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Eigenvalue problems for a cooperative system with a large parameter

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Dedicated to Klaus Schmitt, great teacher and colleague.

Abstract

We consider the principal eigenvalue of a cooperative system of elliptic boundary value problems as a parameter tends to infinity. The main aim is to introduce a new approach to deal with the limit problem by focusing on the resolvent operator corresponding to the system rather than the eigenvalue problem itself. This allows the consistent use of elementary properties of bilinear forms and the semi-groups they induce. At the same time we weaken assumptions in related work.

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1 Introduction

We provide a new approach to deal with the asymptotics of the principal eigenvalue for a cooperative elliptic system

$$\begin{aligned} \mathcal{A}_1 u_1 + \lambda m_1 u_1 - d_1 u_2 &= \mu(\lambda) u_1 && \text{in } \Omega, \\ \mathcal{A}_2 u_2 + \lambda m_2 u_2 - d_2 u_1 &= \mu(\lambda) u_2 && \text{in } \Omega, \\ u_1 = u_2 &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^N$ as $\lambda \to \infty$. Here, $\mathcal{A}_1$, $\mathcal{A}_2$ are uniformly strongly elliptic operators in divergence form with bounded and measurable coefficients,

$m_1, m_2 \in L^\infty(\Omega)$ are non-negative and $d_1, d_2 \in L^\infty(\Omega)$ are positive. The precise assumptions are listed in Section 2. The problem was originally considered in Álvarez Caudevilla & López-Gómez [1, 2] under strong regularity assumptions on the domain and the coefficients, and was substantially generalised in Dancer [7]. The results are motivated by applications to non-linear cooperative systems.

The purpose of this paper is to introduce new techniques to simplify the proofs and at the same time to weaken the assumptions even more. The idea is to make consistent use of the theory of positive irreducible operators, form methods and test function arguments. The framework for that is outlined in Section 2.

Contrary to earlier work we focus on the resolvent operator associated with the linear system corresponding to (1.1) rather than the eigenvalue problem itself. In Theorem 3.1 we prove the convergence of the resolvent operator in the operator norm and infer from that the convergence of the eigenvalues and eigenfunctions. This is quite different from the earlier work in [1, 7] and may be useful for studying the corresponding semi-linear problem by using the ideas from [8, Section 9.2] for instance.

As in [1, 7] we show that $\mu(\lambda) \to \mu_\infty$ converges, and that μ_∞ is a principal eigenvalue of some limit problem. It is quite easy to see that the limit problem has support in U_i, where

$$U_i = \{x \in \Omega \colon m_i(x) = 0\} \qquad (i = 1, 2). \tag{1.2}$$

As shown in [7] the limit problem, at least formally, has the form

$$\begin{aligned} \mathcal{A}_1 u_1 - d_1 P u_2 &= \mu_\infty u_1 && \text{in } \Omega,\\ \mathcal{A}_2 u_2 - d_2 P u_1 &= \mu_\infty u_2 && \text{in } \Omega,\\ u_i &\in H_0^1(U_i) && i = 1, 2, \end{aligned} \tag{1.3}$$

where P is the projection $Pu := 1_{U_1 \cap U_2} u$ on $L^2(\Omega)$. Our aim is to show that

$$\mu_\infty \leq \min\{\mu_1, \mu_2\}, \tag{1.4}$$

where μ_i is the smallest eigenvalue of $\mathcal{A}_i$ on U_i with Dirichlet boundary conditions ($i = 1, 2$). We note that U_i could have many components, so we cannot expect a unique simple principal eigenvalue for the limit problem without making further assumptions. If m_1, m_2 are positive everywhere, then $\mu_\infty = \infty$ since the limit problem is zero. To make sure $\mu_\infty < \infty$ we need to assume that U_1 or U_2 has non-empty interior. If $m_1 = m_2 = 0$, then the limit problem coincides with the original problem, and is therefore not very interesting. The precise results are discussed in Section 4, where we also discuss the case of strict inequality in (1.4). Details are in Remarks 4.2 and 4.3.

There are strong assumptions on the regularity of U_i in [1], and substantially weaker ones in [7]. We remove these regularity conditions almost entirely by taking a different point of view on the limit problem. For non-smooth domains U_i there are several possible choices for the domain of the Laplace operator and more generally an elliptic operator in divergence form. The two are extensively discussed in

[4]. From [7] it is clear that the limit problem as $\lambda \to \infty$ naturally involves the Dirichlet problem on the maximal domain and therefore we can work with weaker assumptions on m_1 and m_2.

We finally note that the techniques we introduce in this paper could be applied to a scalar problem of similar structure such as the ones treated in [15, 6, 11], or the periodic-parabolic case in [10].

2 The eigenvalue problem

We assume that $\mathcal{A}_i$, $i = 1, 2$, are elliptic operators of the form

$$\mathcal{A}_i u = -\operatorname{div}\bigl(A_i(x)\nabla u + a_i(x)u\bigr) + b_i(x)\cdot\nabla u + c_{i0}(x)u$$

with $A_i \in L^\infty(\Omega, \mathbb{R}^{N\times N})$, $a_i, b_i \in L^\infty(\Omega, \mathbb{R}^N)$ and $c_{0i} \in L^\infty(\Omega)$. The matrix A_i is uniformly positive definite, that is, there exists $\alpha > 0$ such that

$$\alpha|\xi|^2 \le \operatorname{Re}\bigl(A_i(x)\xi\cdot\xi\bigr)$$

for all $\xi \in \mathbb{C}^N$ and almost all $x \in \Omega$. We then look at the cooperative system

