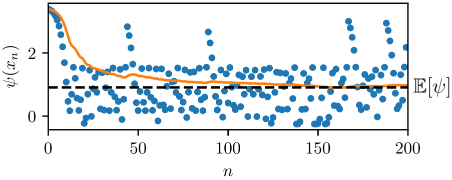
$$\frac{1}{N}\sum_{n=0}^{N-1}\Phi(x_n) \xrightarrow{N\to\infty} \int \Phi(x)\,\mathrm{d}\mu(x) =: \mathbb{E}[\Phi]$$



Setting

Consider a smooth family of mixing chaotic dynamical systems $x_n = T^\varepsilon(x_{n-1})$, with physical invariant measures μ^ε.
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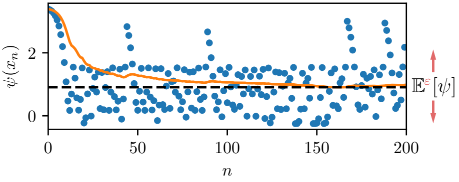

Linear response theory

$$\mathbb{E}^{\varepsilon}[\Phi] := \int \Phi(x)\,\mathrm{d}\mu^{\varepsilon}(x)$$

Linear response theory (LRT) answers: *What is* $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mu^{\varepsilon}$*?*
(e.g. for Taylor approximations)

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(e.g. for Taylor approximations)

. . . supposing $\mathbb{E}^{\varepsilon}[\Phi]$ is differentiable

LRT in practice

The application of linear response theory to climate systems has met with some success:

- Toy models: Majda et al '07, '10, Lucarini & Sarno '11
- Barotropic models: Bell '80, Gritsun & Dymnikov '99, Abramov & Majda '09
- Quasi-geostrophic models: Dymnikov & Gritsun '01
- Atmospheric models: North et al '04, Cionni et al '04, work of Gritsun and others '02, '07, '10, Ring & Plumb '08
- Coupled climate models: Langen & Alexeev '05, Kirk & Davidoff '09, Fuchs et al '14, Ragone et al '15

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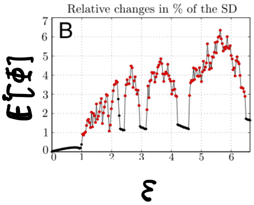
- Rough responses are known in atmospheric and ocean dynamics (e.g. Chekroun et al. ’14)

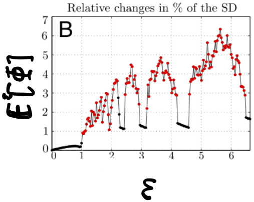


Chekroun et al., 2014

LRT in practice

However:

- Rough responses are known in atmospheric and ocean dynamics (e.g. Chekroun et al. '14)
- The failure of linear response needs very long time series to be visible (Gottwald, W. & Wouters '16)



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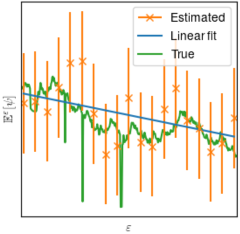

LRT in theory

Analytically, we know LRT works in

- Statistical mechanics: Kubo '66
- Stochastic dynamical systems: Hänggi '78, Hairer & Majda '10
- Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97-8

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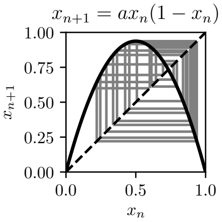
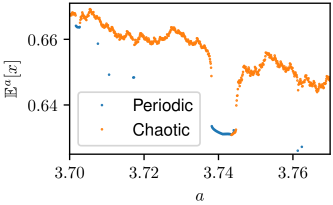
- Statistical mechanics: Kubo '66
- Stochastic dynamical systems: Hänggi '78, Hairer & Majda '10
- Axiom A (uniformly hyperbolic dissipative chaos): Ruelle '97-8
- Other dissipative systems. . . ?

Baladi and others ('08, '10, '14, '15) proved there is no **linear response for quadratic maps, even Whitney differentiability**.





