University of Sydney

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Talented Student Program Report

# Well-posedness of a stochastic PDE governed by the $p$-Laplace operator

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**Abstract**

This report aims to prove the existence of a unique solution to a stochastic nonlinear parabolic boundary-value problem involving the $p$-Laplace operator. We first convert the SPDE into an abstract Cauchy problem, and then to a deterministic problem with a time-dependent operator. Using key operator properties such as monotonicity, as well as the properties of the Gelfand triple on which the problem is constructed, we are able to demonstrate the existence of a unique solution to the deterministic Cauchy problem, which satisfies the SPDE almost everywhere.

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## 1 Introduction and Main Results

Let $p \geq 2$ and $\Omega$ be an open, bounded set in $\mathbb{R}^n$. Our main focus in this paper will be to consider the existence of a unique solution $X = X(t)(x)$ to the stochastic nonlinear parabolic boundary-value problem

$$\begin{aligned} \frac{\partial X(t)(x)}{\partial t} - \Delta_p X(t)(x) &= B\frac{dW(t)}{dt} && \text{for } x \in \Omega, t \in (0,T), \\ X(t)(x) &= 0 && \text{for } x \in \partial\Omega, t \in [0,T], \\ X(0)(x) &= u_0(x) && \text{for } x \in \Omega, \end{aligned} \tag{1.1}$$

for any $T > 0$ and for a given $u_0 \in L^2(\Omega)$, where $L^p(\Omega)$ denotes the space of functions $u : \Omega \to \mathbb{R}$ which are $p$-integrable. In problem (1.1), $B$ belongs to $\mathcal{L}(W_0^{1,p}(\Sigma), W^{1,p}(\Omega))$ where $\Sigma$ is another open, bounded subset of $\mathbb{R}^n$, $\{W(t)\}_{t\geq 0}$ is a family of cylindrical Wiener processes in $W_0^{1,p}(\Omega)$, and $\Delta_p$ is the $p$-Laplace operator, which is defined as follows.

**Definition 1.1** ($p$-Laplace Operator)**.** *The operator $\Delta_p : W^{1,p}_{loc} \to \mathcal{D}'(\Omega)$ given by*

$$\langle \Delta_p u, \phi \rangle = -\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi dx$$

*for every $\phi \in C_c^1(\Omega)$ is called the $p$-Laplace operator on $\Omega$.*

In order to consider the existence of a unique solution to problem (1.1), we want first to transform this SPDE into an abstract Cauchy SDE. We will do this by considering functions in the First Sobolev Space, which for $1 \leq p < \infty$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$$

where

$$\nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n} \right)$$

for $1 \leq i \leq n$. For a function $u \in W^{1,p}(\Omega)$, we define the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_p + ||\nabla u||_p$$

where

$$||u||_p = \left( \int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

We define $W_0^{1,p}(\Omega)$ as the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$, and $W_0^{1,p}(\Omega)$ is endowed with the same norm as for $W^{1,p}(\Omega)$. The space $W_0^{1,p}(\Omega)$ is separable in general, and also reflexive for $p > 1$.

We say that functions $u \in W_0^{1,p}(\Omega)$ vanish on the boundary of $\Omega$ (refer to Brezis [3, Section 9.4 and Remark 2.2] for details on the delicacies of this remark). The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,q}(\Omega)$, where $q = \frac{p}{p-1}$.

In order to reduce problem (1.1), we redefine the function $X$. For any fixed value of $t \in [0,T]$, we can consider the function $x \mapsto X(t)(x)$ and denote this by $X(t)$, where $X$ is an element of $W_0^{1,p}(\Omega)$.

Then we can rewrite problem (1.1) as an abstract stochastic Cauchy problem in $L^2(\Omega)$,

$$\begin{aligned} dX(t) - \Delta_p^D X(t)dt &= BdW(t) \quad \text{for } t \in (0,T), \\ X(0) &= u_0, \end{aligned} \tag{1.2}$$

where $\Delta_p^D : \mathcal{D}(\Delta_p^D) \subseteq L^2(\Omega) \to L^2(\Omega)$ is known as the Dirichlet $p$-Laplace operator, because the Dirichlet boundary conditions have been "absorbed" and are automatically satisfied for all $X \in W_0^{1,p}(\Omega)$.

We will now leave our example with the Dirichlet $p$-Laplace operator, and consider the more general case. Consider the abstract stochastic Cauchy problem in the Hilbert space $H$:

$$\begin{aligned} dX(t) + AX(t)dt &= BdW(t) \quad \text{for all } t > 0, \\ X(0) &= u_0, \end{aligned} \tag{1.3}$$

for a given $u_0 \in H$. We assume in problem (1.3) that $A : \mathcal{D}(A) \subseteq H \to H$ is a $m$-accretive operator (refer to Section 2) on a separable Hilbert space, $B$ belongs to $L(V,H)$ (where $V$ is another Banach space), and $W(t)$ is a cylindrical Wiener process in $V$, defined on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.

We will now consider the notion of solution to this problem (1.3).

**Definition 1.2** (Notion of Solution). *A solution to problem (1.3) is a continuous $H$-valued stochastic process $X = X(t)$ on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, is measurable with respect to the filtration $\mathcal{F}_t$, satisfies*

$$X(t) = u_0 - \int_0^t AX(s)ds + \int_0^t BdW(s), \quad \mathbb{P}\text{-a.s. for all } t > 0, \tag{1.4}$$

*and*

$$\mathbb{E}\int_0^T \left(||X(t)||_V^{\alpha+1} + ||X(t)||_H^2\right) dt \text{ is finite for all } T > 0, \tag{1.5}$$

*where $\mathbb{E}$ denotes the expected value and $\alpha$ is a positive constant (refer to Wang [8, Definition 2.1.1] for details).*

We can consider $\mathcal{F}_t$ to be a family of $\sigma$-algebras which depend on the time $t$, and are generated by the Wiener process $W(s)$ for $0 \leq s \leq t$. The Wiener process is defined in Section 2.

