# An explicit symplectic integrator for the zero angular momentum 3-body problem in regularised coordinates

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# The 3-body problem

- Problem of three bodies moving under mutual gravity: Hamiltonian is

$$H = \frac{1}{2}\sum \frac{|P_j|^2}{m_j} - \sum \frac{m_k m_l}{a_j}, \text{ where } a_j = |X_l - X_k|$$

(summation is over cyclic permutations of $(1, 2, 3)$, denoted by $(j, k, l)$).
- Explicit solution in closed form cannot be written, but a lot is open to enquiry.
- e.g. numerical integration and [Waldvogel, 1982] regularisation of binary collisions.
- Families of collision orbits bound regions of certain dynamics.
- Relative periodic orbits in Cartesian coordinates are exactly periodic in these regularised ones.

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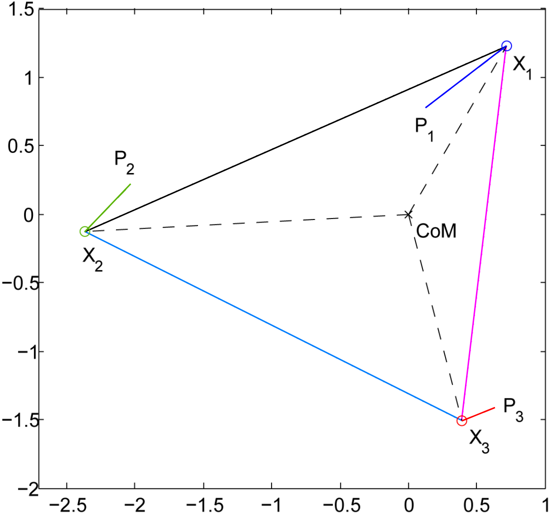


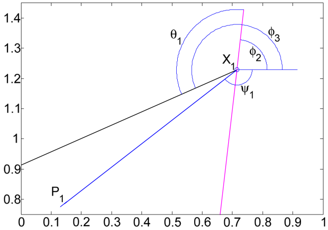

Example 3-body configuration in complex Cartesian coordinates, showing positions and physical momenta. Centre of mass at origin, $\sum_{j=1}^{3} P_j = 0$, $Im \sum_{j=1}^{3} \bar{X}_j P_j = 0$.

# Symmetry-reduced coordinates

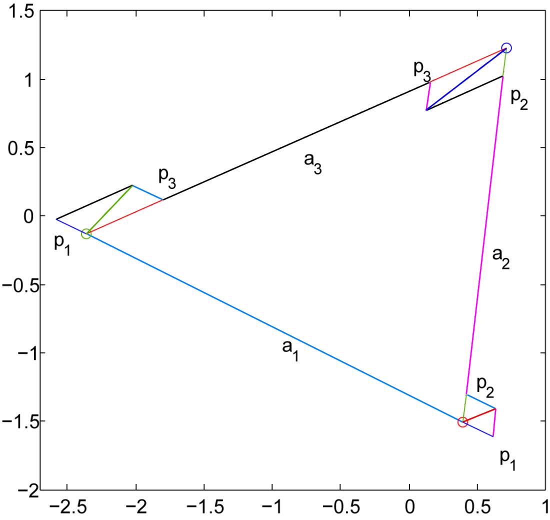
Coordinate transformation from (complex) Cartesian to symmetry-reduced coordinates by translations and rotations

$$\{X_j\} \rightarrow \{a_j, \phi\} \qquad a_j = |X_l - X_k|$$
$$\phi = \tfrac{1}{3}(\phi_1 + \phi_2 + \phi_3)$$
$$\{P_j\} \rightarrow \{p_j, p_\phi\} \qquad p_j = |P_l| \frac{\sin(\phi_j - \psi_l)}{\sin(\theta_j)} = \ldots$$
$$p_\phi = Im \sum_{j=1}^{3} \bar{X}_j P_j = \text{const}$$

$$\phi_j = \arg(X_l - X_k),\ \psi_j = \arg(P_j),\ \theta_j = \phi_l - \phi_k \text{ mod } 2\pi$$




Graphic of the angles $\phi_2$, $\phi_3$, $\psi_1$ and $\theta_1$.




Example 3-body configuration showing geometric interpretation of reduced coordinates $a_j$ and conjugate momenta $p_j$ as projections.