The question

In this talk we will address the following question:

> *When and why does linear response occur at macroscopic scales in high-dimensional systems?*

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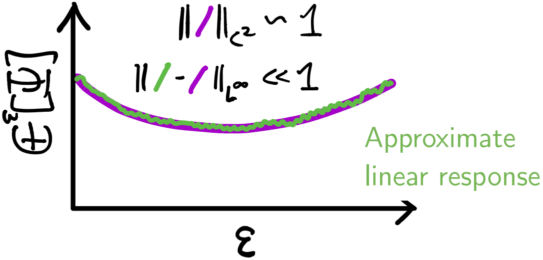
> *When and why does linear response occur (for all practical purposes) at macroscopic scales in high-dimensional systems?*

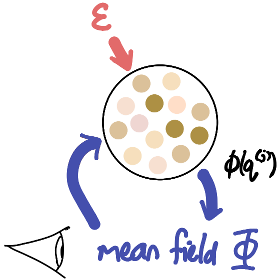


The model

We study a reasonably simple multiscale system:

M subsystems $q^{(j)} \circlearrowleft f(\cdot\,; \Phi, a^{(j)}, \varepsilon)$

Parameters $a^{(j)} \sim \nu$



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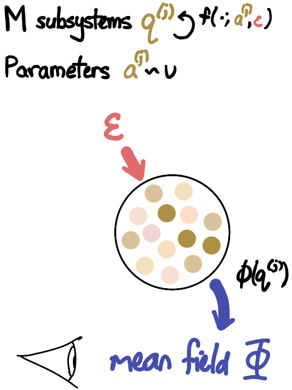

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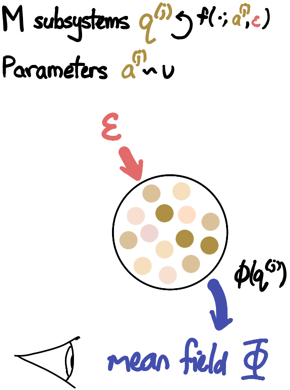

We will derive reductions for mean-field dynamics Φ, and discuss (very rich) LRT properties of these systems.

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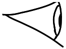
ε

microscopic subsystem		macroscopic observables: uncoupled	macroscopic observables: coupled
f satisfies LRT	finite M	✓	✓
	$M \to \infty$	✓	✯
f violates LRT with smooth $\frac{d\nu}{da}$	finite M	(✓)	(✓)
	$M \to \infty$	✓	✯
f violates LRT with non-smooth $\frac{d\nu}{da}$	finite M	✗	(✓)
	$M \to \infty$	✗	✗



mean field Φ

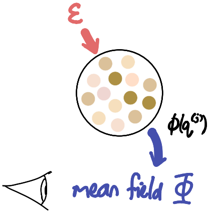
We will derive reductions for mean-field dynamics Φ, and discuss (very rich) LRT properties of these systems.

Uncoupled case

System parameters: $a^{(j)}$, $j = 1, \ldots, M$ sampled from measure ν

Microscopic dynamics:

$$q_n^{(j)} = f(q_{n-1}^{(j)}; a^{(j)}, \varepsilon), \; j = 1, \ldots, M$$



Mean-field observable:

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

Each subsystem $q^{(j)}$ evolves independently: suppose they have physical measures $\mu^{a^{(j)},\varepsilon}$ and are mixing.

Uncoupled case: expectations

Two (nested) ways to take expectations:

- Over dynamics, i.e. initial conditions: $\mathbb{E}^{\varepsilon}[\cdots]$
- Over dynamical systems, i.e. selection of parameters $a^{(j)}$ (if relevant): $\langle \mathbb{E}^{\varepsilon}[\cdots] \rangle$

Uncoupled case: expectations

relevant for LRT

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LRT of mean-field Φ

We are interested in behaviour with respect to ε of

$$\mathbb{E}^{\varepsilon}[\Phi] = \frac{1}{M}\sum_{j=1}^{M}\mathbb{E}^{\varepsilon}[\phi(q^{(j)})]$$

