Projective functors and their applications

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1 Introduction and motivation

• Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic 0. Set $U = U(\mathfrak{g})$ and let $Z = Z(U)$ be its centre (the ring of Laplace operators).

• Following Bernstein–Gelfand's classic paper [1], we define and investigate *projective functors* arising from finite-dimensional $\mathfrak{g}$-modules V. These are endofunctors of the category $\mathscr{M}_{Zf}$ of Z-finite $\mathfrak{g}$-modules, occurring as direct summands of the functor

$$F_V : \mathscr{M}_{Zf} \to \mathscr{M}_{Zf}, \quad M \mapsto V \otimes M.$$

When restricted to a category $\mathscr{M}(\theta)$ of $\mathfrak{g}$-modules with fixed central character θ, projective functors and their morphisms are well behaved, and admit easy classifications.

• **Goals:** See/prove the main theorems on projective functors, then apply them in two directions: finding equivalences $\mathscr{M}(\theta) \cong \mathscr{M}(\theta')$ for certain pairs (θ, θ'), and producing an easy proof of Duflo's theorem.

In the second part, we explore connections to the theory of Harish–Chandra modules and principal series representations, obtaining equivalences to subcategories of $\mathcal{O}$ and Jordan–Holder multiplicities (respectively).

2 Preliminaries

2.1 Category theory

• All categories and functors are assumed to be k-linear, unless otherwise stated.

• If $\mathscr{B}$ is a complete subcategory of the abelian category $\mathscr{A}$, and $\mathscr{B}$ is closed under subquotients, then $\mathscr{B}$ is abelian too.

• Suppose $\mathscr{A}$ is an abelian category containing a class of objects $\mathscr{P}$ closed under direct sums. An object A is $\mathscr{P}$-generated in case there exists an exact sequence

$$P \to A \to 0$$

in $\mathscr{A}$, and $\mathscr{P}$-presented in case there is an exact sequence

$$P' \to P \to A \to 0$$

in $\mathscr{A}$. The full subcategory of $\mathscr{P}$-presentable objects in $\mathscr{A}$ is denoted $\mathscr{A}_{\mathscr{P}}$.

• The opposite algebra of an associative unital k-algebra A is denoted A°. Thus (A, B)-bimodules X may be identified with left $A \otimes B^{\circ}$-modules. Write A^2 for the algebra $A \otimes A^{\circ}$.

• Let us denote by $h(X)$ the functor of tensoring induced by X:

$$h(X) : B\text{-mod} \to A\text{-mod}, \quad M \mapsto X \otimes_B M.$$

Recall that, by definition, a *right continuous* functor is right exact and commutes with inductive limits.

• **Theorem 2.1 (Watt):** Let $\mathscr{C}$ be the full subcategory of right continuous functors within the category of functors B-mod $\to$ A-mod. Then the functor

$$h : (A, B)\text{-bimod} \to \mathscr{C}, \quad X \mapsto h(X)$$

is an equivalence of categories.

2.2 Lie theory

• **Standard notation:**

(i) $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, dual to the space $\mathfrak{h}^*$ of weights of $\mathfrak{g}$.

(ii) R^+ is a choice of positive roots inside the root system R, with half-sum ρ and corresponding nilpotent sublagebra $\mathfrak{n}^+$

(iii) To each $\gamma \in R$ corresponds the dual root $h_\gamma \in \mathfrak{h}$ and the reflection σ_γ,

$$\sigma_\gamma : \mathfrak{h}^* \to \mathfrak{h}^*, \quad \sigma_\gamma(\chi) = \chi - \chi(h_\gamma)\gamma;$$

these generate the Weyl group $W = \langle \sigma_\gamma \rangle$.

(iv) $\Lambda = \{\chi \in \mathfrak{h}^* : \chi(h_\gamma) \in \mathbb{Z} \text{ for all } \gamma \in R\}$ is the lattice of integer weights, containing the sublattice Γ generated by R.

(v) Given $\chi \in \mathfrak{h}^*$, let R_χ denote the set of $\gamma \in R$ for which $\chi(h_\gamma) \in \mathbb{Z}$, and let

$$W_\chi = \mathrm{Stab}_W(\chi), \quad W_{\chi+\Gamma} = \mathrm{Stab}_W(\chi + \Gamma)$$

be stabilisers with respect to the action of W on $\mathfrak{h}^*$ and $\mathfrak{h}^*/\Gamma$. Recall that we call χ regular in case W_χ is trivial.

(vi) $|\chi|$ denotes the length of $\chi \in \mathfrak{h}^*$ with respect to some W-invariant inner product on Λ.

• **A partial order on $\mathfrak{h}^*$:** Given $\gamma \in R^+$, write

$$\psi <_\gamma \chi \quad \text{for } \psi, \chi \in \mathfrak{h}^*$$

whenever $\psi = \sigma_\gamma(\chi)$ and $\chi(h_\gamma) \in \mathbb{Z}^+$. We then let $\psi < \chi$ whenever there exist

$$\psi = \psi_0, \ldots, \psi_n = \chi \in \mathfrak{h}^*, \quad \gamma_1, \ldots, \gamma_n$$

such that $\psi_i <_{\gamma_{i+1}} \psi_{i+1}$ for all i. (So $<$ is the transitive closure of all the $<_\gamma$.) Call χ dominant if it is $<$-maximal.

• **Central characters of $\mathfrak{g}$:** $\Theta = \mathrm{Hom}(Z, k)$. The kernel $J_\theta \subseteq Z$ of $\theta \in \Theta$ is clearly a maximal ideal.

• Denote by $\eta^* : Z \to S(\mathfrak{h})$ the Harish–Chandra homomorphism. Identifying $S(\mathfrak{h})$ with the set of polynomial functions on $\mathfrak{h}^*$, we obtain a dual map

$$\eta : \mathfrak{h}^* \to \Theta, \quad \eta(\chi)(z) = \eta^*(z)(\chi).$$

• **Theorem 2.2 (Harish–Chandra):** η is an epimorphism with fibres

$$\eta^{-1}(\eta(\chi)) = W(\chi).$$

• Any (U, U)-bimodule Y admits an adjoint action of $\mathfrak{g}$ given by

$$X \cdot u = Xu - uX, \quad X \in \mathfrak{g}, u \in U;$$

denote the resulting $\mathfrak{g}$-module by Y^{ad}.

