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Change of stability for Schrödinger semigroups*

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In this paper we analyse the change of stability of Schrödinger semigroups with indefinite potentials when a coupling parameter varies. Generically, the change of stability takes place at a principal eigenvalue associated with the problem. The uniqueness of the principal eigenvalue is shown for several classes of potentials.

1. Introduction

We shall be concerned with the stability of the zero solution of the linear parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \lambda m u & \text{in } \mathbf{R}^N \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbf{R}^N, \\ \lim\limits_{|x| \to \infty} u(x) = 0, \end{cases} \tag{1.1}$$

when the parameter λ varies in the positive real axis. Here, m is some bounded and continuous weight function being positive somewhere and u_0 an initial condition. The stability is understood as stability in the $\|\cdot\|_\infty$-norm. We point out that by duality and interpolation we also get the stability in $L_p(\mathbf{R}^N)$, $1 \leqq p < \infty$. The question of stability of the zero solution is closely related to the existence of a principal eigenvalue for the elliptic eigenvalue problem

$$\begin{cases} -\Delta \varphi = \lambda m \varphi & \text{on } \mathbf{R}^N, \\ \lim\limits_{|x| \to \infty} \varphi(x) = 0. \end{cases} \tag{1.2}$$

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By a principal eigenvalue we mean a $\lambda > 0$ such that (1.2) has a positive solution φ. The function φ is then called principal eigenfunction.

The main goal of this paper is to obtain theorems of the following type under various assumptions on m.

THEOREM 1.1. *There exists a unique principal eigenvalue $\lambda_1 > 0$ for (1.2) and the zero solution of (1.1) is asymptotically stable for $\lambda \in [0, \lambda_1)$ and unstable for $\lambda > \lambda_1$.*

The stability or instability of the zero solution of (1.1) is equivalent to the stability or instability of the semigroup generated by $\Delta + \lambda m$ on the space

$$C_0(\mathbf{R}^N) := \left\{ u \in C(\mathbf{R}^N) : \lim_{|x| \to \infty} u(x) = 0 \right\}.$$

The operator Δ generates an irreducible analytic semigroup on $C_0(\mathbf{R}^N)$ with domain of definition

$$D(\Delta) = \{u \in C_0(\mathbf{R}^N) : \Delta u \in C_0(\mathbf{R}^N)\},$$

where the derivatives have to be understood in the sense of distributions ([**8**]; for the theory of irreducible semigroups, see [**15**]). Throughout this work, we restrict ourselves to considering continuous and bounded weight functions. For these weights the operator $A_\lambda := \Delta + \lambda m$ also generates an irreducible analytic semigroup on $C_0(\mathbf{R}^N)$, with the same domain of definition, $D(\Delta)$.

For analytic semigroups the spectral mapping theorem holds (cf. [**15**, Corollary A-III.6.7]), i.e. $\sigma(e^{tA_\lambda}) \backslash \{0\} = e^{t\sigma(A_\lambda)}$ for $t > 0$. This makes it possible to study the spectrum of

$$S_\lambda := e^{A_\lambda} \tag{1.3}$$

to analyse the stability of the semigroup. Such stability is determined by the spectral radius

$$r(\lambda) := \operatorname{spr} S_\lambda.$$

The semigroup is exponentially stable if $r(\lambda) < 1$ and unstable if $r(\lambda) > 1$. When $r(\lambda) = 1$, extra information is needed to decide if the semigroup is stable or not. Thus the graph of $r(\lambda)$ gives a great deal of information about the stability of the zero solution, and the nature of this graph is constrained by a theorem of Kato [**13**] which asserts that $r(\lambda)$ is a log-convex function.

The stability of the zero solution and the behaviour of $r(\lambda)$ is well understood for bounded domains (see [**3**, **10**]). In the bounded domain case, $\sigma(A_\lambda)$ consists only of eigenvalues and if a principal eigenvalue λ_1 exists then 0 must be the greatest eigenvalue of $\Delta + \lambda_1 m$ and so $r(\lambda_1) = 1$. In the case of Dirichlet or Robin boundary conditions, there exists a principal eigenvalue $\lambda_1 > 0$; also $r(0) < 1$ and the graph of $r(\lambda)$ is as shown in Figure 1.1(a) and so Theorem 1.1 holds. The theorem also holds in the case of Neumann boundary conditions with $\int_D m(x)\,dx < 0$ and the graph of $r(\lambda)$ is as shown in Figure 1.1(b). If, however, we consider Neumann boundary conditions with $\int_D m(x)\,dx \geqq 0$, then no positive principal eigenvalue exists, the graph of $r(\lambda)$ is as shown in Figure 1.1(c) and so Theorem 1.1 does not hold.

The situation is more complicated in the case of unbounded regions. Principal eigenvalues may not exist for (1.2) unless some further restrictions are placed on m.

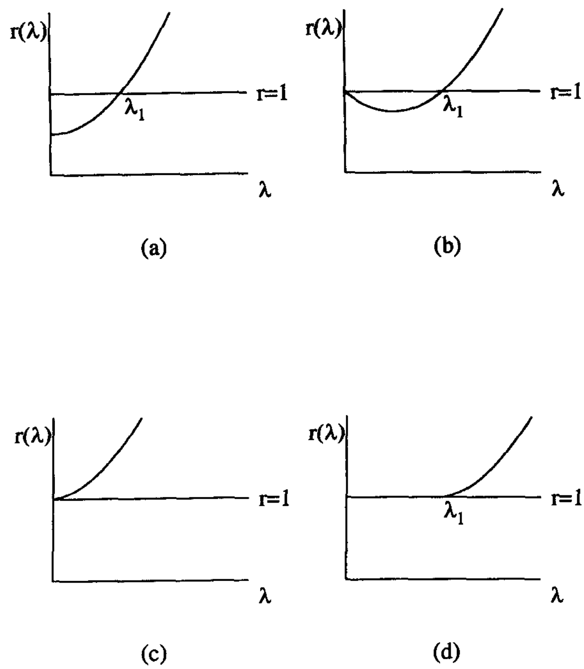

Figure 1.1.

We shall investigate the validity of Theorem 1.1 and the nature of the graph of $r(\lambda)$ when m satisfies assumptions of the following types:

(i) m is sufficiently small at infinity and $N \geqq 3$; in particular, we shall obtain detailed results in the case where m has compact support.

(ii) m is sufficiently negative at infinity.

The existence of principal eigenvalues has been proved under such assumptions in [**1**, **5**, **6**]. Results on the uniqueness of such eigenvalues and on the change of stability have been obtained in [**9**] for some special periodic-parabolic problems. Theorems on change of stability for positive potentials have also been obtained in [**19**, Section B.5].

We now consider the general nature of the function $r(\lambda)$. Since $\sigma(\Delta) = (-\infty, 0]$, $r(0) = 1$. Moreover, since m is somewhere positive, it follows that $\lim_{\lambda \to \infty} r(\lambda) = \infty$ (see [**9**]). In order that Theorem 1.1 holds, it is necessary that there exists a principal eigenvalue λ_1 such that the stability properties of the zero solution change at $\lambda = \lambda_1$ and this indicates that we must have $r(\lambda_1) = 1$. Thus the graph of $r(\lambda)$ in Figure 1.1(b) is compatible with the validity of Theorem 1.1 and we shall show that when m is sufficiently negative at infinity then the theorem holds and $r(\lambda)$ is as shown in Figure 1.1(b). When m is sufficiently small at infinity, however, Theorem 1.1 may still hold but the graph of $r(\lambda)$ has a different nature. If m is sufficiently small at infinity, e.g. m has compact support, then λm is a relatively compact perturbation of Δ and so

$$\sigma_{\mathrm{ess}}(\Delta + \lambda m) = \sigma_{\mathrm{ess}}(\Delta) = (-\infty, 0].$$

Hence $r(\lambda) \geqq 1$ for all $\lambda \geqq 0$. It was shown in [5] that there exists a principal eigenvalue λ_1 in this case; it follows easily that 0 is an extreme point of $\sigma(\Delta + \lambda_1 m)$ and so $r(\lambda_1) = 1$. Since $r(\lambda)$ is log convex, it follows that $r(\lambda) \equiv 1$ for $0 < \lambda < \lambda_1$. Thus the graph of $r(\lambda)$ may be as shown in Figure 1.1(d). In order to prove Theorem 1.1 in this case, it is necessary to prove the stability of the zero solution in the range $0 < \lambda < \lambda_1$ where $r(\lambda) \equiv 1$. Suppose that m is non-negative. Since $e^{t\Delta}$ is neutrally stable and

$$e^{t\Delta} \leqq e^{t(\Delta + \lambda m)},$$

it is surprising that any such stability result should hold. We shall show that such results for non-negative m are possible when $N \geqq 3$ but not when $N = 1, 2$. The deeper reason for that different behaviour lies in the recurrence and nonrecurrence properties of the Brownian motion modelling the diffusion process (e.g. [11]). In fact, the nonrecurrence of the diffusion in dimension $N \geqq 3$ makes heat disappear at infinity, so that in spite of the existence of a heat source even asymptotic stability is possible.