Before stating our general existence result, we want to state the existence and uniqueness theorem relating to problem (1.1).

**Theorem 1.1.** *Let $p \geq 2$, $T > 0$, and $\Omega$ be an open, bounded set in $\mathbb{R}^n$. Assume $B$ belongs to $\mathcal{L}(W_0^{1,p}(\Sigma), W^{1,p}(\Omega))$, where $\Sigma$ is another open, bounded subset of $\mathbb{R}^n$, and that $\{W(t) : t \geq 0\}$ is a family of cylindrical Wiener processes in $W_0^{1,p}(\Omega)$. Then for every $u_0 \in L^2(\Omega)$ and any $T > 0$, problem (1.1) has a unique solution $\{X(t)\}_{t \geq 0}$ which is continuously dependent on the initial data in $L^2(\Omega)$.*

We will return to the proof of this theorem in Section 4. We now want to move towards our primary existence result, and in order to do so, we assume that $V$ is a reflexive Banach space continuously embedded in $H$ such that

$$V \hookrightarrow H \hookrightarrow V^* \tag{1.6}$$

where $\hookrightarrow$ indicates a continuous and dense injection from one Banach space into another. The relationship in equation (1.6) is known as a Gelfand triple (refer to Section 2), and will be assumed throughout this paper.

We want to demonstrate the existence of solutions to problem (1.3) using the following main result.

**Theorem 1.2** (Barbu 2010, Theorem 4.20 [2])**.** *Let $T > 0$ and $V$ be a Banach space which is continuously embedded in the Hilbert space $H$, such that $V \hookrightarrow H \hookrightarrow V^*$ as in equation (1.6). Let $A : V \to V^*$ be a demicontinuous monotone operator satisfying the following conditions:*

$$\langle Au, u \rangle_{V^*,V} \geq \omega ||u||_V^p + C_1, \text{ for all } u \in V, \text{ with } \omega > 0, p > 1, \tag{1.7}$$

$$||Au||_{V^*} \leq C_2(1 + ||u||_V^{p-1}), \text{ for all } u \in V, \text{ with } p > 1. \tag{1.8}$$

*Also assume that $BW \in L^p([0,T]; V) \cap C([0,T]; H)$ $\mathbb{P}$-a.s.. Then for every $u_0 \in H$, problem (1.3) has a unique solution $X = X(t) \in L^p([0,T]; V) \cap C([0,T]; H)$ which is continuously dependent on the initial data.*

Here, we denote by $L^p([a, b]; V)$ the space of functions $u : [a, b] \to V$ which are $p$-integrable. The norm on this $L^p$ space is given by

$$||u||_p = \left( \int_b^a ||u||_V^p \right)^{\frac{1}{p}}.$$

For details on demicontinuity and monotonicity, refer to Section 2. Note that this theorem assumes an operator $A : V \to V^*$, whereas problem (1.3) specified an operator $A : \mathcal{D}(A) \subseteq H \to H$. Theorem 1.2 can be applied in this case because the Gelfand triple allows us to restrict the operator $A : V \to V^*$ to $A_H : D(A_H) \subseteq H \to H$, where we define

$$\mathcal{D}(A_H) = \{u \in V : Au \in H\}.$$

So then $A_H u = Au$ for all $u \in \mathcal{D}(A_H)$, and we consider $A_H$ to be the operator included in problem (1.3).

The proof of Theorem 1.2 depends primarily on the existence of solutions for a deterministic Cauchy problem, and so we want to show that we can reduce problem (1.3) to such a problem. We first apply the substitution

$$u(t) = X(t) - BW(t)$$

and then the problem (1.3) is reduced to

$$\begin{aligned} \frac{du(t, \omega)}{dt} + A\big(u(t, \omega) + BW(t, \omega)\big) = 0, \qquad \text{for all } t > 0, \\ u(0) = u_0. \end{aligned} \tag{1.9}$$

For almost all $\omega \in \Omega$, this corresponds to the time-dependent problem

$$\begin{aligned} \frac{du(t)}{dt} + A(t)u(t) = 0, \qquad \text{for all } t > 0, \\ u(0) = u_0, \end{aligned} \tag{1.10}$$

where for every $t \geq 0$, we define the operator $A(t) : V \to V^*$ by

$$A(t)u = A(u + BW(t)) \tag{1.11}$$

for every $u \in V$.

The existence of solution to problem (1.10) can then be shown using this key theorem.

**Theorem 1.3** (Barbu 2010, Theorem 4.17 [2])**.** *Let $T > 0$ and $V$ be a reflexive, separable Banach space continuously embedded in the Hilbert space $H$ such that $V \hookrightarrow H \hookrightarrow V^*$, and let $2 \leq p < \infty$. Let $\{A(t)\}_{t\in[0,T]}$ be a family of monotone, demicontinuous operators $A(t) : V \to V^*$ satisfying the following conditions:*

$$A(t)u(t) \text{ is measurable from } [0,T] \text{ to } V' \text{ for every measurable } u : [0,T] \to V, \tag{1.12}$$

$$\big\langle A(t)u, u\big\rangle_{V^*,V} \geq \omega||u||_V^p + C_1, \text{ for some } \omega > 0, \text{ for all } u \in V \text{ and all } t \in [0,T], \tag{1.13}$$

$$||A(t)u||_{V^*} \leq C_2(1 + ||u||_V^{p-1}), \text{ for all } u \in V \text{ and for all } t \in [0,T]. \tag{1.14}$$

*Then for every $u_0 \in H$ and $f \in L^q([0,T];V^*), q = \dfrac{p}{p-1}$, there is a unique weak solution $u \in L^p([0,T];V) \cap C([0,T];H)$ that satisfies*

$$\frac{du(t)}{dt} + A(t)u(t) = f(t), \quad a.e. t \in (0,T), \quad u(0) = u_0 \tag{1.15}$$

*and where the solution is continuously dependent on the initial data.*

We want to explain the notion of solution for problem (1.15).