The $q^{(j)}$ will move independently toward statistical equilibrium, so

$$\mathbb{E}^{\varepsilon}[\phi(q^{(j)})] = \underbrace{\int \phi(q)\,\mathrm{d}\mu^{a^{(j)},\varepsilon}(q)}_{\text{function of } \varepsilon \text{ and } a^{(j)} \sim \nu}$$

LRT of mean-field Φ

Because the $a^{(j)}$ are randomly selected, a CLT in $\langle \cdot \rangle$ gives

$$\mathbb{E}^{\varepsilon}[\Phi] = \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}^{\varepsilon}[\phi(q^{(j)})] = \bar{\Phi}^{\varepsilon} + \frac{1}{\sqrt{M}} \eta^{\varepsilon} + o(1/\sqrt{M})$$

where η^{ε} is a mean-zero Gaussian process in ε, and

$$\bar{\Phi}^{\varepsilon} = \langle \mathbb{E}^{\varepsilon}[\phi(q)] \rangle = \iint \phi(q) \, \mathrm{d}\mu^{a,\varepsilon}(q) \, \mathrm{d}\nu(a)$$

So response of mean-field Φ is $\bar{\Phi}^{\varepsilon}$ plus small correction for finite ensemble size.

LRT of $\bar{\Phi}^\varepsilon$

$$\bar{\Phi}^\varepsilon = \langle \mathbb{E}^\varepsilon[\phi(q)] \rangle = \iint \phi(q)\,\mathrm{d}\mu^{a,\varepsilon}(q)\,\mathrm{d}\nu(a)$$

- Clearly if all microscopic subsystems satisfy LRT then so does $\bar{\Phi}^\varepsilon$.

LRT of $\bar{\Phi}^\varepsilon$

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- Clearly if all microscopic subsystems satisfy LRT then so does $\bar{\Phi}^\varepsilon$.
- On the other hand if the microscopic subsystems violate LRT and ν is discrete (e.g. $\nu = \delta_{a_0}$), then $\bar{\Phi}^\varepsilon$ will not have LRT.

LRT of $\bar{\Phi}^\varepsilon$

If ν is smooth (e.g. $\frac{\mathrm{d}\nu}{\mathrm{d}a} \in BV$), then averaging over $\mathrm{d}\nu(a)$ can give *"collective" linear response* of microscopic systems that may violate LRT:

- **An easy case:** If f can be written as $f(\,\cdot\,; a + K\varepsilon)$:

$$\frac{\mathrm{d}\bar{\Phi}^\varepsilon}{\mathrm{d}\varepsilon} = \int \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\int \phi(q)\, \mathrm{d}\mu^{a+K\varepsilon}(q) \right) \mathrm{d}\nu(a)$$

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$$\begin{aligned} \frac{\mathrm{d}\bar{\Phi}^\varepsilon}{\mathrm{d}\varepsilon} &= \int \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\int \phi(q)\,\mathrm{d}\mu^{a+K\varepsilon}(q)\right)\mathrm{d}\nu(a) \\ &= \int K \frac{\mathrm{d}}{\mathrm{d}a}\left(\int \phi(q)\,\mathrm{d}\mu^{a+K\varepsilon}(q)\right)\mathrm{d}\nu(a) \end{aligned}$$

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$\implies$ LRT holds

LRT of $\bar{\Phi}^{\varepsilon}$

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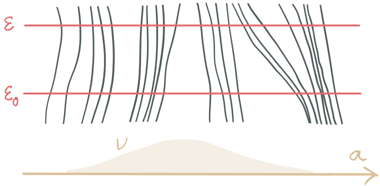
  - These maps obey LRT along topological conjugacy classes (Ruelle '09);



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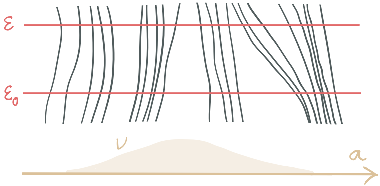

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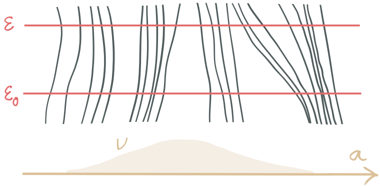
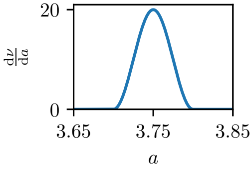
This may imply $\bar{\Phi}^\varepsilon$ has linear response.