$$\begin{aligned} \mathcal{A}_1 u_1 + \lambda m_1 u_1 - d_1 u_2 + \gamma u_1 &= f_1 \quad \text{in } \Omega,\\ \mathcal{A}_2 u_2 + \lambda m_2 u_2 - d_2 u_1 + \gamma u_1 &= f_2 \quad \text{in } \Omega,\\ u_1 = u_2 &= 0 \quad\ \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $m_1, m_2 \ge 0$ and $d_1, d_2 \ge 0$ are bounded and measurable and $\gamma \in \mathbb{R}$. We could replace the Dirichlet boundary conditions by more general variational boundary conditions such as those in [8, Section 2], but that does not change the main idea. We define the form associated with $\mathcal{A}_i$ by

$$\mathfrak{a}_i(u, v) = \int_\Omega \bigl(A_i\nabla u + a_i u\bigr)\cdot\nabla v + \bigl(b_i\cdot\nabla u + c_{0i}u\bigr)v\,dx \tag{2.2}$$

for all $u, v \in H_0^1(\Omega)$. For $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $H_0^1(\Omega, \mathbb{R}^2)$ we define the bilinear form

$$\mathfrak{a}(u, v) := \mathfrak{a}_1(u_1, v_1) + \mathfrak{a}_2(u_2, v_2) - \langle d_1 u_2, v_1\rangle - \langle d_2 u_1, v_2\rangle, \tag{2.3}$$

where $\langle\cdot\,,\cdot\rangle$ is the inner product in $L^2(\Omega)$. We also set

$$\langle f, g\rangle := \langle f_1, g_1\rangle + \langle f_2, g_2\rangle$$

if $f, g \in L^2(\Omega, \mathbb{R}^2)$ and define the multiplication operator

$$\boldsymbol{M}u := \begin{bmatrix} m_1 & 0\\ 0 & m_2 \end{bmatrix}\begin{bmatrix} u_1\\ u_2 \end{bmatrix}.$$

If $f \in L^2(\Omega, \mathbb{R}^2)$ we say $u \in H_0^1(\Omega, \mathbb{R}^2)$ is a weak solution of (2.1) if

$$\mathfrak{a}(u, v) + \lambda\langle \boldsymbol{M}u, v\rangle = \langle f, v\rangle$$

for all $v \in H_0^1(\Omega, \mathbb{R}^2)$. By using test functions of the form $(v_1, 0)$ or $(0, v_2)$ and Green's identity we easily see that classical solutions are weak solution, and sufficiently smooth weak solutions are classical solutions of (2.1) as in the scalar case. We define the positive cone in $H_0^1(\Omega, \mathbb{R}^2)$ and $L^2(\Omega, \mathbb{R}^2)$ to be the set (f_1, f_2), where f_1 and f_2 are non-negative almost everywhere. It is known that both spaces with the order defined by this cone are ordered Banach spaces. As usual in ordered Banach spaces we write $u \geq 0$ if u is non-negative and $u > 0$ if u is non-negative and non-zero. We note that $H_0^1(\Omega, \mathbb{R}^2)$ as well as $L^2(\Omega, \mathbb{R}^2)$ are Banach lattices, that is $u^+ := \max\{u, 0\} \in H_0^1(\Omega, \mathbb{R}^2)$ for all $u \in H_0^1(\Omega, \mathbb{R}^2)$ (see for instance [12, Lemma 7.6]).

It is also useful sometimes to work with the adjoint problem associated with the form

$$\mathfrak{a}^\sharp(u, v) := \mathfrak{a}(v, u).$$

The corresponding elliptic operators have the same structure as $\mathcal{A}_1, \mathcal{A}_2$ and the operator induced is the dual of $\boldsymbol{A}$ (see for instance [8, Section 2.3]).

The eigenvalue problem (1.1) in the weak form consists of finding $\mu(\lambda)$ and a non-negative non-zero $(u_1, u_2) \in H_0^1(\Omega, \mathbb{R}^2)$ so that

$$\mathfrak{a}(u, v) + \lambda\langle \boldsymbol{M}u, v\rangle = \mu(\lambda)\langle u, v\rangle$$

for all $v \in H_0^1(\Omega, \mathbb{R}^2)$. This is equivalent to

$$\mathfrak{a}(u, v) + \lambda\langle \boldsymbol{M}u, v\rangle + \gamma\langle u, v\rangle = (\gamma + \mu(\lambda))\langle u, v\rangle,$$

so for the purpose of finding properties of the principal eigenvalue $\mu(\lambda)$ and its asymptotic behaviour as $\lambda \to \infty$ we can replace $\mathcal{A}_i$ by $\mathcal{A}_i + \gamma I$. The following lemma allows us to choose a suitable $\gamma \in \mathbb{R}$.

Lemma 2.1. *The form $\mathfrak{a}(\cdot\,,\cdot)$ is bounded on $H_0^1(\Omega, \mathbb{R}^2)$ and there exists $\gamma_0 \in \mathbb{R}$ such that*

$$\frac{\alpha}{2}\|u\|_{H^1}^2 \leq \mathfrak{a}(u, u) + \gamma\|u\|_2^2 \tag{2.4}$$

for all $\gamma \geq \gamma_0$ and all $u \in H^1(\Omega, \mathbb{R}^2)$.