• **Theorem 2.3 (Kostant):** For any finite-dimensional $\mathfrak{g}$-module U, $\mathrm{Hom}_{\mathfrak{g}}(L, U^{\mathrm{ad}})$ is naturally a free Z-module of rank equal to the multiplicity of the zero weight in L.

• **Some key categories of U-modules:** Full inside of $\mathscr{M} = U$-mod:

$$\mathscr{M}_f = \{\text{finitely generated } U\text{-modules}\}, \quad \mathscr{M}_{Zf} = \{Z\text{-finite } U\text{-modules}\}.$$

For $\theta \in \Theta$ and $n \geq 1$, set $U_\theta^n = U_\theta / J_\theta^n$ and

$$\mathscr{M}^n(\theta) = \{M \in \mathscr{M} : J_\theta^n M = 0\} = U_\theta^n\text{-mod}.$$

$$\mathscr{M}^\infty(\theta) = \{M \in \mathscr{M} : \text{for all } m \in M \text{ there exists } n \geq 1 \text{ such that } J_\theta^n m = 0\},$$

suppressing the superscript for the case $n = 1$.

• **Elementary fact:** each Z-finite module M admits a unique decomposition

$$M = \bigoplus_{\theta \in \Theta} M_\theta, \quad M_\theta \in \mathscr{M}^\infty(\theta).$$

- Hence $\mathscr{M}_{Zf} \cong \prod_\theta \mathscr{M}^\infty(\theta)$ and we obtain projection functors

$$\mathrm{Pr}(\theta) : \mathscr{M}_{Zf} \to \mathscr{M}^\infty(\theta).$$

- Also have subcategory $\mathcal{O} \subseteq \mathscr{M}_{Zf}$, containing the Verma module

$$M_\chi = U/U(I_{\chi-\rho} + \mathfrak{n});$$

$I_{\chi-\rho}$ is the ideal in $U(\mathfrak{h}) \subseteq U$ generated by the elements $h - (\chi - \rho)(h)$.

- **Verma properties to recall:**

  (i) The unique and pairwise non-isomorphic simple quotients L_χ of the M_χ exhaust the simple modules in $\mathcal{O}$.

  (ii) The natural homomorphism $Z \to \mathrm{End}_{\mathfrak{g}}(M_\chi) = k$ coincides with the character $\eta(\chi)$.

  (iii) There is a unique indecomposable projective object $P_\chi \in \mathcal{O}$ mapping onto L_χ; these projective objects admit a filtration by Verma modules. The common value

  $$d_{\chi\psi} = [M_\chi : L_\chi] = \dim\,\mathrm{Hom}(P_\psi, M_\chi)$$

  satisfies $d_{\chi\psi} > 0$ if and only if $\chi > \psi$, and $d_{\chi\chi} = 1$.

  (iv) The classes $\delta_\chi = [M_\chi]$ form an free basis of the Grothendieck group $K(\mathcal{O})$. The unique inner product $\{-, -\}$ on $K(\mathcal{O})$ for which that basis is orthonormal is also clearly W-invariant with respect to the action $w \cdot \delta_\chi = \delta_{w\chi}$.

3 Projective functors

3.1 First properties

- Some of the main actors in our story are the functors

$$F_V : \mathscr{M} \to \mathscr{M}, \quad M \mapsto V \otimes M,$$

where V is a finite-dimensional $\mathfrak{g}$-module.

- **Immediate properties:**

  (i) F_V is exact and commutes with arbitrary direct sums and products.

  (ii) $\mathfrak{g}$-morphisms $\varphi : V_1 \to V_2$ induce natural transformations $F_{V_1} \to F_{V_2}$.

  (iii) We have $F_{V_1} \circ F_{V_2} \cong F_{V_1 \otimes V_2}$ and a biadjunction (F_{V^*}, F_V). (Here V^* is the dual of V, with respect to some anti-involution of $\mathfrak{g}$ fixing points of $\mathfrak{h}$.)

  (iv) Suppose V has weights $\mu_1, \ldots, \mu_n$ (with multiplicty). Then $F_V(M_\chi)$ has a filtration with quotients $M_{\chi+\mu_i}$, $1 \leq i \leq n$.

• To V we also associate the (U,U)-bimodule $\Phi_V = V \otimes U$, where the left and right actions are

$$X(v \otimes u) = Xv \otimes u + v \otimes Xu, \quad (v \otimes u)X = v \otimes uX.$$

• **Lemma 3.1:**

(i) $h(\Phi_V) \cong F_V$.

(ii) $\mathrm{Hom}_{U^2}(\Phi_V, Y) \cong \mathrm{Hom}_{\mathfrak{g}}(V, Y^{\mathrm{ad}})$ for any (U,U)-bimodule Y.

(iii) Φ_V is U-generated on both sides by its subset $V = V \otimes 1$.

• **Corollary 3.2:** F_V preserves the subcategories $\mathscr{M}_f$ and $\mathcal{O}$ in $\mathscr{M}$, and also preserves projective objects in all three categories.

Proof. F_V is exact and $F_V(U) = \Phi_V$ is finitely generated by Lemma 3.1(iii), so $F_V(\mathscr{M}_f) \subseteq \mathscr{M}_f)$. Moreover, if $M \in \mathcal{O}$, then $F_V(M)$ is $\mathfrak{h}$-diagonalisable and $U(\mathfrak{n}^+)$-finite because $V \in \mathcal{O}$, and we have already seen it is finitely generated. So $F_V(\mathcal{O}) \subseteq \mathcal{O}$.

The remaining statement follows from a general fact: functors with exact right adjoints always preserve projectives. □

3.2 Another Kostant theorem

• Have a Z^2-action on the functor F_V, i.e. a ring map $Z^2 \to \mathrm{End}(F_V)$:

$$z \cdot (v \otimes m) = \sum a_i(v \otimes b_i m), \quad \text{for } z = \sum_i a_i \otimes b_i \in Z^2.$$

• This is the action obtained by transport of structure from the action of $Z^2 \subseteq U^2$ on Φ_V to F_V via the equivalence h.