# Regularised coordinates

Another transformation: from symmetry-reduced to regularised

$$\begin{array}{lcl} \{a_j, \phi\} & \to & \{\alpha_j, \phi\} \qquad a_j = \alpha_k^2 + \alpha_l^2 \\ \{p_j, p_\phi\} & \to & \{\pi_j, p_\phi\} \qquad \pi_j = 2\alpha_j(p_k + p_l) \end{array}$$

In $(\alpha_j, \alpha_k, \alpha_l)$-space, each possible triangle is represented four times: if $a, b, c \in \mathbb{R}$ s.t. $a, b, c \geq 0$ and $a, b, c \neq 0$ simultaneously, then

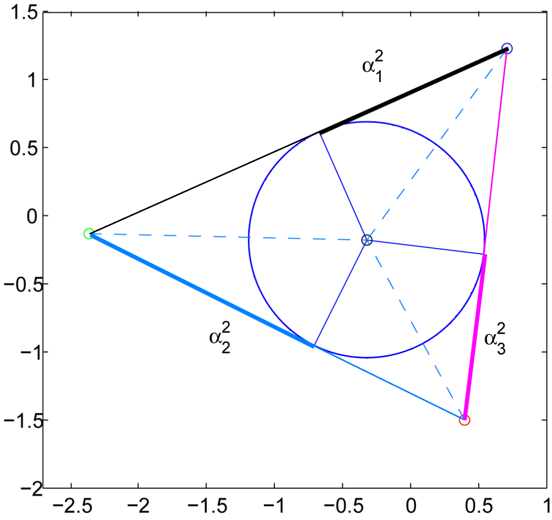
$$(a, b, c) \equiv (a, -b, -c) \equiv (-a, b, -c) \equiv (-a, -b, c)$$

are positively oriented (ordered counterclockwise) and

$$(-a, -b, -c) \equiv (-a, b, c) \equiv (a, -b, c) \equiv (a, b, -c)$$

are negatively oriented (ordered clockwise)

in the space of all triangles.




Example 3-body configuration showing geometric interpretation of regularised coordinates $\alpha_j$. (Conjugate momenta $\pi_j$ not shown.)

Rescale time by $\mathrm{d}t = a_1 a_2 a_3 \mathrm{d}\tau$ and employ Poincaré's trick, considering only surfaces of constant energy $h$.
Now $K = (H - h)\frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv 0$ for physically meaningful orbits.
When $p_\phi = 0$, Hamiltonian becomes polynomial:

$$K = \frac{1}{8} \sum \frac{a_j}{m_j} [\alpha^2 \pi_j^2 + (\alpha_k \pi_l - \alpha_l \pi_k)^2] - \sum m_k m_l a_k a_l - h a_j a_k a_l,$$

where $\alpha^2 := \alpha_j^2 + \alpha_k^2 + \alpha_l^2$ and $a_j = \alpha_k^2 + \alpha_l^2$.
Looks bad, but all binary collisions are regularised simultaneously.

# Some theory: symplectic integration

- We want to look for periodic orbits of the 3BP in these coordinates.
- This involves long time integration that must maintain qualitative accuracy.
- Standard explicit integrators (e.g. Runge-Kutta) won't do. Geometric integration, i.e. symplectic as this is Hamiltonian, must be the way.
- We want an explicit integrator, for the sake of efficiency.
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A monomial Hamiltonian of form

$$H(q_1,\ldots,q_n,p_1,\ldots,p_n) = A\prod_{j=1}^{n} q_j^{m_j} p_j^{n_j}$$

is integrable for $n_j, m_j \in \mathbb{N}$, with integrals $I_j = q_j^{m_j} p_j^{n_j}$ and solutions, when $m_j \neq n_j$,

$$q_j(t) = q_{j,0}(1 + (n_j - m_j)A\prod_{k\neq j} I_k q_{j,0}^{m_j-1} p_{j,0}^{n_j-1} t)^{\frac{n_j}{n_j-m_j}}$$

$$p_j(t) = p_{j,0}(1 + (n_j - m_j)A\prod_{k\neq j} I_k q_{j,0}^{m_j-1} p_{j,0}^{n_j-1} t)^{\frac{m_j}{m_j-n_j}}$$

and, when $m_j = n_j$,

$$q_j(t) = q_{j,0}\exp(m_j A\prod_{k\neq j} I_k q_{j,0}^{m_j-1} p_{j,0}^{m_j-1} t)$$

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# Taking stock

- Let $z \equiv (q, p)$. Given Hamiltonian $H(q, p) = H_1(q) + H_2(p)$, solution can be written

$$z = \Phi(t)z_0 = e^{t\{\cdot, H_1(q)\} + t\{\cdot, H_2(p)\}} z_0$$

- This is approximated to first order in $t$ by the map

$$z = \psi(t)z_0 + \mathcal{O}(t^2) = e^{t\{\cdot, H_1(q)\}} e^{t\{\cdot, H_2(p)\}} z_0 + \mathcal{O}(t^2) \qquad (1)$$

- This is just symplectic Euler.
- The adjoint ($\psi^*(t)$ s.t. if $z_1 = \psi(t)z_0$, then $z_0 = \psi^*(-t)z_1$) is

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- Compose 1 and 2 with “half-steps” to get the familiar symplectic leapfrog:

$$z = \phi(t)z_0 + \mathcal{O}(t^3) = e^{\frac{t}{2}\{\cdot, H_2(p)\}} e^{t\{\cdot, H_1(q)\}} e^{\frac{t}{2}\{\cdot, H_2(p)\}} z_0 + \mathcal{O}(t^3) \qquad (3)$$