Proof. First observe that

$$\begin{aligned}\big|\langle d_1u_2, u_1\rangle + \langle d_2u_1, u_2\rangle\big| &\leq \big(\|d_1\|_\infty + \|d_2\|_\infty\big)\|u_2\|_2\|u_1\|_2\\ &\leq \big(\|d_1\|_\infty + \|d_2\|_\infty\big)\big(\|u_1\|_2^2 + \|u_2\|_2^2\big) = \|d\|_\infty\|u\|_2^2.\end{aligned}$$

If we let

$$\gamma_0 := \sum_{i=1}^{2}\Big(\|c_{0i}^-\|_\infty + \frac{\|a_i + b_i\|_\infty}{2\alpha} + \|d_i\|_\infty\Big),$$

then the above and [8, Proposition 2.1.6] imply that

$$\begin{aligned}\mathfrak{a}(u, u) + \gamma\|u\|_2^2 &\geq \mathfrak{a}_1(u_1, u_1) + \mathfrak{a}_2(u_2, u_2) + (\gamma - \|d\|_\infty)\|u\|_2^2\\ &\geq \frac{\alpha}{2}\|u\|_{H^1}^2 + (\gamma - \gamma_0)\|u\|_2^2 \geq \frac{\alpha}{2}\|u\|_{H^1}^2\end{aligned}$$

for all $\gamma \geq \gamma_0$. The boundedness of the form is easy to see and very similar to the proof of [8, Proposition 2.1.6]. □

By the above lemma the form $\mathfrak{a}(\cdot\,,\cdot)$ induces an operator

$$\boldsymbol{A} \in \mathcal{L}\big(H_0^1(\Omega,\mathbb{R}^2), H^{-1}(\Omega,\mathbb{R}^2))\big)$$

given by

$$\langle \boldsymbol{A}u, v\rangle = \mathfrak{a}(u,v)$$

for all $u, v \in H^1(\Omega,\mathbb{R}^2)$, where $H^{-1}(\Omega,\mathbb{R}^2)$ is the dual of $H_0^1(\Omega,\mathbb{R}^2)$. Its part in $L^2(\Omega,\mathbb{R}^2)$ is a closed operator we also denote by $\boldsymbol{A}$. We use the generalised Beurling-Deny criterion from [18] to prove that $-(\boldsymbol{A}+\lambda\boldsymbol{M})$ generates a positive irreducible semigroup on $L^2(\Omega,\mathbb{R}^2)$.

Proposition 2.2. *If $d_1, d_2 \geq 0$, then for all $\lambda \geq 0$ the operator $-(\boldsymbol{A}+\lambda\boldsymbol{M})$ generates a positive semigroup of compact operators on $L^2(\Omega,\mathbb{R}^2)$. If $d_1, d_2 > 0$ are both not identically zero, then $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is irreducible. Finally, for every $\gamma \geq \gamma_0$ and $\lambda > 0$.*

$$\big\|(\gamma+\boldsymbol{A}+\lambda\boldsymbol{M})^{-1}f\big\|_{H^1} \leq \frac{2}{\alpha}\|f\|_2 \tag{2.5}$$

and

$$\big\|(\gamma-\gamma_0)(\gamma+\boldsymbol{A}+\lambda\boldsymbol{M})^{-1}f\big\|_2 \leq \|f\|_2. \tag{2.6}$$

for all $f \in L^2(\Omega,\mathbb{R}^2)$.

Proof. By a standard result $-(\boldsymbol{A}+\lambda\boldsymbol{M})$ generates an analytic semigroup on $L^2(\Omega,\mathbb{R}^2)$ (see [9, Proposition XVII.6.3] or [18, Theorem 1.52]). From the definition of the form $\mathfrak{a}(\cdot\,,\cdot)$ it is obvious that

$$\mathfrak{a}(u^+,u^-) + \lambda\langle \boldsymbol{M}u^+, u^-\rangle = -\langle d_1u_2^+, u_1^-\rangle - \langle d_2u_1^+, u_2^-\rangle \leq 0$$

for all $u \in H_0^1(\Omega,\mathbb{R}^2)$. Here we use the fact that $\operatorname{supp}(\nabla u^\pm) \subseteq \operatorname{supp}(u^\pm)$ (see [12, Lemma 7.6]). Hence the generalised Beurling-Deny criterion from [18, Theorem 2.2] with $\mathcal{P}(u) := u^+$ and $\mathcal{C}$ the positive cone in $L^2(\Omega,\mathbb{R}^2)$ applies and the semigroup $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is positive. The semigroup is compact since the embedding $H_0^1(\Omega,\mathbb{R}^2) \hookrightarrow L^2(\Omega,\mathbb{R}^2)$ is compact and $D(\boldsymbol{A}) \subseteq H_0^1(\Omega,\mathbb{R}^2)$.

Assuming that $(d_1, d_2) > 0$ with both components not identically zero we prove that $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is irreducible. This is equivalent to $(\gamma I+\boldsymbol{A}+\lambda\boldsymbol{M})^{-1}$ being positive and irreducible for γ large enough. Let $f = (f_1, f_2) > 0$ and fix $\lambda > 0$. Assume that $u = (u_1, u_2)$ is the unique weak solution of $(\gamma I + \boldsymbol{A} + \lambda\boldsymbol{M})u = f$. We know from what we have already proved that $u \geq 0$ and therefore d_1u_2 and d_2u_1 are non-negative. Hence

$$\begin{aligned}\mathcal{A}_1u_1 + (\gamma+\lambda m_1)u_1 &= f_1 + d_1u_2 \geq 0\\ \mathcal{A}_2u_2 + (\gamma+\lambda m_2)u_2 &= f_2 + d_2u_1 \geq 0.\end{aligned}$$

If $f_1 \neq 0$, then by properties of the scalar equation $u_1(x) > 0$ for almost every $x \in \Omega$. By our assumption on d_1 we have $d_2u_1 \geq 0$ is non-trivial. Hence $f_2 + d_2u_1 \geq 0$

is non-trivial and so by a similar argument as before $u_2 > 0$ almost everywhere. Hence $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is irreducible. It is also compact since $H_0^1(\Omega,\mathbb{R}^2)$ is compactly embedded into $L^2(\Omega,\mathbb{R}^2)$.