**Definition 1.3** (Weak Solution in $V^*$)**.** *Let $V$ be a reflexive, separable Banach space and $H$ be a Hilbert space such that $V \hookrightarrow H$. Let $\{A(t)\}_{t\geq 0}$ be a family of operators $A(t) : V \to V^*$ which satisfy conditions (1.12) and (1.14). Then for any given $u_0 \in H$, we call a function $u \in L^p([0,T];V) \cap C([0,T];H)$ a* ***weak solution*** *of the Cauchy problem (1.15) if $u(0) = u_0$ in $H$ and*

$$-\int_0^T \left\langle u(t), \frac{d\xi}{dt}(t) \right\rangle_{V^*,V} dt + \int_0^T \langle A(t)u(t), \xi(t)\rangle_{V^*,V}\, dt = \int_0^T \langle f(t), \xi(t)\rangle_{V^*,V}\, dt$$

*for all $\xi \in C_c^1([0,T];V)$.*

We also want to make the following remark, which allows us to derive an upper bound for

$$||u(t)||_H^2 + ||u(t)||_{L^p([0,t];V)}^p$$

which importantly does not depend on $t$.

**Remark 1.1** (Important Property of Cauchy Problem Solution)**.** *By Theorem 1.3 we know that for given $u_0 \in H$, the weak solution of problem (1.15) is unique and $\dfrac{\partial u}{\partial t} \in L^q([0,T];V^*)$, and so in fact $u$ is a "strong solution"!*

*Thus, multiplying the equation from problem (1.15) by* $u(t)$ *w.r.t* $\langle \cdot, \cdot \rangle_{V^*,V}$*, we have*

$$\left\langle \frac{du(t)}{dt}, u(t) \right\rangle_{V^*,V} + \langle A(t)u(t), u(t) \rangle_{V^*,V} = \langle f(t), u(t) \rangle_{V^*,V}$$

*From Showalter [7, Proposition 3.1.2], we have that*

$$2\langle \frac{du}{dt}, u \rangle_{V^*,V} = \frac{d}{dt}||u||_H^2 \text{ almost everywhere for } t \in [0,T] \tag{1.16}$$

*and using this, we have*

$$\frac{d}{dt}\frac{1}{2}||u(t)||_H^2 + \langle A(t)u(t), u(t) \rangle_{V^*,V} = \langle f(t), u(t) \rangle_{V^*,V} .$$

*Now, integrating over* $(0,t)$ *for any* $0 < t \leq T$ *and applying condition (1.13), one gets that*

$$\frac{1}{2}||u(t)||_H^2 - \frac{1}{2}||u_0||_H^2 + C + \omega \int_0^t ||u(s)||_V^p ds \leq \int_0^t \langle f(s), u(s) \rangle_{V^*,V} ds. \tag{1.17}$$

*Using the Cauchy-Schwartz inequality, we have*

$$\left| \int_0^t \langle f(s), u(s) \rangle_{V^*,V} ds \right| \leq \int_0^t ||f(s)||_{V^*} \cdot ||u(s)||_V ds$$

*and then by Hölder's inequality*

$$\left| \int_0^t \langle f(s), u(s) \rangle_{V^*,V} ds \right| \leq ||f||_{L^q([0,t];V^*)} \cdot ||u||_{L^p([0,t];V)}.$$

*Using Young's inequality, we have that*

$$\left| \int_0^t \langle f(s), u(s) \rangle_{V^*,V} ds \right| \leq \frac{\omega^{-\frac{q}{p}}}{q}||f||_{L^q([0,T];V^*}^q + \frac{\omega}{p}||u||_{L^p([0,t];V)}^p \tag{1.18}$$

*and so applying equation (1.18) to equation (1.17) gives us*

$$||u(t)||_H^2 + \frac{2\omega}{q} \int_0^t ||u(s)||_V^p ds \leq ||u_0||_H^2 + \frac{2\omega^{-\frac{q}{p}}}{q}||f||_{L^q([0,T];V^*}^q - 2C. \tag{1.19}$$

*As equation (1.19) holds for any choice of* $t$*, we can say that*

$$\sup_{t\in[0,T]} \left[ ||u(t)||_H^2 + \frac{2\omega}{q} \int_0^t ||u(s)||_V^p ds \right] \leq ||u_0||_H^2 + \frac{2\omega^{-\frac{q}{p}}}{q}||f||_{L^q([0,T];V^*}^q - 2C \tag{1.20}$$

*and we can see that the right-hand side is independent of* $t$ *as desired.*

In the case of problem (1.10), we only need to consider the case $f(t) \equiv 0$, but we will prove this theorem in the more general case. Note also that we cannot provide a complete theory of existence using this method.

## 2 Preliminaries

We need to introduce some mathematical tools and preliminary results. All lemmas are stated here without proof, which can be found in Barbu [2]. We will begin with the formal definition of a *dual space*.