• Let I_V denote the kernel of the action:

$$I_V = \{z \in Z^2 : z(V \otimes M) = 0 \text{ for all } M \in \mathscr{M}\}.$$

• Note the embedding

$$\eta^* \otimes \eta^* : Z^2 \hookrightarrow S(\mathfrak{h}) \otimes S(\mathfrak{h}) = S(\mathfrak{h} \oplus \mathfrak{h}) = P(\mathfrak{h}^* \oplus \mathfrak{h}^*);$$

since η^* identifies Z with $S(\mathfrak{h})^W$, the image of $\eta^* \otimes \eta^*$ consists of polynomials $Q(\psi, \chi)$ which are W-invariant in each variable.

• **Theorem 3.3 (Kostant):** Let Q be the image of some $z \in Z^2$. Then $z \in I_V$ if and only if $Q(\chi + \mu, \chi)$ is the zero polynomial for any weight $\mu \in P(V)$.

• **Corollary 3.4:**

(i) Z^2/I_V is finitely generated over Z.

(ii) $F_V(\mathscr{M}_{Zf}) \subseteq \mathscr{M}_{Zf}$.

Proof. Define $A = S(\mathfrak{h})$, $B = S(\mathfrak{h})^W$, and

$$J = \{Q \in A^2 : Q(\chi + \mu, \chi) = 0 \text{ for any } \mu \in P(V)\}.$$

Then J is an ideal in A^2 and $J_V = J \cap B^2$ is an ideal in B^2. Claim (i) is equivalent to saying B^2/J_V is finitely generated over B.

By the theorem, there is a B-module embedding

$$i = \oplus_\mu i_\mu : B^2/J_V \to \oplus_{\mu \in P(V)} A,$$

where $i_\mu(Q)(\chi) = Q(\chi + \mu, \chi)$. But A is finitely generated as a B-module because W is finite, so by Noetherianity of B we conclude B^2/J_V is finitely generated over B.

It remains to prove (ii). Exercise from (i): Given a $\mathfrak{g}$-module with $JM = 0$ for some finite-codimension ideal $J \subseteq Z$, cook up a finite-codimension ideal $J' \subseteq Z$ with $J'(V \otimes M) = 0$. Then since F_V commutes with direct limits, we get $F_V(\mathscr{M}_{Zf}) \subseteq \mathscr{M}_{Zf}$. □

3.3 Functor decomposition and the main results

• We have seen that F_V preserves $\mathscr{M}_{Zf}$; let $F_{V,Zf}$ denote its restriction to this subcategory.

• **Definition 3.5:** Direct summands of $F_{V,Zf}$ are known as *projective functors.*

• Every projective functor decomposes into a direct sum of indecomposable projective functors; ultimately we will describe these indecomposables.

• **Proposition 3.6:** Let F, G be projective functors.

(i) F is exact and preserves direct sums and products.

(ii) Direct summands of F are projective; the functors $F \oplus G$ and $F \circ G$ are projective.

(iii) F has projective right and left adjoints.

(iv) $F = \oplus_{\theta,\theta'} \mathrm{Pr}_{\theta'} \circ F \circ \mathrm{Pr}_\theta$ and each of these summands are projective.

• To parametrise projective functors, we require the sets

$$\Xi^0 = \{(\psi, \chi) \in (\mathfrak{h}^*)^2 : \psi - \chi \in \Lambda\}, \quad \Xi = \Xi^0/W,$$

where the quotient is by the component-wise W-action.

• Every element $\xi \in \Xi$ has a *proper representative* (ψ, χ), by which we mean that χ is dominant and $\psi \leq W_\chi(\psi)$. There is a well-defined map

$$\eta^r : \Xi \to \Theta, \quad \eta^r(\psi, \chi) = \eta(\chi).$$

• **Theorem A:**

(i) Each projective functor decomposes into a direct sum of indecomposable projective functors.

(ii) To each $\xi \in \Xi$ there corresponds an indecomposable projective functor F_ξ, unique up to isomorphism with the following properties:

- $F_\xi(M_\varphi) = 0$ if $\eta^r(\xi) \neq \eta(\varphi)$, $\varphi \in \mathfrak{h}^*$.
- If $\xi = (\psi, \chi)$ is written properly, then $F_\xi(M_\chi) = P_\psi$.

(iii) $\xi \mapsto F_\xi$ defines a bijection from Ξ to the set of isomorphism classes of indecomposable projective functors.

Among other things, the next result reveals the remarkable fact that projective functors are determined by their induced action on $K(\mathcal{O})$.

• **Theorem B:** Suppose F, G are projective functors. Then:

(i) If $[F] = [G]$, then F is naturally isomorphic to G.

(ii) If (F, G) is an adjoint pair, then $([F], [G])$ is a conjugate pair on the inner product space $K(\mathcal{O})$.

(iii) $[F]$ is W-equivariant.

• Theorems A and B allow us to compute $[F_\xi]$ explicitly. In particular, $[F_\xi](\delta_\varphi) = 0$ if $\varphi \notin W(\chi)$ and $[F_\xi](\delta_{w(\chi)}) = \sum_{\varphi > \psi} d_{\varphi,\psi}\delta_{w\varphi}$, so understanding F reduces to knowledge of the $d_{\varphi\psi}$.

• **Definition 3.7:** Let $\theta \in \Theta$ and let $F(\theta)$ denote the restriction of a projective functor to $\mathscr{M}(\theta)$. A *projective θ-functor* $F : \mathscr{M}(\theta) \to \mathscr{M}$ is any direct summand of a functor $F_V(\theta)$.

• The third and final theorem in this section underpins the proofs of the previous two.

• **Theorem C:** Let F, G be projective θ-functors, $\chi \in \eta^{-1}(\theta)$. Then

$$i_\chi : \mathrm{Hom}(F, G) \to \mathrm{Hom}(FM_\chi, GM_\chi), \quad i_\chi(\varphi) = \varphi_{M_\chi}$$

is a monomorphism, and an isomorphism if χ is dominant.

Proof sketch. By considering decompositions $F_V(\theta) = F \oplus F', G_L(\theta) = G \oplus G'$, we reduce to the case $F = F_V(\theta)$ and $G = G_L(\theta)$.

To prove injectivity of i_χ, need the following fact: If $\chi \in \eta^{-1}(\theta)$ is a weight and $u \in U_\theta$, then $uM_\chi = 0$ implies $u = 0$.

The isomorphism for χ dominant is proven by counting dimensions using Kostant's theorem 2.3.