The plan of the paper is as follows. In Section 2, we consider the case where $N \geqq 3$ and m is radially symmetric and has compact support. We prove that $r(\lambda)$ is as shown in Figure 1.1(d) and that Theorem 1.1 holds for such weight functions. In Section 3, we obtain general results about the case where m has compact support and show that in general, whether or not there exists a principal eigenvalue, $r'(0) = 0$ and that, whenever $\int_{\mathbf{R}^N} m(x)\,dx < 0$, $r(\lambda) \equiv 1$ for small λ. The results obtained for the compact support case are useful tools for studying the case where m is sufficiently negative. In Section 3, we use them to get an easy proof of Theorem 1.1 in the case where m is negative and bounded away from zero at infinity and, in Section 4, to prove Theorem 1.1 for functions m which are sufficiently negative in the sense that $e^{t(\Delta - \lambda m^-)}$ is exponentially stable, where $m^- \geqq 0$ denotes the negative part of m. A characterisation of such functions m^- was obtained recently by Arendt and Batty [2]. In Section 5, we return to the case where m is small at infinity but may not have compact support. We give a new proof of the existence of a principal eigenvalue λ_1, in this case showing that λ_1 is the limit of the principal eigenvalues for Dirichlet problems on large balls so that λ_1 has the variational characterisation

$$\lambda_1 = \inf_{\substack{\psi \in \mathscr{D}(\mathbf{R}^N) \\ \int_{\mathbf{R}^N} m\psi^2 > 0}} \frac{\int_{\mathbf{R}^N} |\nabla \psi|^2}{\int_{\mathbf{R}^N} m\psi^2}$$

and are thus are able to establish the uniqueness of the principal eigenvalue and the instability of the zero solution when $\lambda > \lambda_1$.

2. The case *m* is radially symmetric with compact support

Throughout this section, we shall assume that $N \geqq 3$ and that $m(x)$ is continuous and radially symmetric with compact support. First we give some definitions and collect some results on spectral theory which will be used later. Let X be a Banach space and $T \in \mathscr{L}(X)$. A $\lambda \in \mathbf{C}$ is said to belong to the Browder *essential spectrum* of

T, denoted by $\sigma_{\text{ess}}(T)$, if and only if one of the following conditions is satisfied: λ is a limit point of $\sigma(T)$, or the image of $\lambda - T$ is not closed, or the space $\bigcup_{k \geqq 1} \ker [(\lambda - T)^k]$ is infinite dimensional. The *essential spectral radius* of T is defined as

$$\operatorname{spr}_{\text{ess}}(T) \equiv \sup \{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\}.$$

REMARKS 2.1. (i) If $\lambda \in \sigma(T) - \sigma_{\text{ess}}(T)$, then λ is a pole of the resolvent of T. Hence, it is an eigenvalue of T (cf. [**7**, Theorem A.3.3]).

(ii) Let T be a closed densely defined operator on the Banach space X and $K \in \mathscr{L}(X)$ such that $K(\mu - T)^{-1}$ is compact for some $\mu \in \rho(T)$. If $\rho(T)$ is connected and dense in $\mathbf{C}$ and $\rho(T+K)$ is not empty, then

$$\sigma_{\text{ess}}(T+K) = \sigma_{\text{ess}}(T),$$

(cf. [**4**, Lemma 18]).

(iii) If X is a Banach lattice and T is positive, then

$$\operatorname{spr}(T) \in \sigma(T).$$

Therefore, whenever $\operatorname{spr}_{\text{ess}}(T) < \operatorname{spr}(T)$, part (i) implies that $\operatorname{spr}(T)$ is an eigenvalue. If that is the case and in addition T is irreducible, then it follows from [**17**, Theorem 5.2] that $\operatorname{spr}(T)$ is an algebraically simple eigenvalue with associated eigenvector lying in the quasi-interior of X_+. Moreover, there is no other eigenvalue with associated positive eigenfunction.

In particular, these properties hold for $X = C_0(\mathbf{R}^N)$ and $T = e^{\Delta + \lambda m}$ when m is continuous and bounded as the following remarks show.

REMARK 2.2. (i) Remark 2.1(ii) may be applied to determine the essential spectrum of $\Delta + m$ as an operator on $C_0(\mathbf{R}^N)$ if $m \in C_0(\mathbf{R}^N)$. Observe first that $D(\Delta)$ is continuously embedded in $C^1(\mathbf{R}^N)$. If we show that $u \mapsto mu$ is a compact operator from $C^1(\mathbf{R}^N)$ to $C_0(\mathbf{R}^N)$, then the hypotheses of Remark 2.1(ii) are obviously satisfied and hence

$$\sigma_{\text{ess}}(\Delta + \lambda m) = \sigma_{\text{ess}}(\Delta) = (-\infty, 0].$$

Using embedding theorems on bounded domains, it is clear that $u \mapsto mu$ is compact from $C^1(\mathbf{R}^N)$ to $C_0(\mathbf{R}^N)$ when m has compact support. The norm of the multiplication operator defined above is $\|m\|_\infty$. Since the continuous functions having compact support are dense in $C_0(\mathbf{R}^N)$ and the ideal of compact operators is closed, it follows that multiplication by any C_0-function is a compact operator.

(ii) The essential spectrum of the operators $e^{t(\Delta + \lambda m)}$ can be determined from the result above. By (i) for any $m \in C_0(\mathbf{R}^N)$ we can apply [**9**, Proposition 5.4] to see that

$$e^{t(\Delta + \lambda m)} = e^{t\Delta} + K_\lambda(t),$$

where $K_\lambda(t)$ is a compact operator on $C_0(\mathbf{R}^N)$. Hence we obtain that

$$\sigma_{\text{ess}}(e^{t(\Delta + \lambda m)}) = \sigma_{\text{ess}}(e^{t\Delta}) = [0, 1]$$

for all $t > 0$ and $\lambda \in \mathbf{R}$. Applying Remark 2.1(iii) to this situation we see that if $r(\lambda) = \operatorname{spr}(e^{\Delta + \lambda m}) > 1$, then $r(\lambda)$ is the only eigenvalue having positive eigenfunction.

Now we have all the ingredients to begin the results of this section. The following theorem holds:

THEOREM 2.3. *Equation* (1.2) *has a unique principal eigenvalue* $\lambda_1 = \lambda_1(m) > 0$. *Moreover,* $r(\lambda) = 1$ *for all* $\lambda \in [0, \lambda_1]$.