Let $\gamma \geq \gamma_0$. By the Lax-Milgram theorem $u := (\gamma + \boldsymbol{A} + \lambda\boldsymbol{M})^{-1} f$ exists for all $f \in L^2(\Omega,\mathbb{R}^2)$. By the coercivity estimate (2.4) we see that

$$\begin{aligned}\frac{\alpha}{2}\|u\|_{H^1}^2 + (\gamma-\gamma_0)\|u\|_2^2 &\leq \mathfrak{a}(u,u) + \lambda\langle \boldsymbol{M}u,u\rangle + \gamma_0\|u\|_2^2\\ &\leq \|f\|_2\|u\|_2 \leq \|f\|_2\|u\|_{H^1}.\end{aligned}$$

Dividing by $\|u\|_{H^1}$ or $\|u\|_2$ we get (2.5) and (2.6), respectively. □

Amongst other facts we next prove the existence of a unique principal eigenvalue for (1.1). The results complement earlier results in [7, 16, 21] and others, who considered operators with more regularity on the coefficients, or operators of a different structure. Our proof is based on abstract results on the spectral radius of compact irreducible operators rather than the construction of sub- and super-solutions.

Theorem 2.3. *Suppose that $m_1, m_2 \geq 0$ and that $d_1, d_2 > 0$ are both non-zero. Then the following assertions are true.*

(i) *If $\gamma \geq \gamma_0$, then $(\boldsymbol{A} + \lambda\boldsymbol{M})^{-1}$ is a positive irreducible operator decreasing in $\lambda\boldsymbol{M}$.*

(ii) *The eigenvalue problem (1.1) has a unique principal eigenvalue $\mu(\lambda)$. It is algebraically simple and the corresponding eigenfunction can be chosen positive.*

(iii) *The adjoint problem has the same principal eigenvalue.*

(iv) *The eigenvalue $\mu(\lambda)$ is strictly increasing as a function of $\lambda\boldsymbol{M} > 0$.*

Proof. As $\boldsymbol{M}$ is a non-negative multiplication operator, standard perturbation theorems show that

$$\left|e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}u\right| \leq e^{-t\boldsymbol{A}}|u| \tag{2.7}$$

for all $u \in L^2(\Omega,\mathbb{R}^2)$, and that $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is decreasing in $\lambda\boldsymbol{M} > 0$. We can also see this from the Trotter-Kato formula

$$e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})} = \lim_{n\to\infty}\left(e^{-\frac{t}{n}\boldsymbol{A}}e^{-\frac{t}{n}\lambda\boldsymbol{M}}\right)^n \tag{2.8}$$

which shows that $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is decreasing in $\lambda\boldsymbol{M}$ since $e^{-t\boldsymbol{A}}$ is a positive semigroup and $e^{-t\lambda\boldsymbol{M}}$ is obviously decreasing as $\lambda\boldsymbol{M}$ increases. As $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ is positive and irreducible [17, Theorem 4.2.2] it has a positive spectral radius $e^{-t\mu(\lambda)}$ which is algebraically simple, and the unique eigenvalue having a positive eigenfunction. Moreover, the adjoint semigroup has the same spectral radius. Note that $\mu(\lambda)$ is a principal eigenvalue of (1.1) with the same eigenfunction. As the resolvent inherits the above properties of the semigroup this proves (i)–(iii).

To prove (iv) note that by (2.8) the semigroup $e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ of positive operators is decreasing in $\lambda\boldsymbol{M}$. This means that also its spectral radius $e^{-t\mu(\lambda)}$ is decreasing, that is, $\mu(\lambda)$ is increasing.

From (2.6) the semigroups $e^{-\gamma_0 t}e^{-t(\boldsymbol{A}+\lambda\boldsymbol{M})}$ are mean ergodic for all $\lambda \geq 0$ with spectral radius $e^{-t(\gamma_0+\mu(\lambda))}$. As the adjoint problem has the same principal eigenvalue, [3, Theorem 1.3] implies that $\mu(\lambda)$ is strictly increasing in λ and $\boldsymbol{M}$ as otherwise the semigroups are the same. □

3 The limit problem

In this section we determine properties of the limit problem of (1.1) as $\lambda \to \infty$. Unlike [1, 7] we do not directly deal with the eigenvalue problem, but get the result via convergence properties of the resolvent. We let U_i be the set where m_i vanishes as defined in (1.2). It is quite a natural assumption to require U_i to be closed sets as this is the case for continuous m_1, m_2. The limit problem as $\lambda \to \infty$ naturally involves the spaces

$$H_0^1(U_i) := \{u \in H_0^1(\Omega)\colon u = 0 \text{ almost everywhere on } \Omega \setminus U_i\}, \tag{3.1}$$

$i = 1, 2$. Note that in general, $H_0^1(U_i) \neq H_0^1(\operatorname{int}(U_i))$. If the spaces are equal, then U_i is said to be stable for the Dirichlet problem. As is clear from [13, 20, 14, 5, 4], the stability of U_i is a mild regularity assumption on ∂U_i, and is closely related to properties of harmonic functions. On bad domains there are several possibilities to define the Dirichlet Laplacian or other elliptic operator. Usually, the minimal form domain is used to define the operator, namely the closure of the test functions on the interior of U_i, that is, the space $H_0^1(\operatorname{int}(U_i))$. The alternative is to use the maximal domain, namely (3.1). The two alternatives are extensively discussed and compared in [4]. It is really an arbitrary choice which one to work with. In the context of our limit problem, the maximal domain $H_0^1(U_i)$ is the natural one. More precisely, the form associated with the limit problem turns out to be the form $\mathfrak{a}(\cdot\,,\cdot)$ restricted to

$$H_0^1(U_1) \times H_0^1(U_2) = H_0^1(\Omega, \mathbb{R}^2) \cap \ker \boldsymbol{M}.$$