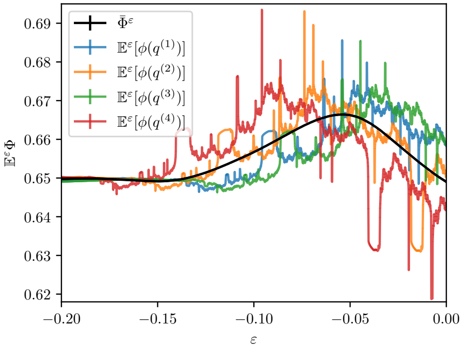


LRT of $\bar{\Phi}^\varepsilon$

Smooth family of unimodal maps:

$$f(q; a, \varepsilon) = (a + 4\varepsilon q(1-q))q(1-q),$$
$$\nu \sim \mathrm{Cosine}(3.75, 0.05)$$



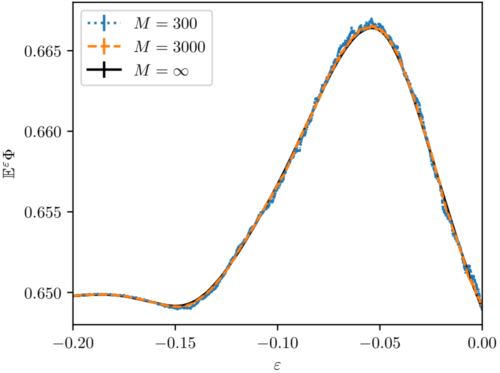


LRT of η^ε

$$\mathbb{E}^\varepsilon[\Phi] = \bar{\Phi}^\varepsilon + \frac{1}{\sqrt{M}}\eta^\varepsilon + o(1/\sqrt{M})$$

Finite M correction η^ε is almost surely as rough as the individual $q^{(j)}$ responses.

Thus, for finite M, Φ may only have "approximate" LRT:



Macroscopic reduction

What about the *dynamics* of Φ_n?

Macroscopic reduction

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The $q^{(j)}$s are independent of each other, so for any n

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

is a sum of independent random variables.
Thus

$$\Phi_n = \mathbb{E}^{\varepsilon}[\Phi] + \frac{1}{\sqrt{M}} \zeta_n + o(1/\sqrt{M})$$

where $\zeta_n, n \in \mathbb{N}$ are mean-zero Gaussian random variables.

Macroscopic reduction

When $M \gg 1$, ζ *appears* to converge to a stationary Gaussian *process*.

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When $M \gg 1$, ζ *appears* to converge to a stationary Gaussian *process*.
The autocorrelation function is the average over ν of the microscopic autocorrelations:

$$\mathrm{Cov}[\zeta_m, \zeta_n] = \langle \mathrm{Cov}[\phi(q_m), \phi(q_n)] \rangle .$$

Hence ζ has decay of correlations and can be approximated by e.g. an AR process.

Mean-field coupled case

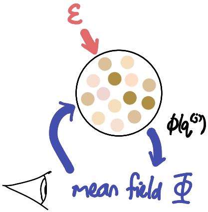
System parameters: $a^{(j)}$, $j = 1, \ldots, M$ sampled from measure ν

Microscopic dynamics:

$$q_n^{(j)} = f(q_{n-1}^{(j)}; \Phi_{n-1}, a^{(j)}, \varepsilon), \; j = 1, \ldots, M$$

Mean-field driver/observable:

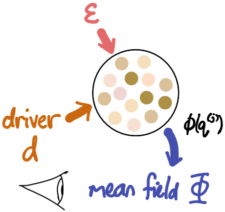
$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$



Externally-coupled system

System parameters: $a^{(j)}$, $j = 1, \ldots, M$ sampled from measure ν
External driver: d_n
Microscopic dynamics:

$$q_n^{(j)} = f(q_{n-1}^{(j)}; d_{n-1}, a^{(j)}, \varepsilon), \; j = 1, \ldots, M$$



Mean-field observable:

$$\Phi_n = \frac{1}{M} \sum_{j=1}^{M} \phi(q_n^{(j)})$$

Suppose $q^{(j)}$ have time-dependent physical measures $\mu_n^{d, a^{(j)}, \varepsilon}$ with decay of correlations.