• We need some subsidiary information before we can proceed to the proofs of the other two theorems. Namely, we will need to see that the restriction

$$F^\infty(\theta) : \mathscr{M}^\infty(\theta) \to \mathscr{M}$$

of a projective F is determined by the restrictions $F^n(\theta) : \mathscr{M}^n(\theta) \to \mathscr{M}$.

• **Proposition 3.8:** Suppose F, G are projective functors. Then any natural transformation

$$\varphi : F(\theta) \to G(\theta)$$

admits a lift $\widehat{\varphi} : F^\infty(\theta) \to G^\infty(\theta)$. If φ is an isomorphism, then so is $\widehat{\varphi}$; if $F = G$, then any idempotent φ can be lifted to an idempotent $\widehat{\varphi}$.

Proof. Let $H^n = \operatorname{Hom}(F^n(\theta), G^n(\theta))$, $1 \leq n \leq \infty$, and let $r_{nm} : H^n \to H^m$ denote the obvious restriction maps, $m \leq n$, so we have an inverse system.

Firstly, we have that $H^\infty = \varprojlim H^n$. This is because F commutes with direct limits and modules $M \in \mathscr{M}^\infty(\theta)$ can be expressed as follows:

$$M = \varinjlim M^n, \quad M^n = \{m \in M : J_\theta^n m = 0\} \in \mathscr{M}^n(\theta).$$

As in the sketch of Theorem C, we may assume $F = F_V$, $G = F_L$. Then, exercise (use Watt's theorem and Lemma 3.1):

$$H^n = (\operatorname{Hom}_{\mathfrak{g}}(L^* \otimes V, U^{\mathrm{ad}}))/J_\theta^n.$$

So H^∞ is a J_θ-adic completion. Then $H^n = H^\infty / J_\theta^n$, so in particular $\varphi \in H^1$ can always be lifted to some $\widehat{\varphi} \in H^\infty$.

Suppose φ is an isomorphism, inverse ψ. To prove $\widehat{\varphi}$ is an isomorphism, it suffices to prove $\widehat{\varphi}\widehat{\psi}$ and $\widehat{\psi}\widehat{\varphi}$ are invertible, so for that reason we can assume $F = G$ and $\varphi = 1$. But then $\widehat{\varphi} = 1 - \alpha$ for some $\alpha \in J_\theta$, which is a unit in H^∞.

We omit the proof that an idempotent φ has an idempotent lift. □

• **Theorem C + Proposition 3.8 = Corollary 3.9:** Suppose F, G are projective functors, χ a dominant weight with $\theta = \eta(\chi)$. Any isomorphism $FM_\chi \cong GM_\chi$ lifts to an isomorphism $F^\infty(\theta) \cong G^\infty(\theta)$, and any $\mathfrak{g}$-module decomposition $FM_\chi \cong \oplus_i M_i$ lifts to a decomposition $F^\infty(\theta) = \oplus F_i$ with $F_i M_\chi = M_i$.

• If F is a projective functor, then F is the direct sum of its restrictions to the subcategories $\mathscr{M}^\infty(\theta)$; that is,

$$F = \bigoplus_\theta F \circ \operatorname{Pr}(\theta).$$

• Now, by the corollary, $F \circ \mathrm{Pr}(\theta)$ splits into a direct sum of (finitely many) indecomposable projective functors, according to the direct sum decomposition of FM_χ. Thus we obtain Theorem A(i).

• **Remark 3.10:** If F is an indecomposable projective functor, then $F = F \circ \mathrm{Pr}(\theta)$ for some $\theta \in \Theta$. Thus $FM_\chi = 0$ whenever $\eta(\chi) \neq \theta$. On the other hand, if $\chi \in \eta^{-1}(\theta)$ is dominant, then $M_\chi = P_\chi$ is an indecomposable projective and hence $FM_\chi = P_\psi$ for some $\psi \in \mathfrak{h}^*$.

• *Proof of Theorem B.* For the first point, suppose $[F] = [G]$. By the previous discussion, it is equivalent to prove $FM_\chi \cong GM_\chi$ for any dominant weight χ. But FM_χ and GM_χ are projective objects in $\mathcal{O}$, whose isomorphism classes are recoverable from their images in $K(\mathcal{O})$.

For the second point, we need to prove $\{[F]x, y\} = \{x, [G]y\}$ for all $x, y \in K(\mathcal{O})$. We can assume $x = [P]$ is the class of a projective, since the classes of projective objects span $K(\mathcal{O})$. Then use the assumed adjunction and the formula

$$\{[P], [M]\} = \dim \mathrm{Hom}(P, M), \quad P \text{ projective, } M \text{ arbitrary in } \mathcal{O}.$$

We omit the rather lengthy proof of $[F]$'s W-equivariance.

• All that remains is to prove the classification results of Theorem A(ii),(iii).

Proof. Given a projective functor F, we define a quantity

$$a_F : (\mathfrak{h}^*)^2 \to \mathbb{Z}, \quad a_F(\psi, \chi) = \{d_\psi, [F]\delta_\chi\}.$$

In fact a_F lands in $\mathbb{N}$. Indeed, if χ is dominant, then FM_χ is projective and $a_F(\psi, \chi) \geq 0$ for any ψ (consider an appropriate Hom space); then use W-equivariance of $[F]$ to deduce that $a_F(\psi, \chi) \geq 0$ always.

Next consider the subsets

$$S(F) = \{(\psi, \chi) : a_F(\psi, \chi) > 0\},$$
$$S^{\max}(F) = \{(\psi, \chi) \in S(F) : |\psi - \chi| \text{ maximal}\}.$$

By non-negativity of a_F, we get that

$$F = \oplus_i F_i \quad \Rightarrow \quad S(F) = \cup_i S(F_i)$$

so that, since $S(F_V) \subseteq \Xi^0$, the same is true for $S(F)$. (Similarly $S^{\max}(F) \subseteq \cup_i S^{\max}(F_i)$.) Both $S(F)$ and $S^{\max}(F)$ are preserved by W, due to the W-equivariance of $[F]$.

Suppose F is indecomposable. Then $S^{\max}(F)/W$ consists of a single point. Indeed, if $F = F \circ \mathrm{Pr}(\theta)$ and $\chi \in \eta^{-1}(\theta)$ is dominant, then $FM_\chi = P_\psi$ and we get $S^{\max}(F) = W(\psi, \chi)$ (exercise).