We denote the form $\mathfrak{a}(\cdot\,,\cdot)$ restricted to that space by $\mathfrak{a}_\infty(\cdot\,,\cdot)$. It is given by

$$\mathfrak{a}_\infty(u, v) = \mathfrak{a}(u, v)$$

for all $u, v \in D(\mathfrak{a}_\infty) := H_0^1(\Omega, \mathbb{R}^2) \cap \ker \boldsymbol{M}$. For now we do not make any assumptions on U_1, U_2 because in any case $\mathfrak{a}_\infty(\cdot\,,\cdot)$ is a closed form on $L^2(U_1) \times L^2(U_2) \subseteq L^2(\Omega, \mathbb{R}^2)$ with domain $D(\boldsymbol{A}_\infty)$. We denote the operator $\mathfrak{a}_\infty(\cdot\,,\cdot)$ induces on $L_2(U_1) \times L_2(U_2)$ by $\boldsymbol{A}_\infty$. Because $\mathfrak{a}_\infty(\cdot\,,\cdot)$ is the restriction of $\mathfrak{a}(\cdot\,,\cdot)$ to $D(\mathfrak{a}_\infty)$ Lemma 2.1 implies the coercivity estimate

$$\frac{\alpha}{2}\|u\|_{H^1}^2 \leq \mathfrak{a}_\infty(u, u) + \gamma\|u\|_2^2$$

for all $u \in D(\mathfrak{a}_\infty)$. Note that $D(\mathfrak{a}_\infty)$ is a Banach lattice with positive cone induced by that of $H_0^1(\Omega, \mathbb{R}^2)$.

Theorem 3.1. *Assume that $f_n \in L^2(\Omega,\mathbb{R}^2)$ is such that $f_n \rightharpoonup f$ weakly in $L^2(\Omega,\mathbb{R}^2)$ and that $\lambda_n \to \infty$. If $\gamma \geq \gamma_0$ and $u_n \in H_0^1(\Omega,\mathbb{R}^2)$ is the unique solution of*

$$\boldsymbol{A}u_n + \lambda_n \boldsymbol{M} u_n + \gamma u_n = f_n, \tag{3.2}$$

then the following assertions are true.

(i) $\|u_n\|_{H^1} \leq \dfrac{2}{\alpha}\|f_n\|_2$ *for all* $n \in \mathbb{N}$.

(ii) $(\gamma - \gamma_0)\|u_n\|_2 \leq \|f_n\|_2$ *for all* $n \in \mathbb{N}$.

(iii) $u_n \to u$ *in* $H_0^1(\Omega,\mathbb{R}^2)$ *and* $\lambda_n\langle \boldsymbol{M}u_n, u_n\rangle \to 0$ *as* $n \to \infty$.

(iv) $u \in D(\mathfrak{a}_\infty) = H_0^1(\Omega) \cap \ker \boldsymbol{M}$.

Moreover, if $\gamma \in \varrho(\boldsymbol{A}_\infty)$, then for λ large enough, $\gamma \in \varrho(\boldsymbol{A} + \lambda \boldsymbol{M})$ and

$$\lim_{\lambda\to\infty} (\gamma I + \boldsymbol{A} + \lambda \boldsymbol{M})^{-1} = (\gamma I + \boldsymbol{A}_\infty)^{-1}$$

in $\mathcal{L}\big(L^2(\Omega,\mathbb{R}^2), H_0^1(\Omega,\mathbb{R}^2)\big)$ in the operator norm.

Proof. Assertions (i) and (ii) follow from (2.5) and (2.6), respectively. As the sequence (f_n) is weakly convergent in $L^2(\Omega,\mathbb{R}^2)$ it is bounded in $L^2(\Omega,\mathbb{R}^2)$. Hence by (i) u_n is bounded in $H_0^1(\Omega,\mathbb{R}^2)$ and therefore a sub-sequence converges weakly in $H_0^1(\Omega,\mathbb{R}^2)$. If u_{n_k} is such a sequence we get from the weak formulation

$$\mathfrak{a}(u_{n_k}, v) + \lambda_{n_k}\langle \boldsymbol{M}u_{n_k}, v\rangle + \gamma\langle u_{n_k}, v\rangle = \langle f_{n_k}, v\rangle \tag{3.3}$$

for all $v \in H_0^1(\Omega,\mathbb{R}^2)$. As all terms remain bounded we conclude that $\langle \boldsymbol{M}u_{n_k}, v\rangle \to \langle \boldsymbol{M}u, v\rangle = 0$ for all $v \in H_0^1(\Omega,\mathbb{R}^2)$. Hence $u \in H_0^1(\Omega,\mathbb{R}^2) \cap \ker \boldsymbol{M}$. Now if $v \in H_0^1(\Omega,\mathbb{R}^2)$, then from (3.3)

$$\mathfrak{a}(u_{n_k}, v) + \gamma\langle u_{n_k}, v\rangle = \langle f_{n_k}, v\rangle$$

since $\langle \boldsymbol{M}u_{n_k}, v\rangle = \langle \boldsymbol{M}v, u_{n_k}\rangle = 0$ if $v \in \ker \boldsymbol{M}$. Passing to the limit we get

$$\mathfrak{a}(u, v) + \gamma\langle u, v\rangle = \langle f, v\rangle.$$

As $\mathfrak{a}(u,v)$ and $\langle f, v\rangle$ are finite for all $v \in H_0^1(\Omega,\mathbb{R}^2)$ we conclude that

$$0 = \lim_{k\to\infty} \langle \boldsymbol{M}u_{n_k}, v\rangle = \langle \boldsymbol{M}u, v\rangle$$