Externally-coupled system

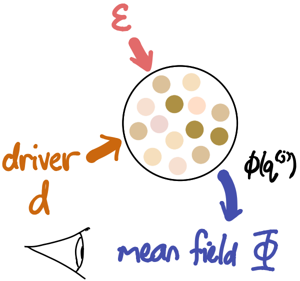
We can make the same CLT reduction as before,

$$\Phi_n = \langle \mathbb{E}^{\varepsilon}[\Phi_n | d] \rangle + \frac{1}{\sqrt{M}} \eta_n^{d,\varepsilon} + \frac{1}{\sqrt{M}} \zeta_n^{d} + o(1/\sqrt{M}),$$

Parameters of this reduction are now time-dependent and depend on past history of d.

Macroscopic reduction of coupled system

Ansatz: if $M \gg 1$, the coupled system can be approximated by setting $d_n \equiv \Phi_n$.



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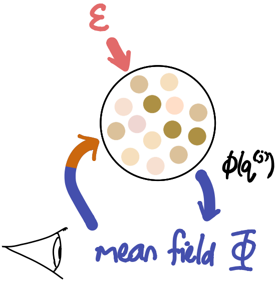

Macroscopic reduction of coupled system

This gives the macroscopic reduction:

$$\Phi_n = \langle \mathbb{E}^\varepsilon[\Phi_n|\Phi] \rangle + \frac{1}{\sqrt{M}}\eta_n^{\Phi,\varepsilon} + \frac{1}{\sqrt{M}}\zeta_n^{\Phi} + o(1/\sqrt{M})$$

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$=: F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon)$

self-generated noise

usually smaller than $\tilde{\zeta}$

LRT of coupled system: finite M

$$\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon) + \frac{1}{\sqrt{M}} \eta_n^{\Phi,\varepsilon} + \frac{1}{\sqrt{M}} \tilde{\zeta}_n^{\Phi} + o(1/\sqrt{M})$$

defines a stochastic dynamical system in Φ.

LRT of coupled system: finite M

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defines a stochastic dynamical system in Φ.
Modulo η's:

LRT of coupled system: finite M

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defines a stochastic dynamical system in Φ.
Modulo η's:

- The noise $\tilde{\zeta}^{\Phi}$ generates (annealed) LRT in the microscopic particles, so this noisy system is $\sim$smooth in Φ and ε.

LRT of coupled system: finite M

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- So Φ obeys LRT for finite M.

LRT of coupled system: finite M

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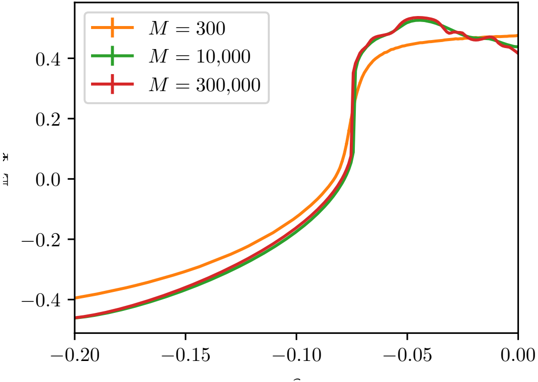
Modulo η's:

- The noise $\tilde{\zeta}^{\Phi}$ generates (annealed) LRT in the microscopic particles, so this noisy system is $\sim$smooth in Φ and ε.
- So Φ obeys LRT for finite M.
- Thus so does Φ.