**Definition 2.1** (Dual Space)**.** *The dual space of a vector space $V$ is a vector space consisting of all linear operators which map from $V$ into $\mathbb{R}$. The dual space of $V$ is denoted by $V^*$.*

For a given normed space $V$, it makes sense then to consider the associated *duality mapping* $J$, which maps elements from $V$ to the power set of its dual space $V^*$; that is, $J : V \to 2^{V^*}$ is defined by

$$J(v) = \left\{v^* \in V^* : \left\langle v^*, v\right\rangle_{V^*,V} = ||v||_V^2 = ||v^*||_{V^*}^2\right\} \text{ for all } v \in V.$$

The duality brackets $\left\langle v^*, v\right\rangle_{V^*,V}$ used above (and throughout this paper) denote the value of the functional $v^* \in V^*$ at $v \in V$.

In order to state a useful result pertaining to the duality mapping, we need to introduce the notions of *weak convergence* and *demicontinuity.*

**Definition 2.2** (Weak Convergence, Demicontinuity)**.** *A sequence $\{x_n\}$ in $V$ is weakly convergent to $x \in V$ if $\lim_{n\to\infty} f(x_n) = f(x)$ for every $f \in V^*$. This is denoted as $x_n \rightharpoonup x$.*

*Let $V$ and $W$ be Banach spaces. An operator $A : V \to W$ is demicontinuous if it is continuous from $V$ (with strong convergence) to $W$ (with weak convergence). That is, $A$ is demicontinuous if for every strongly convergent sequence $x_n \to x$ in $V$, we have that $Ax_n \rightharpoonup Ax$ in $W$.*

We can now state the following lemmas from Barbu [2].

**Lemma 2.1** (Barbu 2010, Theorem 1.1 [2])**.** *Let $V$ be a reflexive Banach space with norm $||\cdot||_V$. Then there exists an equivalent norm $||\cdot||_0$ on $V$ such that $V$ is strictly convex in this norm, and $V^*$ is strictly convex in the dual norm $||\cdot||_0^*$.*

**Lemma 2.2** (Barbu 2010, Theorem 1.2 [2])**.** *Let $V$ be a Banach space. If $V^*$ is strictly convex, then the duality mapping $J : V \to V^*$ is single-valued and demicontinuous.*

We now want to address some of the concepts mentioned in Section 1. The Wiener process was introduced as the stochastic element in problems (1.1), (1.2), and (1.3), and is defined below.

**Definition 2.3** (Cylindrical Wiener Process). *Let $V$ be a separable Banach space, and $\{e_n\}_{n\geq 1}$ a sequence of linearly independent vectors such that for $V_n := span\{e_1, ..., e_n\}$, one has that $V_n \subsetneq V_{n+1}$ and $\overline{\bigcup_{n\geq 1} V_n} = V$. A Wiener process $\{W(t)\}_{t\geq 0}$ (also called standard Brownian motion) is a mapping $W : \Omega \times \mathbb{R}_+ \to V$, where $\{\Omega, \mathcal{F}, \mathbb{P}\}$ is a probability space, which satisfies the following two properties:*

1. $\mathbb{P}\big[W(0) = 0\big] = 1$;
2. *Over disjoint intervals $(t_1, t_2) \subset [0, \infty)$, the increments of $W$ given by $W(t_2) - W(t_1)$ are independent, and are normally distributed with mean $= 0$ and variance $= t_2 - t_1$.*

**Remark 2.1** (Properties of the Wiener Process). *We emphasise the following two properties of Wiener processes.*

*(A) For a cylindrical Wiener Process $\{W(t)\}_{t\geq 0}$ in a separable Hilbert space $H$, we can write*

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \ \text{for } t \in [0, \infty)$$

*where $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $H$ and $\{\beta_k(t)\}_{k\geq 1}$ is a family of one-dimensional Brownian motions (refer to Ahmed [1, Definition 1.2.1, Section 1.2]).*

*(B) A cylindrical Wiener Process $\{W(t)\}_{t\geq 0}$ in a separable Banach space $V$ satisfies*

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \ W(\omega, \cdot) \in C^{\alpha}([0, T]; V)$$

*for some $\alpha \in \left(0, \frac{1}{2}\right)$. In the case $V = \mathbb{R}^d$, see Evans [5, Chapter 3, Part C].*

The derivative of $W(t)$ provides a good approximation for the "white noise", which we incorporate into the model. See Evans [5, Chapter 3] for details on the construction of a Wiener process in $\mathbb{R}$, and an explanation of the notion of (Ito's) stochastic integral.

We also want to define *m-accretive operators* and elaborate on the concept of a *Gelfand triple*, both introduced in Section 1.

**Definition 2.4** (Accretive Operators). *An operator $A : V \to V$ is accretive if for any choice of elements $u, v \in V$, there exists an element $w \in J(u - v)$ such that $\langle w, Au - Av \rangle_{V^*, V} \geq 0$. An accretive operator $A$ is $m$-*accretive *if the range of $A + I$ is equal to $V$ i.e. $R(A + I) = V$, where $I$ is the unity operator.*

It is the assumption that the operator $A$ is $m$-accretive in problem (1.3) which allows us to make the necessary assumptions in Theorem 1.2 (namely, that $A$ is monotone and demicontinuous). We turn now to Gelfand Triples, which provide a key foundation for our existence theorems.

**Definition 2.5** (Gelfand Triples)**.** *Suppose we have a reflexive, separable Banach space $V$, and that $V$ is dense and continuously embedded in the Hilbert space $H$. That is, we have $V \subset H$, and the injection map $i : V \to H$ is continuous and has a dense image.*

*Given that $V$ is reflexive, we also have that $H^*$ is continuously embedded in $V^*$. Using the Riesz-Fréchet Representation Theorem (see Brezis [3, Theorem 5.5 and following remarks]), we can identify $H = H^*$. Thus, we have the continuous embedding $V \hookrightarrow H \hookrightarrow V^*$, and $(V, H, V^*)$ is known as a Gelfand triple. In particular, we have that the inner products coincide; that is,*

$$\langle f, v \rangle_{V^*,V} = (f, v)_H \text{ for all } v \in V \text{ and for all } f \in H.$$

Another important concept in the existence theorems is that of *monotone* operators.