for all $v \in H_0^1(\Omega,\mathbb{R}^2) \cap \ker \boldsymbol{M}$. Hence $\boldsymbol{M}u = 0$ and so $u \in D(\mathfrak{a}_\infty) = H_0^1(\Omega) \cap \ker \boldsymbol{M}$, proving (iv). We also conclude that $u = (\gamma I - \boldsymbol{A}_\infty)^{-1} f$ is the unique limit point of the sequence (u_n). Hence the whole sequence converges, that is, $u_n \to u$ weakly in $H_0^1(\Omega,\mathbb{R}^2)$ and strongly in $L^2(\Omega,\mathbb{R}^2)$. To prove strong convergence in $H_0^1(\Omega)$ note that

$$\begin{aligned}
\mathfrak{a}(u_n - u, u_n - u) &= \mathfrak{a}(u_n, u_n) - \mathfrak{a}(u_n, u) - \mathfrak{a}(u, u_n - u)\\
&= \langle f_n, u_n\rangle - \lambda_n\langle \boldsymbol{M}u_n, u_n\rangle - \gamma\|u_n\|_2^2 - \mathfrak{a}(u_n, u) - \mathfrak{a}(u, u_n - u)\\
&\leq \langle f_n, u_n\rangle - \gamma\|u_n\|_2^2 + \mathfrak{a}(u_n, u) - \mathfrak{a}(u, u_n - u)
\end{aligned}$$

for all $n \in \mathbb{N}$. Recall that $\mathfrak{a}(u,u)+\gamma\|u\|_2^2 = \langle f,u\rangle$. As $u_n \to u$ weakly in $H_0^1(\Omega,\mathbb{R}^2)$ and strongly in $L^2(\Omega,\mathbb{R}^2)$ we conclude that

$$\limsup_{n\to\infty} \mathfrak{a}(u_n-u,u_n-u) \leq \langle f,u\rangle - \bigl(\mathfrak{a}(u,u)+\gamma\|u\|_2^2\bigr) = 0.$$

Now (2.4) implies

$$0 \leq \frac{\alpha}{2}\limsup_{n\to\infty}\|u_n-u\|_{H^1}^2 \leq \limsup_{n\to\infty}\bigl(\mathfrak{a}(u_n-u,u_n-u)+\gamma\|u_n-u\|_2^2\bigr) \leq 0$$

and therefore $u_n \to u$ in $H_0^1(\Omega,\mathbb{R}^2)$. In particular,

$$\begin{aligned}\lambda_n\langle \boldsymbol{M}u_n,u_n\rangle &= \langle f_n,u_n\rangle - \gamma\|u_n\|_2^2 - \mathfrak{a}(u_n,u_n)\\ &\xrightarrow{n\to\infty} \langle f,u\rangle - \gamma\|u\|_2^2 - \mathfrak{a}(u,u) = 0.\end{aligned}$$

Hence (iii) follows. The assertion on the convergence of the resolvents in the operator norm now follows from [8, Proposition 4.1.1 and Theorem 4.3.1]. □

From the above we immediately get the following consequence.

Corollary 3.2. *Suppose that $\gamma \geq \gamma_0$, that $p > N/2$ and that $f \in L^p(\Omega)$ is non-negative. If u_λ is the solution of (2.1), then $u_\lambda \downarrow u$ converges monotonically as $\lambda \uparrow \infty$ increases. Moreover, the components of u_λ converge to zero uniformly on every compact subset of $\operatorname{int}(U_1^c)\cup\operatorname{int}(U_1)$ and $\operatorname{int}(U_2^c)\cup\operatorname{int}(U_2)$, and respectively.*

Proof. By standard regularity theory $u_\lambda \in C(\Omega)$ if $f \in L^p(\Omega,\mathbb{R}^2)$ with $p > N/2$, and therefore looking at pointwise properties makes sense. By Theorem 3.1 we know that the components of $u=(u_1,u_2)$ are zero on U_1^c and U_2^c. Moreover u_1 are bounded solutions of $-\mathcal{A}_1u_1 + d_1Pu_2 + \gamma u_1 = f_1$ and similarly for u_2, where P is the projection introduced just after (1.3). The boundedness is comes from the fact that $0 \leq u \leq u_\lambda$ and u_λ is bounded for all $\lambda \geq 0$. By standar regularity theory u_i is continuous in $\operatorname{int}(U_i)$. Since the convergence is monotone and the limit function u continuous, Dini's theorem (see for instance [19, Proposition 9.2.11]) implies uniform convergence on compact subsets on every compact subset of $\operatorname{int}(U_1^c)\cup\operatorname{int}(U_1)$ and $\operatorname{int}(U_2^c)\cup\operatorname{int}(U_2)$, respectively. □

Note that if the limit function u in the above corollary is continuous on $\bar{\Omega}$, then the convergence is uniform on $\bar{\Omega}$.

4 Eigenvalue estimates for the limit problem

We can now prove the main estimate (1.4). The idea to obtain (1.4) is to apply the comparison results from Theorem 2.3 to various modifications of $\boldsymbol{M}$. We note that $\mu(\lambda)$ is increasing in λ, so $\mu(\lambda) \to \mu_\infty$. In the extreme case m_1 and m_2 are strictly positive on Ω. Then the limit problem is trivial and therefore $\mu_\infty = \infty$. To avoid this situation we need to make some assumption on m_1 and m_2. As in (1.2) we let U_1 and U_2 be the set where m_1 and m_2 vanish. As in (3.1), for $i = 1, 2$, we let

$$V_i := \{u \in H_0^1(\Omega) \colon m_i u = 0\}.$$

Then $V_1 \times V_2 = H_0^1(\Omega, \mathbb{R}^2) \cap \ker \boldsymbol{M}$ is the domain $D(\mathfrak{a}_\infty)$ introduced in the previous section. Clearly $\mathfrak{a}_i(\cdot\,,\cdot)$ defined in (2.2) is a closed form on $L^2(U_i)$ with domain V_i for $i = 1, 2$ (see for instance [8, Proposition 2.1.6]). We could also consider it as a closed form on $L^2(\Omega)$ with non-dense domain. In any case the operator associated with $\mathfrak{a}_i(\cdot\,,\cdot)$ has a smallest principal eigenvalue we denote by μ_i. If $V_i = \{0\}$ we set $\mu_i = \infty$. With these definitions of μ_1, μ_2, our precise result is the estimate (1.4).