Proof. We seek a positive radially symmetric eigenfunction. Thus problem (1.2) reduces to

$$\varphi''(r) + \frac{N-1}{r}\varphi'(r) + \lambda m(r)\varphi(r) = 0, \quad \varphi'(0) = 0. \tag{2.1}$$

Let $R > 0$ be such that $\operatorname{supp}(m) \subset B_R := \{x : |x| < R\}$. Then for $r \geqq R$, (2.1) reduces to

$$\varphi''(r) + \frac{N-1}{r}\varphi'(r) = 0,$$

whose general solution is $C_1 + C_2 r^{-(N-2)}$ where C_1, C_2 are constants. Since $\varphi(r) \to 0$ as $r \to \infty$ and $N \geqq 3$, we must have $\varphi(r) = Cr^{-(N-2)}$, $r \geqq R$, for some $C > 0$. Hence

$$\varphi'(R) = -\frac{N-2}{R}\varphi(R),$$

which gives us the boundary condition to be satisfied by the eigenfunction on the boundary of the ball B_R. It is well known that the boundary value problem

$$\begin{cases} -\Delta\varphi = \lambda m\varphi & \text{for } |x| < R, \\ \dfrac{\partial}{\partial n}\varphi = -\dfrac{N-2}{R}\varphi, & \text{for } |x| = R, \end{cases} \tag{2.2}$$

has a unique principal eigenvalue $\lambda_1 > 0$ with a corresponding unique positive radially symmetric eigenfunction, say $\varphi_0(r)$. Let $C > 0$ be such that $\varphi_0(R) = CR^{-(N-2)}$. Then the function

$$\varphi(x) := \begin{cases} \varphi_0(|x|), & \text{for } |x| \leqq R, \\ C|x|^{-(N-2)}, & \text{for } |x| > R, \end{cases} \tag{2.3}$$

is a principal eigenfunction of (1.2) associated with λ_1.

Remark 2.2(ii) excludes the possibility that $r(\lambda_1) > 1$ since we know that the eigenvalue 1 has a positive eigenfunction. This shows that $r(\lambda_1) = 1$.

Since $r(0) = r(\lambda_1) = 1$ and $\lambda \to r(\lambda)$ is log-convex, it follows that $r(\lambda_1) = 1$ for all $\lambda \in [0, \lambda_1]$. □

REMARK 2.4. Since the decay of the principal eigenfunction φ at infinity is of order $r^{-(N-2)}$, it follows readily that $\varphi \in L_p(\mathbf{R}^N)$ provided $p > (N/(N-2))$. In particular, $\varphi \notin L_1$ for any N and $\varphi \in L_2$ if and only if $N \geqq 5$. This fact was observed originally in [**18**] for the special case $m \geqq 0$.

The stability of the semigroup generated by $\Delta + \lambda m$, $\lambda \in [0, \lambda_1(m))$, on $C_0(\mathbf{R}^N)$ will be obtained from the following result:

PROPOSITION 2.5. *Given* $\lambda \in [0, \lambda_1(m))$ *and* $c_1 > 0$, *there exists* $\psi \in C_0(\mathbf{R}^N)$ *radially sym-*

metric such that

$$\varphi = c_1 + \psi$$

satisfies

$$-\Delta\varphi = \lambda m \varphi$$

and $\varphi(x) > 0$ *for all* $x \in \mathbf{R}^N$.

Proof. We define

$$\psi(x) = c_2 |x|^{-(N-2)}, \quad |x| > R,$$

where c_2 is a constant to be chosen later and $R > 0$ is such that $\operatorname{supp} m \subset B_R$. In B_R, the function ψ is defined as the unique solution of the boundary value problem

$$\begin{cases} -\Delta\psi - \lambda m \psi = \lambda c_1 m & \text{in } B_R, \\ \dfrac{\partial}{\partial n}\psi + \dfrac{N-2}{R}\psi = 0 & \text{on } \partial B_R. \end{cases}$$

Since $\lambda < \lambda_1$, this boundary value problem has a unique radially symmetric solution ψ. If we choose c_2 so that $c_2|x|^{-(N-2)} = \psi(x)$ for $|x| = R$, then $\varphi := c_1 + \psi$ satisfies $-\Delta\varphi = \lambda m \varphi$ on $\mathbf{R}^N$.

It remains to show that φ is positive. In B_R, φ satisfies

$$\begin{cases} -\Delta\varphi - \lambda m \varphi = 0 & \text{in } B_R, \\ \dfrac{\partial}{\partial n}\varphi + \dfrac{N-2}{R}\varphi = c_1 \dfrac{N-2}{R} > 0 & \text{on } \partial B_R. \end{cases} \tag{2.4}$$

Since $\lambda < \lambda_1$, the problem

$$\begin{cases} -\Delta\varphi - \lambda m \varphi = \mu\varphi & \text{in } B_R, \\ \dfrac{\partial}{\partial n}\varphi + \dfrac{N-2}{R}\varphi = 0 & \text{on } \partial B_R \end{cases}$$

has a positive principal eigenvalue μ with corresponding positive principal eigenfunction w. Then using this function w as the auxiliary function in the generalized maximum principle (see [**16**, p. 73, Theorem 10]), it follows that, since φ satisfies (2.4), $\varphi(x) > 0$ for any $x \in \bar{B}_R$. Since $\varphi = c_1 + \psi$ and $|\psi|$ is a decreasing function for $|x| > R$, it is easy to see that $\varphi(x) > 0$ for all $x \in \mathbf{R}^N$ and the proof of the proposition is complete. □

COROLLARY 2.6. *For any* $\lambda \in [0, \lambda_1(m))$, *the Schrödinger semigroup* $e^{t(\Delta + \lambda m)}$ *is bounded.*

Proof. Note first that $\| e^{t(\Delta+\lambda m)} \|_{\infty,\infty} = \| e^{t(\Delta+\lambda m)} 1 \|_\infty$ for all $t \geqq 0$. Let φ be the bound state given by Proposition 2.5. Then

$$1 \leqq \frac{\varphi}{\inf_{x \in \mathbf{R}^N} \varphi(x)} < \infty$$

and hence,

$$\| e^{t(\Delta + \lambda m)} \|_{\infty,\infty} \leqq \left\| e^{t(\Delta + \lambda m)} \frac{\varphi}{\inf_{x \in \mathbf{R}^N} \varphi(x)} \right\|_{\infty} = \frac{\| \varphi \|_{\infty}}{\inf_{x \in \mathbf{R}^N} \varphi(x)} < \infty$$

for all $t \geqq 0$. This proves the assertion. □

We remark that, for $\lambda = \lambda_1(m)$, the semigroup is not necessarily bounded [**18**]. Now we are able to state and prove our stability result.

THEOREM 2.7. *For all $\lambda \in [0, \lambda_1(m))$, the semigroup $e^{t(\Delta + \lambda m)}$ is stable on $C_0(\mathbf{R}^N)$, i.e.*

$$\lim_{t \to \infty} \| e^{t(\Delta + \lambda m)} u_0 \|_{\infty} = 0$$

for all $u_0 \in C_0(\mathbf{R}^N)$.

Proof. Let $R > 0$ be such that $\operatorname{supp} m \subset B_R$ and consider the eigenvalue problem

$$\begin{cases} -\Delta\varphi - \lambda m\varphi = \mu m^+ \varphi & \text{in } B_R, \\ \dfrac{\partial}{\partial n} \varphi + \dfrac{N-2}{R} \varphi = 0 & \text{on } \partial B_R, \end{cases} \tag{2.5}$$

where m^+ is the positive part of m and $\mu > 0$ will be regarded as a new parameter. Since $\lambda < \lambda_1(m)$, there exists a unique $\mu_1 > 0$ such that (2.5) has a positive eigenfunction φ_0, which is radially symmetric. Choose $c > 0$ so that $c|x|^{-(N-2)} = \varphi_0(x)$ for $|x| = R$. Then the function

$$\varphi(x) := \begin{cases} \varphi_0(x) & \text{for } |x| \leqq R, \\ c|x|^{-(N-2)} & \text{for } |x| > R \end{cases}$$

satisfies

$$-\Delta\varphi - \lambda m\varphi = \mu_1 m^+ \varphi > 0$$

on $\mathbf{R}^N$. Since φ is a supersolution for the elliptic problem, it follows that $e^{t(\Delta + \lambda m)}\varphi$ is a decreasing function of t and so

$$\lim_{t \to \infty} \| e^{t(\Delta + \lambda m)} \varphi \|_{\infty} = 0.$$

By comparison, this holds for every $u_0 \in C_0(\mathbf{R}^N)$ having compact support. Given $u_0 \in C_0(\mathbf{R}^N)$ and fixed $\varepsilon > 0$, there exists a decomposition $u_0 = u_1 + u_2$ such that $\| u_1 \|_{\infty} < \varepsilon$ and $u_2 \in C_0(\mathbf{R}^N)$ having compact support. Hence, by the boundedness of the semigroup, we find that

$$\| e^{t(\Delta + \lambda m)} u_0 \|_{\infty} \leqq M\varepsilon + \| e^{t(\Delta + \lambda m)} u_2 \|_{\infty},$$

for some constant $M > 0$, which completes the proof. □

An alternative proof of the above theorem may be based on [**15**, Corollary A-IV.1.14], by showing that zero lies in the continuous spectrum of $\Delta + \lambda m$.