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- This result doesn't depend on the forms of $H_1$ and $H_2$ or even that the system is Hamiltonian.
- So it's trivial to extend the result to a Hamiltonian $H = H_1 + \cdots + H_N$, where each $H_i$ in the sum can be solved explicitly.
- The generalised midpoint rule is thus

$$e^{t\{\cdot,H_1\}+\cdots+t\{\cdot,H_N\}} =$$

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# Going further

- Define $z_0 = \frac{1}{2 - 2^{1/(2n-1)}}$ and $z_1 = -\frac{2^{1/(2n-1)}}{2 - 2^{1/(2n-1)}}$ for some $n \in \mathbb{Z}^+$.
- Given a map $\phi(t) = e^{t\{\cdot, H\}} + \mathcal{O}(t^{2n+1})$
- and $\phi(t)\phi(-t) = Id$ (i.e. a reversible map),
- $\phi(z_0 t)\phi(z_1 t)\phi(z_0 t) = e^{t\{\cdot, H\}} + \mathcal{O}(t^{2n+3})$.
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- A Hamiltonian of form $H = H_1 + \cdots + H_N$,
- each $H_i$ is monomial,
- a way of building a first order map for the flow of H by composing the flows of each $H_i$,
- an adjoint map obtained by reversing the order of composition,
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- Newton's method on a Poincaré section to look for periodic orbits.
- Well suited to picking events such as collinearities ($\alpha_j = 0$, $\alpha_k, \alpha_l \neq 0$), binary collisions ($\alpha_j = \alpha_k = 0$, $\alpha_l \neq 0$).
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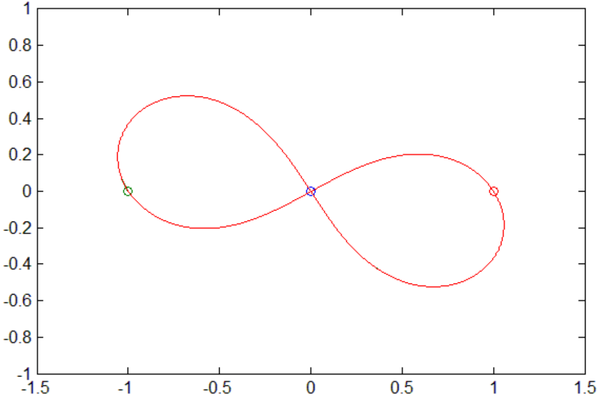
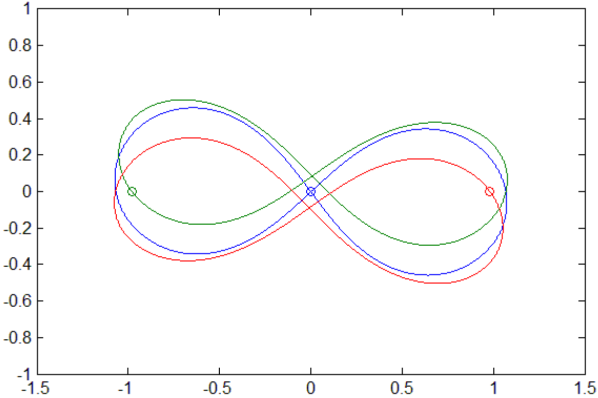



Figure-8 choreography obtained by integrating in regularised coordinates.

# Example 2




Periodic orbit obtained by continuation of the figure-8 with $m_1 = .95$ (blue).

# Energy behaviour

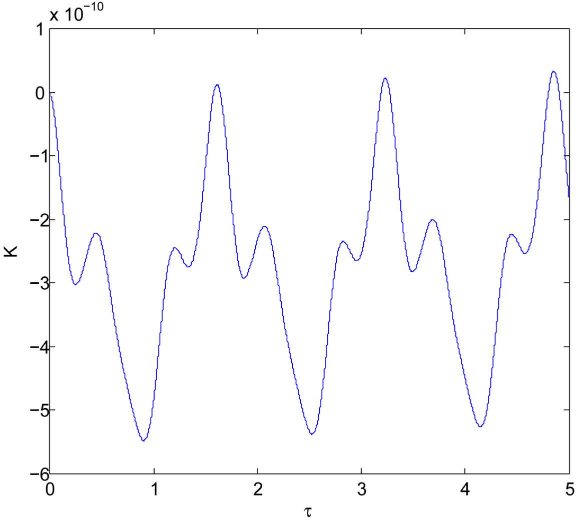



Figure: Absolute energy error vs scaled time $\tau$ for the figure-8 choreography. Time step $\delta\tau = 10^{-5}$ for $5 \times 10^5$ steps.

# Conclusion

In summary,

- binary collisions of the 3-body problem can be regularised,
- the resulting Hamiltonian is polynomial,
- monomial Hamiltonians can be integrated exactly,
- the flow of a Hamiltonian that is a sum of integrable Hamiltonians can be approximated explicitly numerically such that symplecticity is preserved, and
- the resulting explicit integrator is well behaved over large numbers of sufficiently small time steps.

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