In case U_i is the closure of its interior then μ_i is associated with the eigenvalue problem

$$\begin{aligned} \mathcal{A}_i \varphi &= \mu_1 \varphi && \text{in } \operatorname{int}(U_i), \\ \varphi &= 0 && \text{on } \partial U_i \end{aligned}$$

on a maximal domain. As said in the introduction, if U_i is stable for the Dirichlet problem, then the above is the usual eigenvalue problem with the domain of the form $\mathfrak{a}_i(\cdot\,,\cdot)$ being $V_i = H_0^1(\operatorname{int}(U_i))$. For less regular domains our form approach still gives a well defined limit problem having a principal eigenvalue in V_i.

If U_i has non-empty interior we could work with the minimal domain since $H_0^1(\operatorname{int}(U_i))$ is an ideal of V_i. This means that $u \in V_i$ is such that if $u \in V_i$ and $|u| \leq |v|$ for some $v \in H_0^1(\operatorname{int}(U_i))$, then $u \in H_0^1(\operatorname{int}(U_i))$. If $\bar{\mu}_1$ is the corresponding principal eigenvalue for $\mathcal{A}_i$ on the minimal domain, then the domination results [18, Corollary 2.22] imply that $\mu_1 \leq \bar{\mu}_1$. Hence (1.4) always implies the possibly weaker result

$$\mu_\infty \leq \min\{\bar{\mu}_1, \bar{\mu}_2\}.$$

We now state and prove one of our main results.

Theorem 4.1. *Suppose that $\boldsymbol{M} > 0$, and that U_1 or U_2 has non-empty interior. Under the above assumptions*

$$\mu_\infty \leq \min\{\mu_1, \mu_2\} < \infty.$$

Moreover, μ_∞ is the smallest eigenvalue of $\boldsymbol{A}_\infty$ and μ_∞ has a positive eigenfunction.

Proof. We know from Theorem 2.3 that $\mu(\lambda)$ is increasing in λ and therefore converges to $\mu_\infty \leq \infty$. We furthermore know from Theorem 3.1 that $(\boldsymbol{A} + \lambda \boldsymbol{M} + \gamma_0 I)^{-1} \to (\boldsymbol{A}_\infty + \gamma_0 I)^{-1}$ in the operator norm. Hence by [8, Theorem 4.3.1] μ_∞ is the smallest eigenvalue of $\boldsymbol{A}_\infty$. Assuming that U_1 has non-empty interior we can choose an open ball $B \subseteq U_1$. If we define

$$\tilde{\boldsymbol{M}} := \begin{bmatrix} 1_{B^c}(1 + m_1) & 0 \\ 0 & 1 + m_2 \end{bmatrix},$$

then $\boldsymbol{M} \leq \tilde{\boldsymbol{M}}$ and therefore by the monotonicity of the principal eigenvalue (Theorem 2.3) we get

$$\mu(\lambda) \leq \tilde{\mu}(\lambda)$$

for all $\lambda > 0$. Here $\tilde{\mu}(\lambda)$ is the principal eigenvalue of $\boldsymbol{A} + \lambda\tilde{\boldsymbol{M}}$. According to Theorem 3.1 the limit problem as $\lambda \to \infty$ is associated with the form $\mathfrak{a}(\cdot\,,\cdot)$ on the domain $H_0^1(\Omega, \mathbb{R}^2) \cap \ker \tilde{\boldsymbol{M}}$. From the definition of $\tilde{\boldsymbol{M}}$ we have

$$H_0^1(\Omega, \mathbb{R}^2) \cap \ker \tilde{\boldsymbol{M}} = H_0^1(B) \times \{0\}.$$

Here we also use that a ball is a stable set for the Dirichlet problem as discussed in Section 3, that is, $H_0^1(\bar{B}) = H_0^1(B)$. The form (2.3) reduces to

$$\mathfrak{a}(u, v) = \mathfrak{a}_1(u_1, v_1)$$

for all $u = (u_1, 0)$ and $v = (v_1, 0)$ in $H_0^1(B) \times \{0\}$. This means that $\tilde{\mu}(\lambda) \to \tilde{\mu}_1$, where μ_1 is the principal eigenvalue of $\mathcal{A}_1$ on B with Dirichlet boundary conditions. In particular

$$\mu_\infty = \lim_{\lambda\to\infty} \mu(\lambda) \leq \lim_{\lambda\to\infty} \tilde{\mu}(\lambda) = \tilde{\mu}_1 < \infty.$$