It is also possible to prove that $\lambda_1(m)$ is the unique principal eigenvalue (our construction in Theorem 2.3 shows that $\lambda_1(m)$ is the only eigenvalue corresponding to a positive radially symmetric eigenfunction, but does not preclude the existence of another eigenvalue corresponding to a positive nonradially symmetric eigenfunction) and that the zero solution is unstable for $\lambda > \lambda_1(m)$. We defer the proofs of these results to Section 5, where they are given for more general m.

Thus Theorem 1.1 holds in the case where m is continuous and radially symmetric with compact support.

3. General properties for continuous weights with compact support

Throughout this section, it is assumed that $m(x)$ is a continuous weight function with compact support. Our goal is to obtain some general properties for this class of potentials which are of interest in themselves but which we shall also use later to study the case where m is sufficiently negative to ensure that Theorem 1.1 holds. Some open problems arising in [**18**, Section 5] will also be partially answered.

THEOREM 3.1. *Suppose $N \geqq 1$ and that m is continuous and has compact support. Then $r'(0)=0$, where $'$ stands for differentiation with respect to the parameter λ.*

Notice that if $N \geqq 3$ and m is radially symmetric, the above result is an immediate consequence of Theorem 2.3.

Proof of Theorem 3.1. Since $r(\lambda)$ is log-convex and $r(0)=1$, either (i) there exists $\varepsilon > 0$ such that $r(\lambda)=1$ for $\lambda \in [0,\varepsilon]$, or (ii) $r(\lambda) > 1$ for all $\lambda > 0$. If (i) occurs, then there is nothing to show. Suppose (ii) holds. Since $r(\lambda) > 1$, it follows from Remark 2.2(ii) that

$$1 = \mathrm{spr}_{\mathrm{ess}}\,(e^{(\Delta+\lambda m)}) < \mathrm{spr}\,(e^{(\Delta+\lambda m)}) = r(\lambda)$$

and that $r(\lambda)$ is an algebraically simple eigenvalue whose corresponding positive eigenfunction we denote by φ_λ. Then φ_λ is an eigenfunction of the operator $-(\Delta+\lambda m)$ corresponding to the eigenvalue

$$\mu(\lambda) = -\log r(\lambda).$$

Hence, we obtain

$$(-\Delta - \mu(\lambda))\varphi_\lambda = \lambda m \varphi_\lambda. \tag{3.1}$$

Since $\mu(\lambda) < 0$ lies in the resolvent set of $-\Delta$ considered in $C_0(\mathbf{R}^N)$ as well as $L_2(\mathbf{R}^N)$ and the right-hand side of (3.1) lies in $L_2(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$, we see that φ_λ has to be in the domain of definition of Δ as an operator in L_2, which is $H_2^2(\mathbf{R}^N)$. In particular, φ_λ lies in $H_2^1(\mathbf{R}^N)$.

From standard analytical perturbation results (see [**12**]), the function $\mu(\cdot)$ is analytic in $(0,\infty)$ and $\lambda \to \varphi_\lambda$ may be chosen analytic and normalised so that $\|\varphi_\lambda\|_{L_2} = 1$ for all $\lambda > 0$. By differentiating (3.1) with respect to λ, we obtain

$$-(\Delta+\lambda m)\varphi'_\lambda - m\varphi_\lambda = \mu'(\lambda)\varphi_\lambda + \mu(\lambda)\varphi'_\lambda.$$

Multiplying this relation by φ_λ and integrating over $\mathbf{R}^N$, using the fact that $\varphi_\lambda \in H_2^1(\mathbf{R}^N)$, it follows that

$$\mu'(\lambda) = -\int_{\mathbf{R}^N} m\varphi_\lambda^2\,dx. \tag{3.2}$$

In a similar way, multiplying (3.1) by φ_λ and integrating, we obtain

$$\int_{\mathbf{R}^N} \|\nabla\varphi_\lambda\|^2\,dx = \mu(\lambda) + \lambda \int_{\mathbf{R}^N} m\varphi_\lambda^2\,dx. \tag{3.3}$$

In particular, (φ_λ) is bounded in H_2^1, uniformly on bounded intervals of λ and hence we can find a sequence, say $(\varphi_{\lambda_k})_{k\geqq 1}$, $\lambda_k \to 0$, as $k\to\infty$, converging to some φ in $L_{2,\mathrm{loc}}$. Since $\mu(\lambda_k)\to 0$ as $k\to 0$, it follows from (3.3) that $\nabla\varphi_{\lambda_k}$ is a Cauchy sequence in $L_{2,\mathrm{loc}}$ and so φ_{λ_k} is convergent to φ in $H_2^1(\Omega)$ on any bounded set Ω. Letting $k\to\infty$ in (3.3) shows that $\int_{\mathbf{R}^N}\|\nabla\varphi\|^2\,dx = 0$ and so φ is a constant function. Since (φ_{λ_k}) converges to φ in L_2 on any bounded set and $\|\varphi_{\lambda_k}\|_{L_2} = 1$ for all k, it follows that $\varphi\equiv 0$. Hence (φ_k) converges to the zero function in L_2 on the support of m and so by (3.2) $\mu'(0) = 0$. Thus $r'(0) = 0$ and the proof is complete. □

We now use Theorem 3.1 to show that Theorem 1.1 holds for potentials negative and bounded away from zero at infinity.

COROLLARY 3.2. *Let $m\in C(\mathbf{R}^N)$ be such that there exist $c>0$ and $R>0$ satisfying $m(x)\leqq -c<0$ for all $|x|>R>0$. Then the problem (1.2) has a unique principal eigenvalue, denoted by $\lambda_1 = \lambda_1(m) > 0$, which is algebraically simple. Moreover, the zero solution of (1.1) is exponentially stable if $\lambda\in(0,\lambda_1)$ and unstable for $\lambda>\lambda_1$. In addition, we have that $r'(0)\leqq -c$.*

Proof. Let $m^*(x)$ be continuous radially symmetric such that $m(x)\leqq m^*(x)$, $x\in\mathbf{R}^N$, and $m^*(x) = -c$ for $|x|>2R$. Then m^*+c is continuous radially symmetric and has compact support. Set $r(\lambda) = \mathrm{spr}\,(e^{(\Delta+\lambda m)})$ and $r^*(\lambda) = \mathrm{spr}\,(e^{(\Delta+\lambda(m^*+c))})$. Then

$$e^{(\Delta+\lambda m)} \leqq e^{(\Delta+\lambda m^*)} = e^{(\Delta+\lambda(m^*+c)-\lambda c)} = e^{-\lambda c}e^{(\Delta+\lambda(m^*+c))} \tag{3.4}$$

and so

$$r(\lambda)\leqq e^{-\lambda c}r^*(\lambda)$$

By Lemma 3.1, $dr^*/d\lambda = 0$ and so $(d/d\lambda)(e^{-\lambda c}r^*(\lambda))|_{\lambda=0} = -c$. It follows that $r'(0)\leqq -c$ and so $r(\lambda)<1$ for λ small enough. On the other hand, it follows from [**9**, Lemma 7.2] that

$$\lim_{\lambda\to\infty} r(\lambda) = \infty,$$

and this, together with the log-convexity of $r(\lambda)$, shows the existence of a unique $\lambda_1 = \lambda_1(m)$ such that $r(\lambda)<1$ for $\lambda\in(0,\lambda_1)$, $r(\lambda_1) = 1$ and $r(\lambda)>1$ for $\lambda>\lambda_1$. But $\mathrm{spr}_{\mathrm{ess}}\,(\Delta+\lambda m)\subseteq(-\infty,-\lambda c]$, see [**5**, Theorem 2.1], and so $\mathrm{spr}_{\mathrm{ess}}\,(e^{(\Delta+\lambda m)})<1$. Hence it follows from Remark 2.1(iii) that λ_1 is a principal eigenvalue. It is easy to see that the stability results in the statement of the theorem hold.