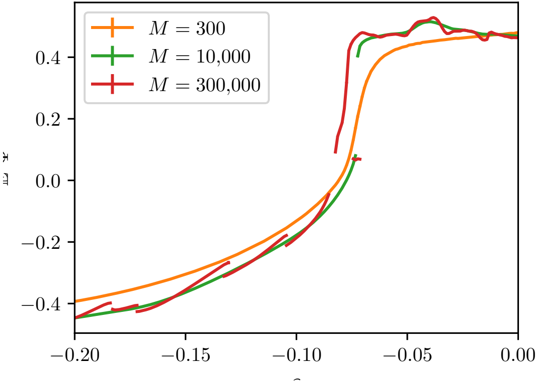
LRT of coupled system: finite M

LRT for unimodal microscopic components, $\nu \sim \mathrm{Cosine}$:



LRT of coupled system: finite M

LRT for unimodal microscopic components, ν discrete:



Thermodynamic limit

As $M \to \infty$ the CLT reduction reduces to the law of large numbers:

$$\Phi_n = F(\Phi_{n-1}, \Phi_{n-2}, \ldots; \varepsilon).$$

If we have LRT without coupling, this defines a smooth dynamical system.
External forcing washes out over time because of microscopic mixing, so

$$\Phi_n \approx F(\Phi_{n-1}, \Phi_{n-2}, \ldots, \Phi_{n-K}; \varepsilon),$$

i.e. emergent dynamics of Φ_n are low-dimensional.

Thermodynamic limit

If dynamics converges to equilibrium $\Phi_n \equiv \bar{\Phi}^\varepsilon$ we have

$$\bar{\Phi}^\varepsilon = F(\bar{\Phi}^\varepsilon, \bar{\Phi}^\varepsilon, \dots ; \varepsilon) := F_0(\bar{\Phi}^\varepsilon; \varepsilon),$$

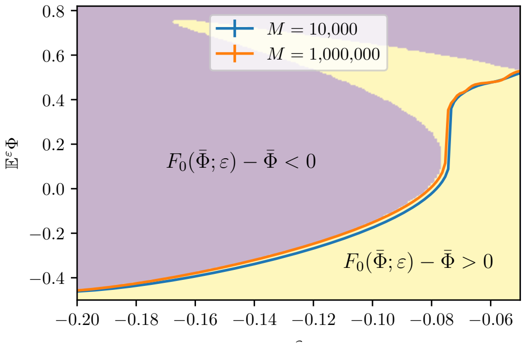
which is a smooth function if the microscopic subsystems have "collective" linear response. Then,

$$\frac{\mathrm{d}\bar{\Phi}^\varepsilon}{\mathrm{d}\varepsilon} = \left(1 - \frac{\partial F_0}{\partial \bar{\Phi}^\varepsilon}\right)^{-1} \frac{\partial F_0}{\partial \varepsilon}$$

(+ stability) and hence Φ has LRT.

Thermodynamic limit

For unimodal microscopic component example, $\frac{\mathrm{d}\nu}{\mathrm{d}x} \in C^3$, we see saddle-node bifurcation:



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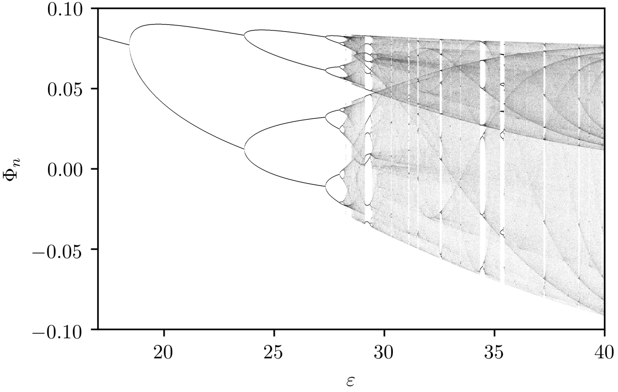
- However, we can use transfer operator methods to approximate the dynamics of the microscopic distributions $\mu_n^{\Phi,\varepsilon}$.
- For uniformly expanding f, Chebyshev spectral methods are very good at this (W. '19).