**Definition 2.6** (Monotonicity)**.** *An operator $A : V \to V^*$ is monotone if for any elements $u, v \in V$, we have*

$$\langle Au - Av, u - v \rangle_{V^*,V} \geq 0.$$

*Note that we can equivalently express $A$ as a subset of $V \times V^*$, defined by*

$$A = \{[u, u^*] \in V \times V^* : u^* \in Au\}$$

*$A$ is maximal monotone if $A$ is not properly contained in any other monotone subset of $V \times V^*$.*

The first of the following lemmas provides a necessary and sufficient condition for an operator to be maximal monotone, and the second lemma gives a sufficient condition for maximal monotonicity of a sum of operators.

**Lemma 2.3** (Barbu 2010, Theorem 2.3 [2])**.** *Let $V$ and $V^*$ be a reflexive and strictly convex Banach spaces, and let $A : V \to V^*$ be a monotone operator. Let $\Phi_p(v) = J(v)||v||_V^{p-1}$ for $v \in V$. Then $A$ is maximal monotone if and only if, for some $\lambda > 0$ and some $p > 1$, we have $R(A + \lambda\Phi_p) = V^*$.*

**Lemma 2.4** (Barbu 2010, Corollary 2.6 [2])**.** *Let $V$ be a reflexive Banach space, let $A : V \to V^*$ be a maximal monotone operator, and let $B : V \to V^*$ be a demicontinuous monotone operator. Then $A + B$ is maximal monotone.*

Finally, we want to state a sufficient condition for a maximal monotone operator to be surjective. For this, we need to introduce coercive operators.

**Definition 2.7** (Coercivity)**.** *An operator $A : V \to V^*$ is coercive if*

$$\frac{\langle Au, u\rangle_{V^*,V}}{||u||_V} \to \infty \text{ as } ||u||_V \to \infty.$$

We can now refer to the following lemma.

**Lemma 2.5** (Barbu 2010, Corollary 2.2 [2])**.** *Let $V$ be a reflexive Banach space and let $A : V \to V^*$ be a coercive maximal monotone operator. Then $A$ is surjective i.e. $R(A) = V^*$.*

# 3 Existence of Solutions

In this section, we first want to show that we can find a unique solution to problem (1.10) using Theorem 1.3, and then use this to prove Theorem 1.2 find a solution to problem (1.3). To do this, we will continue to refer to the Gelfand triple (1.6) introduced in Section 1.

Recall that $V$ is a reflexive and separable Banach space, and $H$ is a Hilbert space. We will also consider the spaces

$$\mathcal{V} = L^p([0, T]; V), \qquad \mathcal{H} = L^2([0, T]; H), \qquad \mathcal{V}^* = L^q([0, T]; V^*),$$

which form another Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$.

Before proving Theorem 1.3, we need to introduce the following lemma, which demonstrates the monotonocity of an operator which will appear in the proof.

**Lemma 3.1.** *Consider the operator $B : \mathcal{V} \to \mathcal{V}^*$, which is defined by $Bu = \frac{du}{dt}$ for functions $u \in D(B)$, where*

$$D(B) = \{u \in \mathcal{V} : \frac{du}{dt} \in \mathcal{V}^*, u(0) = u_0 \in H\}.$$

*Then $B$ is a monotone operator.*

*Proof:* We want to show that $\langle Bv - Bu, v - u\rangle_{\mathcal{V}^*,\mathcal{V}} \geq 0$ for any choice of functions $u, v \in D(B)$. To see this, let $u, v \in D(B)$. Then, we see that

$$\begin{aligned}\langle Bv - Bu, v - u\rangle_{\mathcal{V}^*,\mathcal{V}} &= \int_0^T \langle Bv(t) - Bu(t), v(t) - u(t)\rangle_{V^*,V} dt \\ &= \int_0^T \langle \frac{dv(t)}{dt} - \frac{du(t)}{dt}, v(t) - u(t)\rangle_{V^*,V} dt\end{aligned}$$

We can use the equality from Showalter [7, Proposition 3.1.2] quoted in equation (1.16) to see that

$$\begin{aligned}\langle Bv - Bu, v - u\rangle_{\mathcal{V}^*,\mathcal{V}} &= \int_0^T \langle \frac{dv(t)}{dt} - \frac{du(t)}{dt}, v(t) - u(t)\rangle_{V^*,V} dt \\ &= \int_0^T \frac{d}{dt}\frac{1}{2}||v(t) - u(t)||_H^2 dt \\ &= \frac{1}{2}||v(T) - u(T)||_H^2 - \frac{1}{2}||v(0) - u(0)||_H^2 \\ &= \frac{1}{2}||v(T) - u(T)||_H^2 \\ &\geq 0 \text{ for any functions } u, v \in D(B).\end{aligned}$$

So $B$ is a monotone operator. □

We need to introduce the following theorem from Barbu [2], which is the autonomous operator equivalent of Theorem 1.3. We will state this first theorem without proof, and it will be used in the proof for the case of the non-autonomous operator.