Finally we prove that λ_1 is the unique principal eigenvalue. If $\lambda<\lambda_1$, then $r(\lambda)<1$ and a principal eigenvalue cannot exist. If $\lambda>\lambda_1$, then according to Remark 2.2(ii) $r(\lambda)>1$ is the only eigenvalue having positive eigenfunction so that 1 cannot be a principal eigenvalue of $e^{(\Delta+\lambda m)}$, i.e. λ is not a principal eigenvalue of (1.2.). □

This corollary extends related theorems in [**5**] and [**9**] by guaranteeing the uniqueness of the principal eigenvalue obtained in [**5**] and by supplementing the results in [**9**] by estimating the slope of $r(\lambda)$ at $\lambda = 0$; this slope gives information on the rate of convergence to zero of any positive solution of (1.1) if λ is small.

Corollary 3.2 also enables us to prove the following result about the case where m has compact support.

THEOREM 3.3. *Assume that m is continuous with compact support and*

$$\int_{\mathbf{R}^N} m(x)\,dx < 0.$$

For any ball $B \subset \mathbf{R}^N$ such that $\int_B m < 0$, *let* $\lambda_{N,B}(m)$ *denote the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi = \lambda m\varphi & \text{in } B, \\ \dfrac{\partial}{\partial n}\varphi = 0 & \text{on } \partial B. \end{cases}$$

Define

$$\lambda_{N,\infty}(m) = \sup_B \{\lambda_{N,B}(m) : m(x) \leqq 0 \quad \forall\, x \in \mathbf{R}^N \backslash B\}.$$

Then

$$r(\lambda) = 1 \quad \textit{for all } \lambda \in [0, \lambda_{N,\infty}].$$

REMARK 3.4. The assumption $\int_{\mathbf{R}^N} m < 0$ guarantees the existence of a ball B such that $\int_B m < 0$ and $m(x) \leqq 0$ for any $x \notin B$.

Proof of Theorem 3.3. For any $\varepsilon > 0$, consider the potentials m_ε defined by

$$m_\varepsilon = m - \varepsilon.$$

As there is $x_0 \in \mathbf{R}^N$ such that $m(x_0) > 0$, $m_\varepsilon(x_0) > 0$ if ε is small enough. Let $\lambda_1(m_\varepsilon)$ denote the principal eigenvalue associated with m_ε, whose existence and uniqueness is guaranteed by Corollary 3.2. It follows from the same corollary that $\operatorname{spr}(e^{(\Delta + \lambda m_\varepsilon)}) < 1$ for all $\lambda \in (0, \lambda_1(m_\varepsilon))$. Hence

$$r(\lambda) = \operatorname{spr}(e^{(\Delta + \lambda m)}) = \operatorname{spr}(e^{(\Delta + \lambda m_\varepsilon + \lambda\varepsilon)}) = e^{\lambda\varepsilon} \operatorname{spr}(e^{(\Delta + \lambda m_\varepsilon)}) < e^{\lambda\varepsilon} \tag{3.5}$$

for $\lambda \in (0, \lambda_1(m_\varepsilon))$. Let $B \subset \mathbf{R}^N$ be an arbitrary ball such that $\int_B m < 0$ and $m(x) \leqq 0$ whenever $x \notin B$. Then, $\int_B m_\varepsilon < 0$ for ε small enough. For such ε, let $\lambda_{N,B}(m_\varepsilon)$ denote the principal eigenvalue of the Neumann problem associated with m_ε in the ball B. By [**5**, Theorem 2.3],

$$\lambda_{N,B}(m_\varepsilon) \leqq \lambda_1(m_\varepsilon).$$

As $m_\varepsilon \leqq m$, $\lambda_{N,B}(m) \leqq \lambda_{N,B}(m_\varepsilon)$. Thus,

$$\lambda_{N,B}(m) \leqq \lambda_1(m_\varepsilon)$$

and it follows from (3.5) that

$$r(\lambda) < e^{\lambda\varepsilon}, \quad \lambda \in (0, \lambda_{N,B}(m)).$$

Therefore, passing to the limit as $\varepsilon \to 0$,

$$r(\lambda) \leqq 1, \quad \lambda \in (0, \lambda_{N,B}(m)).$$

Finally, as m has compact support, it follows from Remark 2.2 that $r(\lambda) \geqq 1$ for all $\lambda \geqq 0$, which completes the proof. □

The following result shows that when $N \geqq 3$ the condition $\int_{\mathbf{R}^N} m\,dx < 0$ is not necessary to ensure that $r(\lambda) = 1$ for $\lambda > 0$ close to zero.

THEOREM 3.5. *Assume that m is continuous with compact support and* $N \geqq 3$. *Let* m^* *be an arbitrary continuous radially symmetric potential with compact support such that*

$$m(x) \leqq m^*(x), \quad x \in \mathbf{R}^N.$$

Let $\lambda_1(m^*)$ *denote the principal eigenvalue associated with* m^*, *whose existence and uniqueness is guaranteed by Theorem 2.3. Then*

$$r(\lambda) = \operatorname{spr}(e^{(\Delta + \lambda m)}) = 1, \quad \textit{for all } \lambda \in [0, \lambda_1(m^*)].$$

Proof. As $m \leqq m^*$,

$$r(\lambda) \leqq \operatorname{spr}(e^{(\Delta + \lambda m^*)}).$$

Moreover, it follows from Theorem 2.3 that

$$\operatorname{spr}(e^{(\Delta + \lambda m^*)}) = 1$$

for all $\lambda \in [0, \lambda_1(m^*)]$, and hence $r(\lambda) \leqq 1$ for such range of λ. Finally, it follows from Remark 2.2 that $r(\lambda) \geqq 1$ for any $\lambda \geqq 0$, which completes the proof. □

However, as the following theorem shows, for the cases $N = 1$ or $N = 2$ the condition $\int_{\mathbf{R}^N} m\,dx < 0$ is essential so that $r(\lambda) = 1$ for $\lambda > 0$ close to zero. Before proving this result, we need a simple lemma.

LEMMA 3.7. *Let* $m \in C(\mathbf{R}^N)$ *and B be an arbitrary ball in* $\mathbf{R}^N$. *Denote by* $T_{\lambda,B}(t)$ *the semigroup generated by* $\Delta + \lambda m$ *in* $C_0(B)$, *i.e. by the Dirichlet problem on B, and by* $T_\lambda(t)$ *the Schrödinger semigroup* $e^{t(\Delta + \lambda m)}$. *Then*

$$r_B(\lambda) := \operatorname{spr}(T_{\lambda,B}(t)) \leqq \operatorname{spr}(T_\lambda(t)) = r(\lambda)$$

for all λ *in* $\mathbf{R}$.