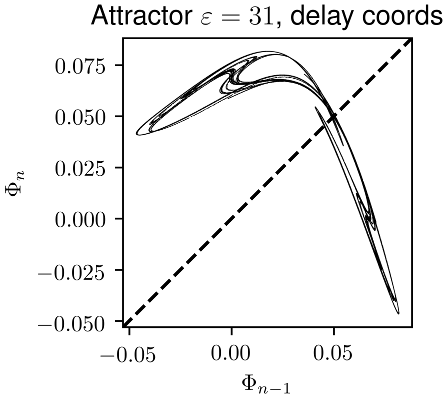
Macroscopic dynamics in thermo. limit

We choose an f uniformly expanding with perturbation parameter ε regulating the strength of an appropriate mean-field coupling.
For large ε we see period doubling bifurcation to chaos:



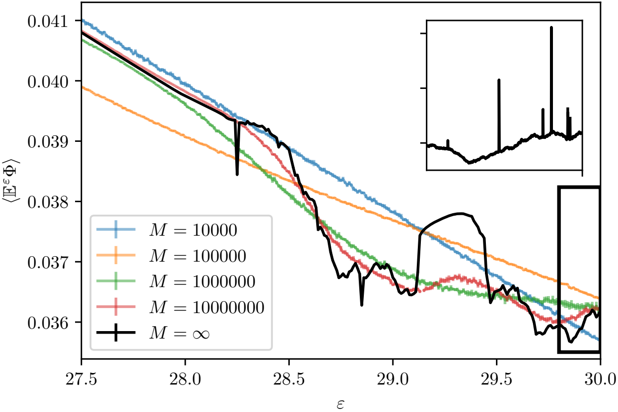
Macroscopic dynamics in thermo. limit

The attracting Φ dynamics look unimodal:



LRT in thermodynamic limit

We have breakdown of LRT in the thermodynamic limit:



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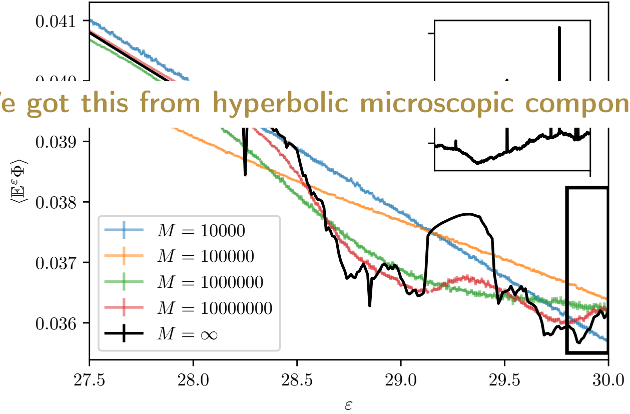

We got this from hyperbolic microscopic components!

Macroscopic dynamics in thermo. limit

Side question:
Chaotic hypothesis says macroscopic dynamics should be hyperbolic (i.e. splitting between stable and unstable directions). Is this the case?

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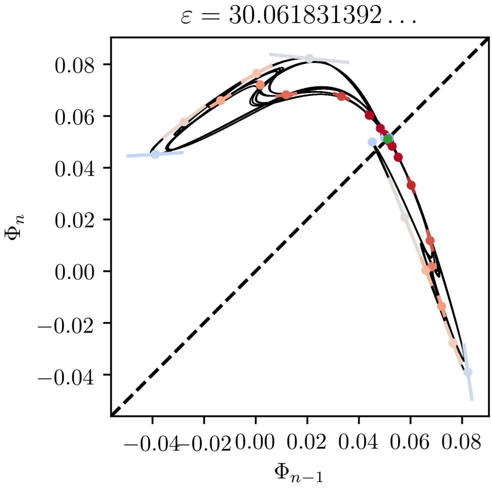
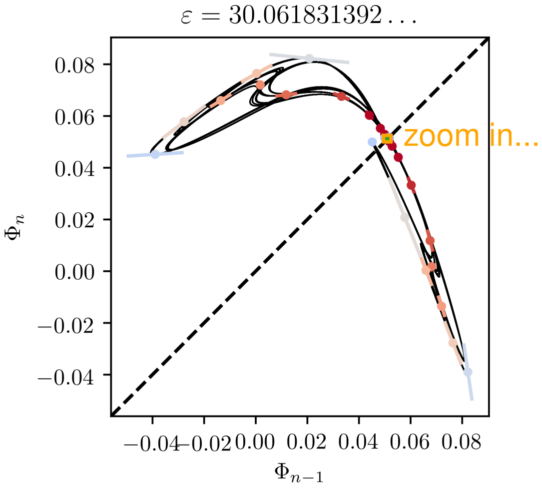
Answer: No. There are homoclinic tangencies.