**Theorem 3.2** (Barbu 2010, Theorem 4.10 [2])**.** *Let $V$ be a reflexive, separable Banach space continuously embedded in the Hilbert space $H$ such that $V \subset H \subset V^*$. Let $A : V \to V^*$ be a demicontinuous monotone operator, satisfying the following conditions:*

$$\langle Au, u\rangle_{V^*,V} \geq \omega||u||_V^p + C_1, \text{ for some } \omega > 0 \text{ and some } p > 1, \text{ for all } u \in V \tag{3.1}$$

$$||Au||_{V^*} \leq C_2(1 + ||u||_V^{p-1}), \text{ for all } u \in V \tag{3.2}$$

*Then given $u_0 \in H$ and $f \in L^q([0,T]; V^*), q = \dfrac{p}{p-1}$, there is a unique weak solution $u \in L^p([0,T]; V) \cap C([0,T]; H)$ that satisfies*

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad a.e.t \in (0,T), \quad u(0) = u_0, \tag{3.3}$$

*and where the solution is continuously dependent on the initial data.*

Before constructing the proof for Theorem 1.3, we need to introduce one more lemma.

**Lemma 3.3.** *Define the operator* $A_0 : \mathcal{V} \to \mathcal{V}^*$ *by*

$$(A_0u)(t) = A(t)u(t) \tag{3.4}$$

*for* $t \in [0, T]$*, where* $A(t)$ *belongs to the family of operators from Theorem 1.3. Then* $A_0$ *is monotone, demicontinuous, and coercive.*

*Proof:* First, we want to show that $A_0$ is monotone. We can see that

$$\begin{aligned}\left\langle A_0u - A_0v, u - v\right\rangle_{\mathcal{V}^*,\mathcal{V}} &= \int_0^T \left\langle A(t)u(t) - A(t)v(t), u(t) - v(t)\right\rangle_{V^*,V} dt \\ &=: \int_0^T K(t)dt\end{aligned}$$

where $K(t) \geq 0$ for all $t \in [0, T]$ because $A$ is monotone. So then

$$\left\langle A_0u - A_0v, u - v\right\rangle_{\mathcal{V}^*,\mathcal{V}} \geq 0$$

for all $u, v \in \mathcal{V}$, and so $A_0$ is monotone.

Second, we want to show that $A_0$ is demicontinuous. Let $\{u_n\}$ be a sequence of functions in $\mathcal{V}$ such that $u_n \to u$. We know that $A$ is demicontinuous, and so

$$\left\langle Au_n(t), v\right\rangle_{V^*,V} \to \left\langle Au(t), v\right\rangle_{V^*,V}$$

for all $v \in V$. We want to show that $\left\langle A_0u_n, w\right\rangle_{\mathcal{V}^*,\mathcal{V}} \to \left\langle A_0u, w\right\rangle_{\mathcal{V}^*,\mathcal{V}}$ for all $w \in \mathcal{V}$. We can see that

$$\begin{aligned}\left\langle A_0u_n, w\right\rangle_{\mathcal{V}^*,\mathcal{V}} &= \int_0^T \left\langle Au_n(t), w(t)\right\rangle_{V^*,V} dt \\ &\to \int_0^T \left\langle Au(t), w(t)\right\rangle_{V^*,V} dt \\ &= \left\langle A_0u, w\right\rangle_{\mathcal{V}^*,\mathcal{V}}\end{aligned}$$

for all $w \in \mathcal{V}$. Thus $A_0$ is demicontinuous.

Finally, we want to show that $A_0$ is coercive i.e. $\dfrac{\left\langle A_0u, u\right\rangle_{\mathcal{V}^*,\mathcal{V}}}{||u||_{\mathcal{V}}} \to \infty$ as $||u||_{\mathcal{V}} \to \infty$ for $u \in \mathcal{V}$.

So then, for $u \in \mathcal{V}$,

$$\begin{aligned}\frac{\langle A_0 u, u\rangle_{\mathcal{V}^*,\mathcal{V}}}{||u||_{\mathcal{V}}} &= \frac{1}{||u||_{\mathcal{V}}}\int_0^T \langle A(t)u(t), u(t)\rangle_{V^*,V} dt \\ &\geq \frac{1}{||u||_{\mathcal{V}}}\int_0^T \omega ||u||_V^p + C_1 dt \\ &= T\omega ||u||_{\mathcal{V}}^{p-1} + \frac{TC_1}{||u||_{\mathcal{V}}} \\ &\to \infty \text{ as } ||u||_{\mathcal{V}} \to \infty.\end{aligned}$$

Thus, $A_0$ is coercive. □

We can now commence the proof of Theorem 1.3.

***Proof of Theorem 1.3:*** Consider the operator $B : \mathcal{V} \to \mathcal{V}^*$, defined in Lemma 3.1. From Lemma 3.1, we know that $B$ is a monotone operator. We now want to show that $B$ is maximal monotone. Let $\Phi_p(u) = J(u)||u||_V^{p-2}$ for $u \in \mathcal{V}$, where $J$ is the duality mapping from $V$ to $V^*$, and let $f \in \mathcal{V}^*$. We want to consider the problem,

$$Bu + \Phi_p u = f, \quad u(0) = u_0, \tag{3.5}$$

for $u \in D(B)$. Given that $\mathcal{V}$ and $\mathcal{V}^*$ are reflexive and $B \subset \mathcal{V} \times \mathcal{V}^*$ is monotone, we can use Lemma 2.3 to say that if $R(B + \Phi_p) = \mathcal{V}^*$ for some $p > 1$, then $B$ is maximal monotone. That is, we want to show that there exists a solution $u \in \mathcal{V}$ to problem (3.5) for any choice of $f \in \mathcal{V}^*$.

Note that $\Phi_p$ is an autonomous operator. We want to invoke Theorem 3.2 to show that $B + \Phi_p$ is surjective, and so we need to show that the operator $\Phi_p = J(u)||u||_V^{p-2}$ satisfies the conditions outlined in the theorem.