Proof. Let $u_0 \in C_0(\mathbf{R}^N)$ and let u denote the corresponding solution of (1.1). Set $v_0 := u_0|_B$ and let v denote the solution of the Dirichlet problem

$$\begin{cases} \partial_t \varphi - \Delta\varphi = \lambda m \varphi & \text{on } B \times (0, \infty), \\ \varphi(x) = 0 & \text{on } \partial B \times (0, \infty), \\ v(\cdot, 0) = v_0 & \text{on } B. \end{cases}$$

Then, an easy comparison argument shows that

$$T_{\lambda,B}(t) v_0 = v(\cdot, t) \leqq u(\cdot, t) = T_\lambda(t) u_0$$

on B for all $t \geqq 0$ and, hence,

$$\| T_{\lambda,B}(t) v_0 \|_\infty \leqq \| T_\lambda(t) u_0 \|_\infty$$

for all $t \geqq 0$. But from this it follows that

$$\| T_{\lambda,B}(t) \|_{\infty,\infty} \leqq \| T_\lambda(t) \|_{\infty,\infty}$$

and the assertion follows from the formula for the spectral radius. □

THEOREM 3.7. *Assume that $N \leqq 2$ and that*

$$\int_{\mathbf{R}^N} m(x)\, dx > 0.$$

Then, for any $\lambda > 0$, the Schrödinger equation $-\Delta\varphi = \lambda m\varphi$ does not admit a positive solution in L_∞ (i.e. a 'bound state'). Moreover, the semigroup $e^{(\Delta + \lambda m)}$ is unstable and

$$r(\lambda) > 1 \quad \textit{for all } \lambda > 0.$$

Proof. The first assertion was shown in [**5**, Theorem 3.2]. It remains to prove the instability of the semigroup for all values of $\lambda > 0$. To do this, denote by λ_{D,B_R} the principal eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} -\Delta\varphi = \lambda m\varphi & \text{on } |x| \leqq R, \\ \varphi(x) = 0 & \text{for } |x| = R. \end{cases}$$

It is shown in [**5**, Lemma 3.1] that $\lambda_{D,B_R} \to 0$ as $R \to \infty$. Fix $\lambda > 0$ arbitrary. To prove that $r(\lambda) > 1$, we choose a ball B such that $\lambda_{D,B} < \lambda$. Using the results on bounded domains, we see that $r_B(\lambda) > 1$ [**3**]. By the above lemma, it is now clear that $r(\lambda) > 1$, proving the instability. □

As the previous results show, the qualitative behaviour of the Schrödinger semigroup is heavily dependent on the spatial dimension when m has compact support. When $N \geqq 3$ and m is radially symmetric, there is a unique principal eigenvalue $\lambda_1(m)$ and the Schrödinger equation has a bound state for any $\lambda \in (0, \lambda_1(m))$ and the zero solution is L_∞ stable in this λ range. On the other hand, if $N < 3$ then the equation does not admit a radially symmetric principal eigenfunction in $C_0(\mathbf{R}^N)$ because it is easy to see that any such eigenfunction would have to be identically zero outside the support of m and this is impossible for a classical solution of (1.2). However, if $\int_{\mathbf{R}^N} m < 0$ and $N < 3$, there is one value of $\lambda > 0$ for which the semigroup $e^{t(\Delta + \lambda m)}$ has a bound state, that is, $\lambda = \nu_1$, the principal eigenvalue of the Neumann problem in large balls, the bound state consisting of the principal eigenfunction to the Neumann problem on any ball containing the support of m extended to be identically equal to an appropriate constant outside the ball. This bound state, which does not vanish at infinity, can be regarded as the limit of principal eigenfunctions of Neumann boundary value problems. This is in contrast with the case $N \geqq 3$, where the principal eigenfunction vanishes at infinity and so can be regarded as a solution of a Dirichlet boundary value problem. In fact, when $N \geqq 3$, we shall show in Section 5 that the principal eigenvalue of (1.2) can be obtained as the limit of the principal eigenvalues of the corresponding Dirichlet problems in large balls. The results obtained above have a bearing on some open problems arising in [**18**, Section 5].

4. The case *m* negative at infinity

In this section we prove the exponential stability of the zero solution for a wide class of potentials m which are negative at infinity. A recent result obtained by Arendt and Batty in [**2**] characterises potentials V such that the Schrödinger semigroup associated with $\Delta - V$ is exponentially stable. We shall use this result to prove

Theorem 1.1 for an associated class of potentials m. The following definition from [2] will be useful.

DEFINITION 4.1. Given a subset $G \subset \mathbf{R}^N$, it is said that G contains arbitrarily large balls if for any $r > 0$ there exists $x \in G$ such that G contains the ball centred at x with radius r. By $\mathcal{G}$ we shall denote the class of subsets of $\mathbf{R}^N$ containing arbitrarily large balls.

THEOREM 4.2. *Suppose that m admits a decomposition of the form $m = m_1 - m_2$ with* $\operatorname{supp} m_1$ *compact and $m_2 \geqq 0$ such that $\int_G m_2(x)\,dx = \infty$ for all $G \in \mathcal{G}$. Then, there exists a unique principal eigenvalue $\lambda_1 > 0$ of* (1.2) *and the zero solution of* (1.1) *is exponentially stable for $\lambda \in [0, \lambda_1)$ and unstable for $\lambda > \lambda_1$.*

Proof. As m is positive somewhere,

$$\lim_{\lambda \to \infty} r(\lambda) = \infty. \tag{4.1}$$

We now show that $r(\lambda) < 1$ for $\lambda > 0$ small enough. By the variation-of-constants formula, we can write

$$e^{t(\Delta + m)} = e^{t(\Delta - m_2)} + \int_0^t e^{(t-s)(\Delta - m_2)} m_1 e^{s(\Delta + m)}\,ds.$$

As m_1 has compact support, it follows from [9, Proposition 5.4] that the integral on the right-hand side of the above relation defines a compact operator on $C_0(\mathbf{R}^N)$. Hence

$$S_\lambda = T_\lambda + K_\lambda,$$

where K_λ is a compact operator. Thus, from Remark 2.1, part (ii), we find that

$$\operatorname{spr}_{\text{ess}} S_\lambda = \operatorname{spr}_{\text{ess}} T_\lambda, \quad \lambda \geqq 0.$$

Moreover, it follows from [2, Theorem 1.2], that $\operatorname{spr} T_\lambda \leqq \| T_\lambda \| < 1$ for $\lambda > 0$. Hence

$$\operatorname{spr}_{\text{ess}} S_\lambda < 1, \quad \lambda > 0. \tag{4.2}$$

Let $m = m_1 - m_2$ be a decomposition of m such that m_1 and m_2 satisfy the conditions of the theorem and in addition

$$\int_{\mathbf{R}^N} m_1(x)\,dx < 0.$$

As $m_2 \geqq 0$,

$$\operatorname{spr} S_\lambda \leqq \operatorname{spr} e^{(\Delta + \lambda m_1)}.$$

Moreover, it follows from Theorem 3.3 that

$$\operatorname{spr} e^{(\Delta + \lambda m_1)} = 1, \quad \lambda \in [0, \lambda_{N,\infty}].$$

Thus,

$$\operatorname{spr} S_\lambda \leqq 1, \quad \lambda \in [0, \lambda_{N,\infty}]. \tag{4.3}$$

To show that $\operatorname{spr} S_\lambda < 1$ for $\lambda > 0$ small enough, we argue by contradiction. Suppose that $\operatorname{spr} S_\lambda = 1$ for λ close to zero. As $\lambda \to \operatorname{spr} S_\lambda$ is log-convex, $\operatorname{spr} S_\lambda \geqq 1$ for all $\lambda \geqq 0$

and hence it follows from (4.2) that

$$\operatorname{spr}_{\mathrm{ess}} S_\lambda < \operatorname{spr} S_\lambda, \quad \lambda \geqq 0.$$

So, Remark 2.1(iii) guarantees that $\operatorname{spr} S_\lambda$ is an algebraically simple eigenvalue of S_2 for all $\lambda > 0$. Therefore, $\lambda \to \operatorname{spr} S_\lambda$ is analytic for $\lambda > 0$ and so $\operatorname{spr} S_\lambda \equiv 1$ for all $\lambda \geqq 0$, which is impossible since $\operatorname{spr} S_\lambda > 1$ for λ large enough. This contradiction shows that $r(\lambda) < 1$ for $\lambda > 0$ small. As $r(\lambda)$ is log-convex, there is a unique $\lambda_1 > 0$ such that $r(\lambda_1) = 1$. The uniqueness and stability assertions of the theorem now follow as in the proof of Corollary 3.2. □

Clearly Corollary 3.2 is a special case of Theorem 4.2; the proof we gave, however, in Section 3 is based on much more elementary considerations than the proof above, which depends among other things on Theorem 3.3 which in our development depends in turn on Corollary 3.2.