To each indecomposable projective functor F we have associated a $\xi \in \Xi$, such that if ξ is written properly, then $FM_\chi = P_\psi$. And each $\xi = (\psi, \chi)$ arises thus: If V is a finite-dimensional $\mathfrak{g}$-module with extremal weight $\psi - \chi$, then $(\psi, \chi) \in S^{\max}(F_V)$ and therefore $(\psi, \chi) \in S^{\max}(F)$ for some indecomposable summand F of F_V. $\square$

4 First applications

4.1 Worked example

• To apply this theory in a concrete case, we consider a worked example, following [3].

• Let us take $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with $\Lambda = \mathbb{Z}$, and suppose we are interested in studying the functor $F = F_{L_{m+1}}$.

• We know the module L_{m+1} is $(m+1)$-dimensional with weights

$$P(L_{m+1}) = \{-m, -m+2, \dots, m-2, m\},$$

so from 3.1 we know $L_{m+1} \otimes M_n$ has a Verma filtration with factors M_i for $i \in I = P(L_{m+1}) + n$. Thus in $K(\mathcal{O})$, for dominant $n \geq 0$:

$$[L_{m+1} \otimes M_n] = \sum_{i \in I} [M_i].$$

• We want to rewrite the latter sum in terms of projective classes:

$$\begin{aligned} \sum_{i \in I} [M_i] &= \sum_{i \in I, i < -1} ([M_i + [M_{-i}]) + \sum_{i \in I, i > m-n} [M_i] \\ &= \sum_{i \in I, i < -1} [P_i] + \sum_{i \in I, i > m-n} [P_i]. \end{aligned}$$

• Thus from the theorem we can deduce a decomposition into indecomposables,

$$F^{\infty}(n) = \bigoplus_{i \in I, i < -1} F_{(j,n)} \oplus \bigoplus_{i \in I, i > m-n} F_{(j,n)}.$$

4.2 Equivalences between categories $\mathscr{M}(\theta)$

• **Theorem 4.1:** For $\theta, \theta' \in \Theta$, let

$$F_{\theta', V, \theta} = \Pr(\theta') \circ F \circ \Pr(\theta) : \mathscr{M}^{\infty}(\theta) \to \mathscr{M}^{\infty}(\theta').$$

Suppose we have dominant weights $\chi \in \eta^{-1}(\theta), \psi \in \eta^{-1}(\theta')$ such that $W_\chi = W_\psi$ and $\lambda = \psi - \chi \in \Lambda$. Then

$$F_{\theta', V, \theta} : \mathscr{M}^{\infty}(\theta) \to \mathscr{M}^{\infty}(\theta'), \quad F_{\theta, V^*, \theta'} : \mathscr{M}^{\infty}(\theta') \to \mathscr{M}^{\infty}(\theta),$$

are inverse equivalences of categories, where V is a finite-dimensional $\mathfrak{g}$-module with extremal weight λ.

Proof. Let $F = F_{\theta',V,\theta}$, $G = F_{\theta,V^*,\theta'}$. Remembering λ is an extremal weight of V (so that $-\lambda$ is such for V^*), one can show that (exercise)

$$FM_\chi = M_\psi, \quad GM_\psi = M_\chi.$$

Hence $GFM_\chi = M_\chi$, so the theorem provides that $GF \cong \mathrm{Pr}(\theta)$; similarly $FG \cong \mathrm{Pr}(\theta')$. By restricting F, G to $\mathscr{M}(\theta), \mathscr{M}(\theta')$, we deduce that they are categorical equivalences. □

- The following observations in [1] refine earlier results of Zuckerman [4]:
  (i) Let $\mathscr{H}$ be any complete subcategory of $\mathscr{M}$ preserved by all functors F_V, e.g. $\mathscr{H} = \mathcal{O}$. The same proof method shows that the intersections of $\mathscr{H}$ with $\mathscr{M}^\infty(\theta)$ and $\mathscr{M}^\infty(\theta')$ are equivalent.
  (ii) If we assume just an inequality of stabilisers $W_\psi \subseteq W_\chi$, then (in the notation of the proof) we conclude $GF \cong \mathrm{Id}^{\oplus|W_\chi : W_\psi|}$.

4.3 Lattices of two-sided ideals and submodules

- **Notation:** Suppose χ is a dominant weight with $\eta(\chi) = \theta$. Let Ω_θ be the lattice of two-sided ideals in U_θ; let Ω_χ be the submodule lattice of M_χ.
- **Theorem 4.2:** Let χ be a dominant weight, $\theta = \eta(\chi)$.
  (i) The mapping
  $$\nu : \Omega_\theta \to \Omega_\chi, \quad \nu(J) = JM_\chi$$
  is a lattice isomorphism if χ is regular.
  (ii) In general, ν is an embedding. Let $\mathscr{P}$ be the class of modules isomorphic to direct sums of P_ψ for $\psi < \chi$ and $\psi \leq W_\chi(\psi)$. Then the image of ν consists of the $\mathscr{P}$-generated submodules of M_χ.

Sketch. Note that (i) is a special case of (ii), so it suffices to prove (ii).

For (ii): Let (φ, F) be the pair of a projective θ-functor $F : \mathscr{M}(\theta) \to \mathscr{M}$ and $\varphi : F \to 1_{\mathscr{M}(\theta)}$ a natural transformation. Then let

$$J(\varphi, F) = \operatorname{im} \varphi_{U_\theta} \subseteq U_\theta, \quad M(\varphi, F) = \operatorname{im} \varphi_{M_\chi} \subseteq M_\chi.$$

Then follow these steps:

(a) $J(\varphi, F)$ is a two-sided ideal with $\nu(J(\varphi, F)) = M(\varphi, F)$.

(b) Every two-sided ideal of U_θ has the form $J(\varphi, F)$: take finitely many generators for a given J, belonging to a finite-dimensional ad-invariant subspace $V \subseteq J$; then $F = F_V$ and φ is induced from a bimodule homomorphism $V \otimes U_\theta \to U_\theta$.

(c) If $M(\varphi, F) \subseteq M(\varphi', F')$, then $J(\varphi, F) \subseteq J(\varphi', F')$: because FM_χ is projective, get a morphism $a : FM_\chi \to F'M_\chi$ such that $\varphi'_{M_\chi} \circ a = \varphi_{M_\chi}$. Then use Theorem C to get $\alpha : F \to F'$ lifting a, $\varphi' \circ \alpha = \varphi$, and hence the desired inclusion of images.