How do we know? Continuation with Chebyshev transfer operator methods (`Poltergeist.jl`).

Macroscopic dynamics in thermo. limit



Macroscopic dynamics in thermo. limit



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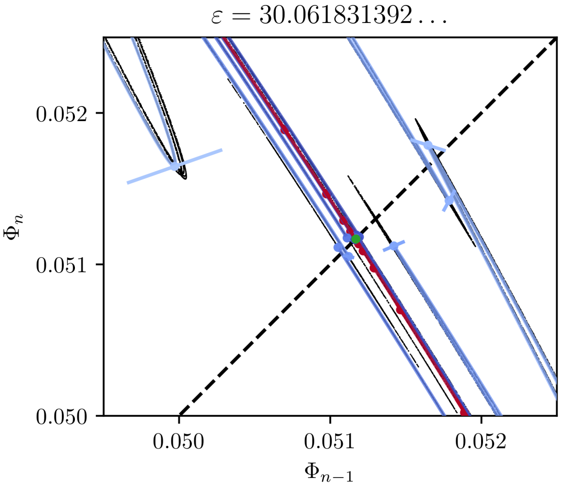

Conclusions

Various mechanisms by which linear response may emerge *and/or break down* in large coupled chaotic systems:

- Macroscopic LRT from inhomogeneous microscopic variables that individually violate LRT
- LRT in large (finite) chaotic systems via feedback of self-generated noise
- In thermodynamic limit LRT may depend on structure of macroscopic dynamics
    - This may be non-hyperbolic chaos, leading to LRT violation

Mostly these depend on the system's network structure!

Further directions

- More rigorous study of some of these phenomena (e.g. Sélley and Tanzi '20)
- Study of chaotic networks beyond global, mean-field couplings

Further details

Wormell, C.L. and Gottwald, G.A., 2019. Linear response for macroscopic observables in high-dimensional systems. *Chaos* 29: 113127.

Wormell, C.L. and Gottwald, G.A., 2018. On the validity of linear response theory in high-dimensional deterministic dynamical systems. *Journal of Statistical Physics* 172: 1479-1498.

Aside on periodic windows

Unimodal maps have periodic dynamics on a dense (but not full measure) parameter set—i.e., non-mixing.
To keep things simple, we avoid this by adding "hidden" dynamics $r_n^{(j)} \in [0,1]$:

$$f(q, r; a, \varepsilon) = \begin{cases} (\tilde{f}(q; a, \varepsilon), 2r), & r \leq 1/2 \\ (q, 2r-1), & r > 1/2. \end{cases}$$

This makes the unimodal $q^{(j)}$ dynamics mixing while retaining the same invariant measures.

(N.B. at statistical equilibrium, $\{r_n \geq 1/2\}_{n\in\mathbb{N}}$ are *i.i.d.* Bernoulli.)

"Mixing"

If dynamical system $x_n = f(x_{n-1})$ is mixing with respect to measure μ then for all $w \in L^2(\mu)$ with $\mathbb{E}[w] = 1$,

$$\mathbb{E}[\phi(x_n)w(x_0)] = \int \phi(x_n)w(x_0)\,\mathrm{d}\mu(x_0) \xrightarrow{n\to\infty} \mathbb{E}[\phi]$$

More generally, are going to assume that if $\tilde{\mu}$ is a "nice" measure,

$$\int \phi(x_n)\,\mathrm{d}\tilde{\mu}(x_0) \xrightarrow{n\to\infty} \mathbb{E}[\phi]$$