It is shown [**2**, Corollary 1.9 and Theorem 1.10] that there exist potentials vanishing in a finite number of strips and satisfying all the requirements of Theorem 4.2. Some characterisations of the weights m_2 satisfying $\int_G m_2 = \infty$ for all $G \in \mathscr{G}$ can also be found in [**2**, Proposition 1.4].

5. The case of potentials with fast decay at infinity

In this section, we show the existence and the uniqueness of the principal eigenvalue of (1.2) for continuous weights m satisfying

$$|m(x)| \leqq c(1 + |x|^2)^{-\alpha} \tag{5.1}$$

for some constants $c > 0$ and $\alpha > 1$. As any continuous potential with compact support satisfies (5.1), Theorem 3.7 shows that under hypothesis (5.1) $N \geqq 3$ is necessary for the existence of a principal eigenvalue. Thus we shall assume that $N \geqq 3$ throughout the section.

To show the existence of a principal eigenvalue, we need some preliminary results, which are of interest by themselves. Consider the weighted Hilbert space

$$H := L_2((1 + |x|^2)^{-\alpha}\, dx) \tag{5.2}$$

and define

$$\|\psi\|_V := \|\nabla\psi\|_{L_2(\mathbf{R}^N)}, \quad \psi \in \mathscr{D}(\mathbf{R}^N) = C_0^\infty(\mathbf{R}^N).$$

When $N \geqq 3$, $\|\cdot\|_V$ defines a norm in $\mathscr{D}(\mathbf{R}^N)$, since by Hardy's inequality

$$\int_{\mathbf{R}^N} |\nabla\psi|^2\, dx \geqq c_0 \int_{\mathbf{R}^N} \frac{|\psi|^2}{|x|^2}\, dx \tag{5.3}$$

for some constant $c_0 > 0$ independent of $\psi \in \mathscr{D}(\mathbf{R}^N)$. Let V denote the completion of $(\mathscr{D}(\mathbf{R}^N), \|\cdot\|_V)$. Then, V is a Hilbert space with inner product

$$(u, v)_V := \int_{\mathbf{R}^N} \langle \nabla u, \nabla v \rangle\, dx.$$

Moreover, since

$$|x|^{-2} \geqq (1 + |x|^2)^{-1} \geqq (1 + |x|^2)^{-\alpha},$$

it follows from (5.3) that the space V is continuously embedded in H. The following result shows that this embedding is compact.

LEMMA 5.1. *The embedding of V in H is compact.*

Proof. Let (u_k) be a bounded sequence in V. We have to show that it has a convergent subsequence in H. As $V \hookrightarrow H$, (u_k) is also bounded in the Fréchet space $H^1_{2,\mathrm{loc}}(\mathbf{R}^N)$. Moreover, since $H^1_{2,\mathrm{loc}}(\mathbf{R}^N)$ is compactly embedded in $L_{2,\mathrm{loc}}(\mathbf{R}^N)$, there exists a subsequence of (u_k), again denoted by (u_k), which converges to some function $u \in L_{2,\mathrm{loc}}(\mathbf{R}^N)$. In other words,

$$\lim_{k\to\infty} \|u_k - u\|_{L_2(\Omega)} = 0 \tag{5.4}$$

for any bounded $\Omega \subset \mathbf{R}^N$.

Given arbitrary $R > 0$, the following estimates hold

$$\begin{aligned}\|u_k - u_l\|_H^2 &= \int_{|x|\geqq R} (1+|x|^2)^{-\alpha}(u_k - u_l)^2\,dx + \int_{B_R} (1+|x|^2)^{-\alpha}(u_k-u_l)^2\,dx \\ &\leqq (1+R^2)^{1-\alpha} \int_{|x|\geqq R} \frac{(u_k-u_l)^2}{|x|^2}\,dx + \int_{B_R} (u_k-u_l)^2\,dx \\ &\leqq (1+R^2)^{1-\alpha}\frac{1}{c_0}\|u_k-u_l\|_V^2 + \|u_k-u_l\|_{L_2(B_R)}^2,\end{aligned} \tag{5.5}$$

where Hardy's inequality is used to get the last estimate.

Consider arbitrary $\varepsilon > 0$. As (u_k) is bounded in V and $1-\alpha<0$, there exists $R>0$ such that

$$(1+R^2)^{1-\alpha}\frac{1}{c_0}\|u_k-u_l\|_V^2 \leqq \frac{\varepsilon}{2}, \quad \forall k, l \in \mathbf{N}. \tag{5.6}$$

Moreover, it follows from (5.4) that there exists $N_0 \in \mathbf{N}$ such that

$$\|u_k-u_l\|_{L_2(B_R)}^2 \leqq \frac{\varepsilon}{2}, \quad \forall k, l \geqq N_0.$$

This estimate together with (5.5) and (5.6) completes the proof. □

Given $R>0$, consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda m u & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{5.7}$$

Where R is large enough, there exists $x_0 \in B_R$ such that $m(x_0)>0$. Hence (5.7) possesses a unique principal eigenvalue, which will be denoted by $\lambda_1(R)$. It was shown in [**8**, Lemma 6.6], that the mapping $R \to \lambda_1(R)$ is strictly decreasing and that the limit

$$\lambda_\infty := \lim_{R\to\infty} \lambda_1(R)$$

exists and is finite. The question of interest is to characterise whether λ_∞ is a principal eigenvalue of (1.2) or not. In fact, this was an open question in [**5**] and [**9**]. The answer is positive as the following theorem shows.

THEOREM 5.2. *$\lambda_\infty > 0$ is the unique principal eigenvalue of* (1.2). *Moreover, the zero solution of* (1.1) *is unstable for $\lambda > \lambda_\infty$.*

We first show that λ_∞ is a principal eigenvalue of (1.2). Let φ_R denote the principal eigenfunction associated with $\lambda_1(R)$. Such eigenfunctions lie in the Sobolev space $H_2^1(B_R)$ vanishing at the boundary and so, extended to equal zero outside B_R, can be regarded as lying in $H_2^1(\mathbf{R}^N)$. As they have compact support, we can also regard them as elements of V. If we assume that the φ_R are normalised so that $\|\varphi_R\|_V = 1$, it follows from Lemma 5.1 that there exists a sequence of positive real numbers (R_k) and a $\varphi \in H$ such that $\lim_{k\to\infty} R_k = \infty$ and

$$\lim_{k\to\infty} \varphi_k = \varphi, \tag{5.8}$$

where φ_k stands for φ_{R_k} and convergence is in H and so in $L_{2,\mathrm{loc}}(\mathbf{R}^N)$. Next, we show that (φ_k) is also a Cauchy sequence in V and therefore converges in V. For simplicity we shall write $\lambda_k := \lambda_1(R_k)$ and $B_k := B_{R_k}$. Without loss of generality, we may assume that $l \leqq k$ and hence $B_l \subset B_k$. This allows us to consider φ_l as a test function. Using in addition that φ_k and φ_l satisfy the elliptic boundary value problem (5.7) and Green's formula, we obtain

$$\begin{aligned}
\|\varphi_k - \varphi_l\|_V^2 &= \int_{B_k} |\nabla\varphi_k|^2 - 2\int_{B_k} \nabla\varphi_k \cdot \nabla\varphi_l + \int_{B_k} |\nabla\varphi_l|^2 \\
&= \int_{B_k} \varphi_k \Delta\varphi_k - 2\int_{B_k} \varphi_l \Delta\varphi_k + \int_{B_k} \varphi_l \Delta\varphi_l \\
&= \lambda_k \int_{\mathbf{R}^N} m\varphi_k^2 - 2\lambda_k \int_{\mathbf{R}^N} m\varphi_k\varphi_l + \lambda_l \int_{\mathbf{R}^N} m\varphi_l^2 \\
&= \lambda_k \int_{\mathbf{R}^N} m\varphi_k(\varphi_k - \varphi_l) + (\lambda_l - \lambda_k)\int_{\mathbf{R}^N} m\varphi_k\varphi_l + \lambda_l \int_{\mathbf{R}^N} m\varphi_l(\varphi_l - \varphi_k) \\
&\leqq \lambda_k \|\varphi_k\|_H \|\varphi_k - \varphi_l\|_H + |\lambda_l - \lambda_k|\, \|\varphi_k\|_H \|\varphi_l\|_H + \lambda_l \|\varphi_l\|_H \|\varphi_l - \varphi_k\|_H.
\end{aligned}$$