From these steps, we get that ν is an embedding. Also, (a) and (b) show that its image consists exactly of submodules having the form $M(\varphi, F)$. But Theorem C provides that these are exactly the submodules of the form $\varphi(FM_\chi)$, where F is a protective θ-functor and $\varphi \in \mathrm{Hom}_{\mathcal{O}}(FM_\chi, M_\chi)$. Now deduce the result from Theorem A, Corollary 3.9, and the Verma properties at the end of Section 2.

4.4 Duflo's theorem

The result in the previous section allows for an easy re-derivation of Duflo's famous theorem [2]

• **Theorem 4.3 (Duflo):** Let $J \in \Omega_\theta$ be a two-sided prime ideal. Then a weight $\psi \in \eta^{-1}(\theta)$ exists such that $J = \mathrm{Ann}\, L_\psi$.

Proof. Take $\chi \in \eta^{-1}(\theta)$ dominant. Let $L_1, \ldots, L_n$ be the composition factors of the module $M = M_\chi / JM_\chi$, with annihilators $I_i \subseteq U_\theta$. Certainly $J \subseteq I_i$ for all i, and the product $I = I_1 \cdots I_n$ annihilates M. It follows from section 4.2 that $I \subseteq J$. Invoking that J is prime gives $J = I_i$ for some i. But now from our knowledge of M_χ, we have that $L_i = L_\psi$ for some $\psi < \chi$, and the result follows. □

5 Classifying Harish–Chandra modules

5.1 Context

• **Extrinsic motivation:** Harish–Chandra modules for semisimple Lie groups over $\mathbb{R}$, which arise when studying the unitary dual of G: c.f. Anna's IFS talks in October last year.

• In this setup, we have a semisimple Lie group G and maximal compact K, with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$.

• Best understood special case: when G is a *complex* semisimple Lie group viewed as real. But to apply our theory above, we have to complexify and return to the algebraically closed $k = \mathbb{C}$. In fact the Lie algebra complexifies to $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{k}$ complexifies to a skew-diagonal copy of $\mathfrak{g}$ (with respect to the fixed anti-involution $t : \mathfrak{g} \to \mathfrak{g}$).

• The objective is then to study Harish–Chandra modules on the Lie algebra side, connecting them with subcategories of $\mathcal{O}$ and principal series representations.

• Throughout, we work instead with $\mathfrak{g}\oplus\mathfrak{g}^0$, because it's more convenient. This is possible because $U \cong U^{\mathrm{op}}$ thanks to the fixed anti-automorphism t of $\mathfrak{g}$.

5.2 Notation and definitions

• We retain the notation from last week, and introduce:

(i) $\mathfrak{g}^2 = \mathfrak{g} \oplus \mathfrak{g}^0$, where $\mathfrak{g}^0$ is the opposite Lie algebra.

(ii) Its subalgebra $\mathfrak{k} = \{(X, -X)\} \cong \mathfrak{g}$.

(iii) $Y|_{\mathfrak{k}}$ denotes the restriction of a $\mathfrak{g}^2$-module Y to a representation of $\mathfrak{k}$; this coincides with the $\mathfrak{g}$-module Y^{ad}.

• **Definition 5.1**: A U^2-module M is $\mathfrak{k}$-algebraic if the module $M|_{\mathfrak{k}}$ is a direct sum of finite-dimensional irreducible $\mathfrak{k}$-modules.

• **Lemma 5.2:** Let $\mathscr{H}$ ($\mathscr{H}_f$) denote the category of (finitely generated) $\mathfrak{k}$-algebraic modules. Then:

(i) $\mathscr{H}, \mathscr{H}_f$ are closed with respect to subquotients.

(ii) Modules Φ_V belong to $\mathscr{H}_f$, and any $Y \in \mathscr{H}_f$ is a quotient of some Φ_V.

(iii) Φ_V are projective objects in $\mathscr{H}, \mathscr{H}_f$.

• **Proposition 5.3:** Suppose $Y \in \mathscr{H}_f$. Then the following are equivalent:

(i) For any finite-dimensional $\mathfrak{g}$-module V, the space

$$\mathrm{Hom}_{\mathfrak{g}}(V, Y|_{\mathfrak{k}}) = \mathrm{Hom}_{\mathscr{H}}(\Phi_V, Y)$$

is finite-dimensional.

(ii) Any irreducible representation of $\mathfrak{g}$ occurs in a decomposition of $Y|_{\mathfrak{k}}$ with finite multiplicity.

(iii) The annihilator of Y in Z^2 has finite codimension.

(iv) The module Y is Z-finite on the right.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is a straightforward exercise.

(iv) $\Rightarrow$ (i): Suppose Y is Z-finite on the right, with generators $Y_1, \ldots, y_n$ over U^2. Let

$$J = \{z \in 1 \otimes Z : zy_i = 0\}, \quad J = \cap J_i.$$

By assumption, J has finite codimension in $1 \otimes Z$ and $JY = 0$. Now Y is a quotient of some Φ_L, hence also of Φ_L/J, so by projectivity of Φ_V we can assume $Y = \Phi_L/J$.

But then

$$\mathrm{Hom}_{\mathfrak{k}}(V,Y) = \mathrm{Hom}_{\mathfrak{k}}(V,\Phi_L)/J = \mathrm{Hom}_{\mathfrak{g}}(V, L\otimes U^{\mathrm{ad}})/J,$$

which is $\mathrm{Hom}(L^*\otimes V, U^{\mathrm{ad}})/J$, and this is finite-dimensional by Kostant's first theorem.

• Denote by $\mathscr{HC}$ the category of *Harish-Chandra modules* satisfying these equivalent conditions.

5.3 Classification of irreducibles

• From now on let $Z^r = 1\otimes Z, Z^\ell = Z\otimes 1 \subseteq Z^2$.

• We can decompose $\mathscr{HC}$ into a sum of categories, each corresponding to characters (θ_1,θ_2) of Z^2.

• So let us put $\mathscr{H}_f^m(\theta)^r = \{Y\in\mathscr{H}_f : (J_\theta^r)^m Y = 0\}$ and also define $\mathscr{H}_f^\infty(\theta)^r$ analogously to $\mathscr{M}^\infty(\theta)$ above.