Clearly we can apply a similar argument if U_2 has a non-empty interior and show that μ_∞ is finite. To prove (1.4) we use a similar argument. We let

$$\boldsymbol{M}_1 := \begin{bmatrix} m_1 & 0 \\ 0 & 1 + m_2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{M}_2 := \begin{bmatrix} 1 + m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

then clearly $\boldsymbol{M} < \boldsymbol{M}_i$ for $i = 1, 2$. If we denote the principal eigenvalue of $(\boldsymbol{A}_\lambda + \boldsymbol{M}_i)$ by $\mu_i(\lambda)$, then the monotonicity results from Theorem 3.1 imply that $\mu_i(\lambda) < \mu(\lambda)$ for all $\lambda > 0$. Clearly the form domain of the limit problem involving $\boldsymbol{M}_1$ is $V_1 \times \{0\}$ and that for $\boldsymbol{M}_2$ is $\{0\} \times V_2$. The corresponding forms are $\mathfrak{a}_1(\cdot\,,\cdot)$ and $\mathfrak{a}_2(\cdot\,,\cdot)$, respectively, so by the convergence results from the first part of this proof

$$\mu_i = \lim_{\lambda\to\infty} \mu_i(\lambda) \leq \lim_{\lambda\to\infty} \mu(\lambda) = \mu_\infty.$$

for $i = 1, 2$. This proves (1.4). We finally want to show that μ_∞ has a positive eigenfunction. Since $\boldsymbol{A}_\infty$ has compact resolvent, μ_∞ is an eigenvalue. If U_i has many connected components μ_∞ does not need to be simple, so could have several eigenfunctions. Let $\lambda_n \to \infty$ and φ_n a positive eigenfunction to $\mu(\lambda_n)$ normalised such that $\|\varphi_n\|_2 = 1$. Then

$$(\boldsymbol{A} + \lambda_n \boldsymbol{M} + \gamma_0 I)\varphi_n = f_n := \big(\mu(\lambda_n) + \gamma_0\big)\varphi_n$$

As $\big(\mu(\lambda_n) + \gamma_0\big)\varphi_n$ is bounded in $L^2(\Omega, \mathbb{R}^2)$ Proposition 2.2 implies that (φ_n) is bounded in $H_0^1(\Omega, \mathbb{R}^2)$. By Rellich's compactness theorem there exists a subsequence (φ_{n_k}) converging to some φ in $L^2(\Omega, \mathbb{R}^2)$. In particular, $\|\varphi\|_2 = 1$ and $\varphi \geq 0$. By Theorem 3.1 applied with $f_n := \big(\mu(\lambda_n) + \gamma_0\big)\varphi_n$ we conclude that $\varphi_{n_k} \to \varphi$ in $H_0^1(\Omega, \mathbb{R}^2)$, and that φ is a positive eigenfunction to μ_∞. $\square$

We have not said anything about equality in (1.4). In general this is a difficult question depending on whether or not U_1 and U_2 intersect and the number and nature of their components. We look at one case of equality and one for strict inequality. The arguments are largely taken from [7].

Remark 4.2. First of all, as noted in [7], if $U_1 \cap U_2 = \emptyset$, then there is equality. This is because then for $u, v \in V_1 \times V_2$, from the definition of the form (2.3), we have

$$\mathfrak{a}_\infty(u, v) = \mathfrak{a}_1(u_1, v_1) + \mathfrak{a}_2(u_2, v_2).$$

Hence the limit problem is completely discoupled and μ_∞ is equal to the minimum of μ_1 and μ_2.

We finally consider a case of strict inequality in (1.4).

Remark 4.3. Another case considered in [7] is where U_1 and U_2 consist of one connected component and d_1, d_2 are strictly positive on $U_1 \cap U_2$. We assume that $U_1 \cap U_2$ have non-empty interior. Let $\varphi = (\varphi_1, \varphi_2)$ be a positive eigenfunction to μ_∞. Using test functions of the form $(v_1, 0)$ and $(0, v_2)$ we note that

$$\begin{aligned} \mathfrak{a}_1(\varphi_1, v_1) &= \mu_\infty\langle\varphi_1, v_1\rangle + \langle d_1\varphi_2, v_1\rangle, \\ \mathfrak{a}_2(\varphi_2, v_2) &= \mu_\infty\langle\varphi_2, v_2\rangle + \langle d_2\varphi_1, v_2\rangle \end{aligned} \tag{4.1}$$

for all $v_1 \in V_1$ and $v_2 \in V_2$. Due to the Harnack inequality (see for instance [12, Theorem 8.18]) at least one of φ_1 or φ_2 is either strictly positive or identically zero in the interior of U_1 and U_2, respectively. We note that there are no assumptions on the regularity of the sets in Harnack's inequality. The inequality works for any non-negative solution on an open set. Assume φ_1 is strictly positive and that $\varphi_2 = 0$. Then from the above $\mathfrak{a}_1(\varphi_1, v_1) = \mu_\infty\langle\varphi_1, v_1\rangle$ for all $v_1 \in V_1$ and $0 = \mathfrak{a}_2(0, v_2) = \langle d_2\varphi_1, v_2\rangle > 0$ for a suitable choice of $v_2 \in V_2$ by assumption on d_2 and φ_1. As this is a contradiction φ_1 and φ_2 must both be strictly positive. The adjoint problem induced by the form $\mathfrak{a}_i^\sharp(u, v) := \mathfrak{a}_i(v, u)$ on V_i has the same principal eigenvalue. Let $\psi_i^\sharp$ be a positive eigenfunction corresponding to μ_i to the adjoint problem. Using it as a test function in (4.1) we get for $i, j = 1, 2$, $i \neq j$,

$$\mu_i\langle\psi_i^\sharp, \varphi_i\rangle = \mathfrak{a}_i(\varphi_i, \psi_i^\sharp) = \mu_\infty\langle\varphi_i, \psi_i^\sharp\rangle + \langle d_i\varphi_j, \psi_i^\sharp\rangle.$$

As d_i, φ_i and $\psi_i^\sharp$ are positive on $U_1 \cap U_2$ we conclude that $\mu_i \neq \mu_\infty$ for $i = 1, 2$, so there is strict inequality in (1.4).

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