As (φ_k) is convergent in H and also (λ_k) has a finite limit, it follows that (φ_k) is a Cauchy sequence in V. Since $V \hookrightarrow H$, its limit is φ. In particular, it follows that $\lim_{k\to\infty} \varphi_k = \varphi$ in $H^1_{2,\mathrm{loc}}$, and that $\|\varphi\|_V = 1$, i.e. $\varphi \neq 0$. We now prove that φ is a principal eigenfunction of (1.2) associated with λ_∞ by showing that φ is a weak solution of (1.2) for $\lambda = \lambda_\infty$. Suppose $\psi \in \mathscr{D}(\mathbf{R}^N)$. Then if k is sufficiently large

$$\int_{\mathbf{R}^N} \nabla\varphi_k \nabla\psi\, dx = \lambda_k \int_{\mathbf{R}^N} m\varphi_k \psi\, dx,$$

and so letting $k \to \infty$

$$\int_{\mathbf{R}^N} \nabla\varphi \nabla\psi\, dx = \lambda_\infty \int_{\mathbf{R}^N} m\varphi\psi\, dx.$$

Hence φ is a weak solution of $-\Delta\varphi = \lambda_\infty m\varphi$ and it follows by a standard regularity argument that φ is a classical solution of the equation.

As $\varphi_k > 0$ for all $k \geqq 1$, $\varphi \geqq 0$. Moreover, it follows from the definition of φ_k, using

Green's Theorem, that

$$\lambda_k\langle m\varphi_k, \varphi_k\rangle = \langle\nabla\varphi_k, \nabla\varphi_k\rangle = \|\varphi_k\|_V = 1 \tag{5.9}$$

for all $k \geqq 1$. Since $\int_{\mathbf{R}^N} m\varphi_k^2\,dx \leqq c\|\varphi_k\|_H^2$ for some appropriate constant c, it follows that $\int_{\mathbf{R}^N} m\varphi_k^2\,dx$ is bounded and so because of (5.9) the sequence (λ_k) must be bounded away from zero. Hence $\lambda_\infty > 0$ and so we have proved the existence of a positive principal eigenvalue.

It is now easy to prove that the zero solution is unstable for all $\lambda > \lambda_\infty$. If $\lambda > \lambda_\infty$, we can find a ball B_R such that $\lambda_1(R) < \lambda$. Instability can now be proved by using exactly the same argument as that used at the end of the proof of Theorem 3.7.

Finally we prove that λ_∞ is the unique principal eigenvalue. In order to do so, we must first prove two technical lemmas.

LEMMA 5.3. *Let m satisfy* (5.1). *Then, any principal eigenfunction φ of* (1.2) *satisfies $\varphi \in L_{2N/(N-2)}(\mathbf{R}^N)$ and $\nabla\varphi \in L_2(\mathbf{R}^N)^N$.*

Proof. Let φ be a principal eigenfunction associated with some eigenvalue $\lambda > 0$. Then

$$\varphi(x) = c_N \int_{\mathbf{R}^N} \frac{m(y)\varphi(y)}{|x-y|^{N-2}}\,dy, \quad x \in \mathbf{R}^N,$$

for some constant $c_N > 0$ depending on the spatial dimension. Hence

$$\nabla\varphi(x) = -(N-2)c_N \int_{\mathbf{R}^N} \frac{m(y)\varphi(y)}{|x-y|^N}\,x\,dy.$$

It follows from [**14**, Lemma 2.3] and by using bootstrapping arguments that

$$|\varphi(x)| \leqq c|x|^{-(N-2)} \quad \text{and} \quad |\nabla\varphi(x)| \leqq c|x|^{-(N-1)},$$

for some constant $c > 0$, which completes the proof. □

LEMMA 5.4. *Let $\varphi \in L_{2N/(N-2)}(\mathbf{R}^N)$ be such that $\nabla\varphi \in L_2$. Then, there exists a sequence of test functions (φ_n) such that $\lim_{n\to\infty}\varphi_n = \varphi$ in $L_{2N/(N-2)}(\mathbf{R}^N)$ and $\lim_{n\to\infty}\nabla\varphi_n = \nabla\varphi$ in L_2.*

Proof. Let $\psi \in \mathscr{D}(\mathbf{R}^N)$ be such that $0 \leqq \psi \leqq 1$ and $\psi(x) = 1$ if $x \in B_1$ and $\psi(x) = 0$ outside B_2. Define $\psi_n(x) := \psi(x/n)$ and set

$$\varphi_n := \varphi\psi_n.$$

It follows easily from the Lebesgue Dominated Convergence Theorem that $\varphi_n \to \varphi$ in $L_{2N/(N-2)}$. Also, applying Hölder's inequality, we obtain

$$\begin{aligned} \|\nabla\varphi_n - \nabla\varphi\|_2 &\leqq \|(\psi_n - 1)\nabla\varphi\|_2 + \|\varphi\nabla\psi_n\|_{2,A_n} \\ &\leqq \|(\psi_n - 1)\nabla\varphi\|_2 + \|\varphi\|_{2N/(N-2),A_n}\|\nabla\psi_n\|_N, \end{aligned} \tag{5.10}$$

where $A_n := \{n \leqq |x| \leqq 2n\}$ and we denote by $\|\cdot\|_p$ the $L_p(\mathbf{R}^N)$-norm and by $\|\cdot\|_{p,A_n}$ the $L_p(A_n)$-norm. Since $\nabla\varphi \in L_2$, the Lebesgue Dominated Convergence Theorem guarantees that the first term in (5.10) tends to zero as n goes to infinity. Moreover, since $\varphi \in L_{2N/(N-2)}$, then $\lim_{n\to\infty}\|\varphi\|_{2N/(N-2),A_n} = 0$. From the definition of ψ_n it follows readily that $\|\nabla\psi_n\|_N = \|\nabla\psi\|_N$. Smoothing out the functions φ_n, the proof is complete. □

We can now proceed with our proof of the uniqueness of the principal eigenvalue. Suppose that $\hat{\lambda}$ is any other principal eigenvalue of (1.2) with corresponding positive eigenfunction ψ. Since $\lambda_1(R)$ has variational characterisation

$$\lambda_1(R)=\inf_{\substack{\psi\in\mathscr{D}(B_R)\\ \int_{B_R} m\psi^2\,dx>0}}\frac{\displaystyle\int_{B_R}|\nabla\psi|^2\,dx}{\displaystyle\int_{B_R} m\psi^2\,dx},$$

it is easy to show that λ_∞ has the variational characterisation

$$\lambda_\infty=\inf_{\substack{\psi\in\mathscr{D}(\mathbf{R}^N)\\ \int_{\mathbf{R}^N} m\psi^2\,dx>0}}\frac{\displaystyle\int_{\mathbf{R}^N}|\nabla\psi|^2\,dx}{\displaystyle\int_{\mathbf{R}^N} m\psi^2\,dx}.$$

From Lemmas 5.3 and 5.4, we see that the principal eigenfunction ψ can be approximated by a sequence of test functions (ψ_n) in such a way that $\nabla\psi_n\to\nabla\psi$ in L_2 and $\psi_n\to\psi$ in $L_{2N/(N-2)}$ so that $\int_{\mathbf{R}^N} m\psi_n^2\to\int_{\mathbf{R}^N} m\psi^2\,dx$. Thus, it follows from (5.11) that $\hat{\lambda}$ must satisfy

$$\lambda_\infty\leqq\hat{\lambda}.$$

The same argument as at the end of the proof of Theorem 3.8 shows that $r(\lambda)>1$ and in particular the instability of the zero solution for all $\lambda>\lambda_\infty$. By Remark 2.2(ii) $r(\lambda)$ is the only eigenvalue having positive eigenfunction. Hence we must have $\hat{\lambda}=\lambda_\infty$ and the proof of Theorem 5.2 is complete. □

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