First, we want to show that $J(u)||u||_V^{p-2}$ is demicontinuous. From Lemma 2.1, we know that every reflexive Banach space $X$ can be renormed such that $X$ and $X^*$ become strictly convex. As $V$ is a reflexive Banach space, we can assume that $V$ and $V^*$ are strictly convex. Then we can use Lemma 2.2 to deduce that the duality mapping $J : V \to V^*$ (single-valued in this case) is demicontinuous. Given that $||u(t)||_V^{p-2}$ is continuous, we have that $J(u)||u||_V^{p-2}$ is demicontinuous.

Second, we want to show that $J(u)||u||_V^{p-2}$ is a monotone operator. Using the definition of the duality mapping, we have that

$$\big\langle Ju||u||_V^{p-2} - Jv||v||_V^{p-2}, u-v\big\rangle_{V^*,V} = ||u||_V^p + ||v||_V^p - ||u||_V^{p-2}\big\langle Ju, v\big\rangle_{V^*,V} - ||v||_V^{p-2}\big\langle Jv, u\big\rangle_{V^*,V}$$

Using the Cauchy-Schwartz inequality, we have that $\big\langle Ju, v\big\rangle_{V^*,V} \leq ||Ju||_{V^*}||v||_V$ for any $u, v \in \mathcal{V}$. Then

$$\big\langle Ju||u||_V^{p-2} - Jv||v||_V^{p-2}, u-v\big\rangle_{V^*,V} \geq ||u||_V^p + ||v||_V^p - ||u||_V^{p-2}||Ju||_{V^*}||v||_V - ||v||_V^{p-2}||Jv||_{V^*}||u||_V$$

and using that $||Ju||_{V^*} = ||u||_V$, we have

$$\big\langle Ju||u||_V^{p-2} - Jv||v||_V^{p-2}, u-v\big\rangle_{V^*,V} \geq (||u||_V - ||v||_V)(||u||_V^{p-1} - ||v||_V^{p-1})$$

for all functions $u, v \in \mathcal{V}$.

If $||u||_V \geq ||v||_V$, then $||u||_V^{p-1} \geq ||v||_V^{p-1}$ and so $(||u||_V - ||v||_V)(||u||_V^{p-1} - ||v||_V^{p-1}) \geq 0$. The same argument can be made if $||v||_V \geq ||u||_V$. Thus the operator $J(u)||u||_V^{p-2}$ is monotone.

Next, we need to show that condition (3.1) from Theorem 3.2 is satisfied. We can see that

$$\big\langle J(u)||u||_V^{p-2}, u\big\rangle_{V^*,V} = ||u||_V^{p-2}||u||_V^2 = ||u||_V^p \text{ for all } u \in \mathcal{V}.$$

So we have satisfied this condition with equality, with $\omega = 1$ and $C_1 = 0$.

Finally, we need to show that condition (3.2) from Theorem 3.2 is satisfied. We can see that

$$||J(u)||u||_V^{p-2}||_{V^*} \leq ||Ju||_{V^*}||u||_V^{p-2} = ||u||_V^{p-1} \text{ for all } u \in \mathcal{V}.$$

If we let $C_2 = 1$, then $||J(u)||u||_V^{p-2}||_{V^*} \leq C_2(1 + ||u||_V^{p-1})$ as required.

Given that the assumptions are satisfied, we can apply Theorem 3.2 and conclude that for any function $f \in \mathcal{V}^*$ and initial value $u_0 \in H$, there exists a unique continuous function $u \in \mathcal{V}$ such that

$$Bu + J(u)||u||_V^{p-2} = f, \quad u(0) = u_0.$$

Thus, $R(B + \Phi_p) = \mathcal{V}^*$, and so $B$ is maximal monotone.

Now, recall the operator $A_0 : \mathcal{V} \to \mathcal{V}^*$ defined in equation (3.4). We want to show that there is a unique solution to the problem

$$Bu + A_0 u = f, \text{ for any } f \in \mathcal{V}^*. \tag{3.6}$$

We will do this by showing that $B + A_0$ is surjective, and then verifying the uniqueness of the solution.

From Lemma 3.3, we know that $A_0$ is monotone, demicontinuous, and coercive. Given that $B$ is maximal monotone, we can deduce that $B + A_0$ is maximal monotone by Lemma 2.4.

We want to use Lemma 2.5, but this requires that $B+A_0$ is coercive. For $u \in \mathcal{V}$, we have

$$\frac{\big\langle (B + A_0)u, u\big\rangle_{\mathcal{V}^*,\mathcal{V}}}{||u||_{\mathcal{V}}} = \frac{\big\langle Bu, u\big\rangle_{\mathcal{V}^*,\mathcal{V}}}{||u||_{\mathcal{V}}} + \frac{\big\langle A_0 u, u\big\rangle_{\mathcal{V}^*,\mathcal{V}}}{||u||_{\mathcal{V}}}$$
$$\to \infty \text{ as } ||u||_{\mathcal{V}} \to \infty$$

given that $A_0$ is coercive. Thus, $B + A_0$ is coercive.

So we can use Lemma 2.5 to deduce that $R(B + A_0) = \mathcal{V}^*$. That is, problem (3.6) has a solution $u \in \mathcal{V}$ for any function $f \in \mathcal{V}^*$, and so $u(t) \in V$ is a solution to problem (1.15).

To see that this solution is *unique*, we use the monotonicity of $B$ and of $A_0$. Suppose we have two functions $u, v \in \mathcal{V}$ which are both solutions to equation (3.6). Then we know that

$$Bu - Bv + A_0 u - A_0 v = 0$$

and so

$$\big\langle Bu - Bv, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}} = -\big\langle A_0 u - A_0 v, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

As $B$ and $A_0$ are monotone, we must have

$$\big\langle Bu - Bv, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}} \geq 0 \text{ and } \big\langle A_0 u - A_0 v, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}} \geq 0,$$

which means

$$\big\langle Bu - Bv, u - v\big\rangle_{\mathcal{V}^*,\mathcal{V}} = 0.$$

We can use the equality from Showalter [7, Proposition 3.1.2] quoted in equation (1.16) to say that this is equivalent to

$$\frac{d}{dt}\frac{1}{2}||u - v||^2_{\mathcal{H}} = 0,$$

that is, $||u - v||^2_{\mathcal{H}} = \alpha$ where $\alpha$ is a constant.