• **First aim:** to study $\mathscr{H}_f(\theta)^r$, for $\theta\in\Theta$ fixed; this is a complete subcategory of (U,U_θ)-bimodules.

• **Proposition 5.4:** Let $\Phi_V(\theta) = \Phi_V/J_\theta^r$.

(i) The modules $\Phi_V(\theta)$ are projective in $\mathscr{H}_f(\theta)^r$, and every module in $\mathscr{H}_f(\theta)^r$ is a quotient of some $\Phi_V(\theta)$.

(ii) All Hom spaces in $\mathscr{H}_f(\theta)^r$ are finite-dimensional.

(iii) A projective object in $\mathscr{H}_f(\theta)^r$ admits a decomposition into a finite direct sum of indecomposable projective objects.

(iv) $\{\xi\in\Xi : \eta^r(\xi)=\theta\} \leftrightarrow \{$indecomposable projective objects $P_\xi\in\mathscr{H}_f(\theta)^r\}/$iso. P_ξ is determined up to isomorphism by the fact that $P_\xi\otimes_U M_\chi \cong P_\psi$ if $\xi = (\psi,\chi)$ is written properly.

Proof. (i) is immediate from Lemma 5.2(ii) and (iii) is immediate from (ii).

For (ii): if $\Phi_V(\theta)\twoheadrightarrow Y$ then $\mathrm{Hom}_U^2(Y,Y') \subseteq \mathrm{Hom}(\Phi_V(\theta),Y) = \mathrm{Hom}(\Phi_V,Y')$; now recall Prop. 5.3(i).

For (iv): notice that $F_V(\theta):\mathscr{M}(\theta)\to\mathscr{M}$ is the projective θ-functor corresponding to the (U,U_θ)-bimodule $\Phi_V(\theta)$ via the equivalence of categories h. So the statements follow from Theorems A and C, since any projective object P is a quotient (and hence a direct summand) of some $\Phi_V(\theta)$.

• **Lemma 5.5:** P_ξ has unique simple quotient L_ξ; dim $\mathrm{Hom}(P_\xi, L_{\xi'}) = \delta_{\xi,\xi'}$.

Proof. First, note End P_ξ is local with a nilpotent maximal ideal.

If P, P' are proper submodules of P_ξ, then $P + P'$ is proper. Indeed, otherwise we would have a surjection

$$\mathrm{Hom}(P_\xi, P) \oplus \mathrm{Hom}(P_\xi, P') \to \mathrm{Hom}(P_\xi, P_\xi),$$

and thus an expression $1 = \varphi + \varphi'$ in End P_ξ. In a local ring, this means one of φ, φ' is invertible, and hence one of $P, P' = P_\xi$. This proves the existence of a unique simple quotient.

By (c), $\mathrm{Hom}(P_\xi, L_\xi) = \mathrm{Hom}(L_\xi, L_\xi) = k$. If $\mathrm{Hom}(P_\xi, L_{\xi'}) \neq 0$, then $L_\xi \cong L_{\xi'}$ by uniqueness. Use projectivity to lift to to morphisms

$$\varphi : P_\xi \to P_{\xi'}, \quad \psi : P_{\xi'} \to P_\xi.$$

Check: $\varphi\psi$, $\psi\varphi$ are invertible, so φ, ψ are isomorphisms: $P_\xi \cong P_{\xi'}$ and $\xi = \xi'$ by Prop. 5.4(iv).

• **Theorem 5.6:** $\xi \mapsto L_\xi$ is a one-to-one correspondence between Ξ and the equivalence classes of simple Harsih–Chandra modules.

Proof. Every simple module of $\mathscr{H}$ belongs to the category $\mathscr{H}_f(\theta)^r$ for some θ. Then use Prop. 5.4.

• **Corollary 5.7:**

(i) If $J \subseteq Z^2$ is an ideal of finite codimension, then there are finitely many simple $L \in \mathscr{HC}$ with $JL = 0$.

(ii) Suppose $Y \in \mathscr{H}$ is annihilated by some such J, and $\mathrm{Hom}(V, Y)$ is finite-dimensional for any finite-dimensional $\mathfrak{g}$-module V. Then Y is of finite length. In particular, every $Y \in \mathscr{HC}$ is of finite length.

5.4 The functor T_χ

• Consider the functors

$$T_M = (-) \otimes_U M : \mathscr{H} \to \mathscr{M},$$

and in particular, $T_\chi = T_{M_\chi}$ for a weight $\chi \in \eta^{-1}(\theta)$.

• We consider T_χ as a functor $\mathscr{H}_f(\theta)^r \to \mathscr{O}$. Let $\mathscr{P}(\chi)$ denote the class of projective objects in category $\mathcal{O}$ formed as direcy sums of modules P_ψ, for $\psi \in \chi + \Lambda$ and $\psi \leq W_\chi(\psi)$. Also let $\mathcal{O}_{\chi+\Lambda}$ consist of modules $M \in \mathcal{O}$ such that $M^\psi = 0$ for $\psi \notin \chi + \Lambda$.

• **Theorem 5.8:** Let χ be a dominant weight, $\theta = \eta(\chi)$.

(i) If χ is regular, then T_χ defines an equivalence $\mathscr{H}_f(\theta)^r \cong \mathcal{O}_{\chi+\Lambda}$.

(ii) In general, T_χ is an equivalence between $\mathscr{H}_f(\theta)^r$ and $\mathcal{O}_{\mathscr{P}(\chi)}$.

• Note that (i) follows from (ii), because for a regular χ we have $\mathcal{O}_{\mathscr{P}(\chi)} = \mathcal{O}_{\chi+\Lambda}$. To prove (ii) we can apply the following general result.

• **Proposition 5.9:** Let $\mathscr{A}, \mathscr{B}$ be abelian categories. Let $\mathscr{P}$ be the class of projective objects in $\mathscr{A}$ and $T : \mathscr{A} \to \mathscr{B}$ be a right exact functor. Assume:

1. $\mathscr{A} = \mathscr{A}_{\mathscr{P}}$.
2. T preserves projective objects.
3. T is fully faithful on $\mathscr{P}$.

Then T is fully faithful and defines an equivalence of $\mathscr{A}$ with $\mathscr{B}_{T(\mathscr{P})}$.

5.5 Corollaries

• **Theorem 5.10:** Suppose $\chi \in \eta^{-1}(\theta)$ is dominant. Let $M = T_\chi(Y)$ for $Y \in \mathscr{H}_f(\theta)^r$, and let Ω_Y, Ω_M denote their respective lattices of submodules. Then

$$\nu : \Omega_Y \to \Omega_M, \quad \nu(Y' \to Y) = \mathrm{im}(T_\chi(Y') \to T_\chi(Y))$$

is an embedding, with image consisting of $\mathscr{P}_\chi$-generated submodules of M. In particular, ν is an isomorphism if χ is regular.

Proof. This theorem is proven in the same way as Theorem 4.2.

• **Proposition 5.11:** Let χ, ψ be weights in the same orbit of W. Then the categories $\mathcal{O}_{\chi+\Lambda}$, $\mathcal{O}_{\psi+\Lambda}$ are equivalent as Z-categories.

Proof. Assume $\psi = w\chi$. Replacing χ by $\chi + \lambda$, ψ by $\psi + w\lambda$ for $\lambda \in \Lambda$, we can assume χ, ψ are regular. Modifying them further by elements of $W_{\chi+\Gamma}$ and $W_{\psi+\Gamma}$, we can assume χ, ψ are dominant. But now

$$\mathcal{O}_{\chi+\lambda} \cong \mathscr{H}_f(\theta)^r \cong \mathcal{O}_{\psi+\Lambda}, \quad \text{for } \theta = \eta(\chi) = \eta(\psi).$$

6 Multiplicities in principal series reps, Verma modules

6.1 The functor H

• **Motivation:** To describe the inverse functor to the equivalence

$$T_\chi : \mathscr{H}_f(\theta)^r \to \mathcal{O}_{\chi+\Lambda}.$$

• For $M, N \in \mathscr{M}$, give $\mathrm{Hom}_k(M, N)$ the structure of a U^2-module by $((u \otimes u^0)\varphi)(m) = u\varphi(u^0 m)$. Then

$$H(M, N) = \{\mathfrak{k}\text{-finite vectors in } \mathrm{Hom}_k(M, N)\}$$

is a U^2-submodule, and thus we obtain a functor $H : \mathscr{M} \times \mathscr{M} \to \mathscr{H}$.

- For $M \in \mathscr{M}$ we get $H_M = H(M, -) : \mathscr{M} \to \mathscr{H}$.
- **Lemma 6.1:**

 (i) The functor H is left exact in each variable.

 (ii) H_M is right adjoint to T_M.

 (iii) The functors $N \mapsto H(M,N)$, $M \mapsto H(M,N)$ are linear over Z^ℓ, Z^r, respectively. If V is finite-dimensional, then it holds that

$$H(M, V \otimes N) = \Phi_V \otimes_U H(M,N), \quad H(V \otimes M, N) = H(M,N) \otimes_U \Phi_{V^*}.$$

- We are interested in category $\mathcal{O}$, and hence **Lemma 6.2:**

 (i) If $M, N \in \mathcal{O}$, then $H(M,N) \in \mathscr{HC}$.

 (ii) If M is a projective object in $\mathcal{O}$, then the functor $H_M : \mathcal{O} \to \mathscr{HC}$ is exact.

Proof. (ii) follows from the fact that H_M has a left adjoint, and so is left exact, but is also clearly right exact.

For (i), it suffices to verify that $\dim \mathrm{Hom}_{\mathfrak{k}}(V, H(M,N)) < \infty$ for any finite-dimensional $\mathfrak{g}$-module V, by the definition of Harish–Chandra modules. But in fact,

$$\mathrm{Hom}_{\mathfrak{k}}(V, H(M,N)) = \mathrm{Hom}_{U^2}(\Phi_V, H(M,N)) = \mathrm{Hom}_U(\Phi_V \otimes_U M, N),$$

and the latter is a Hom space between objects of category $\mathcal{O}$.

- **Proposition 6.3:** Let $\chi \in \eta^{-1}(\theta)$ be dominant. Then $H_\chi = H_{M_\chi}$ is the inverse to the functor T_χ as it appears in (both situations of) Theorem 5.8.

6.2 Representations of principal series

- Suppose $M, N \in \mathcal{O}$ and give $N \otimes M$ the $\mathfrak{g}^2$-module structure

$$(x,y)(n \otimes m) = (-x^t n) \otimes m - n \otimes ym,$$

where t is a anti-automorphism of $\mathfrak{g}$.

- The conugate module $(N \otimes M)^*$ is naturally isomorphic to the space of bilinear forms $B : N \times M \to k$, with action

$$((x,y)B)(n,m) = B(x^t n, m) + B(n, ym).$$

- We let $\mathrm{Dual}(N,M) \subseteq (N \otimes M)^*$ be the submodule of $\mathfrak{k}$-finite vectors, and then get the **representation of principal series**

$$X(\psi, \chi) = \mathrm{Dual}(M_\psi, M_\chi), \quad \psi, \chi \in \mathfrak{h}^*.$$

• **Proposition 6.4:** Let $M, N \in \mathcal{O}$. Then $\mathrm{Dual}(N, M) = H(M, N^\tau)$. Hence $X(\psi, \chi) \cong H_\chi(M_\psi^\tau)$.

• **Theorem 6.5:** Let $\chi, \psi \in \mathfrak{h}^*$ and $\xi \in \Xi$. Suppose χ is a dominant weight.

(i) If ξ cannot be written properly, $\xi = (\varphi, \chi)$, then $[X(\psi, \chi) : L_\xi] = 0$.

(ii) If ξ can be written properly, $\xi = (\varphi, \chi)$, then $[X(\psi, \chi) : L_\xi] = [M_\psi : L_\varphi]$.

Proof. This is a direct calculation:

$$\begin{aligned}
[X(\psi, \chi) : L_\xi] = \dim \mathrm{Hom}_{\mathscr{H}}(P_\xi, X(\psi, \chi)) &= \dim \mathrm{Hom}_{\mathscr{H}}(P_\xi, H_\chi(M_\psi^\tau)) \\
&= \dim \mathrm{Hom}_U(T_\chi(P_\xi), M_\psi^\tau) \\
&= \dim \mathrm{Hom}_u(P_\varphi, M_\psi^\tau) \\
&= [M_\psi^\tau : L_\varphi] = [M_\psi : L_\varphi].
\end{aligned}$$

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