This tells us that $v = u + \beta$, where $\beta$ is a constant. However, recall that $u$ and $v$ must be in $D(B)$, and all functions in $D(B)$ have a fixed root $u(0) = v(0) = u_0 \in H$. So we can conclude that $u = v$, which demonstrates that the solution is unique.

Finally, we want to show that the solution is continuously dependent on the initial data. Consider solutions $u(t)$ and $\hat{u}(t)$, which satisfy problem (1.10) with initial conditions $u_0$ and $\hat{u}_0$ respectively.

Then for any $T' > 0$, we have

$$\int_0^{T'} \left\langle \frac{du}{dt}(t) - \frac{d\hat{u}}{dt}(t), u(t) - \hat{u}(t) \right\rangle_{V^*,V} dt = \int_0^{T'} \frac{1}{2}\frac{d}{dt}||u(t) - \hat{u}(t)||_H^2 dt$$

using the equality from Showalter [7, Proposition 3.1.2] quoted in equation (1.16). However, from equation (1.10) we have

$$\int_0^{T'} \left\langle \frac{du}{dt}(t) - \frac{d\hat{u}}{dt}(t), u(t) - \hat{u}(t) \right\rangle_{V^*,V} dt = -\int_0^{T'} \left\langle A(t)u(t) - A(t)\hat{u}(t), u(t) - \hat{u}(t) \right\rangle_{V^*,V} dt$$

where it is clear that the right-hand side must be non-positive, given that $A(t)$ is monotone. This shows that

$$\int_0^{T'} \frac{1}{2}\frac{d}{dt}||u(t) - \hat{u}(t)||_H^2 dt \leq 0$$

and integrating, we have

$$\frac{1}{2}||u(T') - \hat{u}(T')||_H \leq \frac{1}{2}||u_0 - \hat{u}_0||_H \tag{3.7}$$

which demonstrates that continuous dependence of solutions on the initial data. This concludes the proof. □

We can now prove our main existence result, Theorem 1.2.

***Proof of Theorem 1.2***: Using the assumptions about the autonomous operator $A$ stated in the theorem, we want to demonstrate that the operator introduced in equation (1.11) satisfies the assumptions of Theorem 1.3. We can then apply Theorem 1.3 to prove that a solution exists to problem (1.10), which satisfies problem (1.3) for almost all $\omega \in \Omega$.

We firstly want to confirm that $A(t)$ as defined in equation (1.11) is both monotone and demicontinuous. Monotonicity is easy to see, as for $u, v \in V$ we have

$$\begin{aligned}\langle A(t)u - A(t)v, u - v\rangle_{V^*,V} &= \langle A(u + BW(t)) - A(v + BW(t)), u - v\rangle_{V^*,V} \\ &= \langle A(u + BW(t)) - A(v + BW(t)), u + BW(t) - v - BW(t)\rangle_{V^*,V} \geq 0\end{aligned}$$

as the operator $A$ is monotone by assumption. To check demicontinuity, let $\{u_n\}$ be a sequence in $V$ such that $u_n \to u$ strongly in $V$. Then we know that

$$Au_n \rightharpoonup Au$$

as $A$ is demicontinuous, and so

$$A(u_n + BW(t)) \rightharpoonup A(u + BW(t))$$

which demonstrates that $A(t)$ is demicontinuous. We now need to check that conditions (1.12), (1.13), and (1.14) are satisfied. First, let us check (1.12). Let $\{u_n\}$ be a sequence of step functions such that $u_n(t) \to u(t)$ strongly in $V$.

Since $A(t)$ is demicontinuous, we have that for all $v \in V$,

$$\langle A(t)u_n, v\rangle_{V^*,V} \to \langle A(t)u, v\rangle_{V^*,V}.$$

As $\{A(t)u_n\}$ is a sequence of measurable step functions and $V$ is reflexive, we have that

$$A(t)u_n \rightharpoonup A(t)u,$$

and so we can say that $A(t)u$ is the pointwise limit of measurable functions. Thus $A(t)$ is measurable (see Showalter [7, Lemma 4.1]).

Second, we want to check that condition (1.13) of Theorem 1.3 is satisfied. From our assumptions, we can say that for all $t \in [0, T]$,

$$\begin{aligned}\big\langle A(t)u, u\big\rangle_{V^*,V} &= \big\langle A(u + BW(t)), u + BW(t)\big\rangle_{V^*,V} - \big\langle A(u + BW(t)), BW(t)\big\rangle_{V^*,V} \\ &\geq \omega||u + BW(t)||_V^p - \big\langle A(u + BW(t)), BW(t)\big\rangle_{V^*,V}\end{aligned}$$

for some $\omega > 0$, given that $u+BW(t) \in V$. Using the reverse triangle inequality, we have

$$\begin{aligned}\big\langle A(t)u(t), u(t)\big\rangle_{V^*,V} &\geq \omega\left(||u||_V^p - ||BW||_V^p\right) - \big\langle A(u(t) + BW(t)), BW(t)\big\rangle_{V^*,V} \\ &\geq \omega||u||_V^p + C_1\end{aligned}$$

for all $u \in V$, and for $t \in [0, T]$, and where

$$C_1 = -\sup_{t\in[0,T]} \Big(\omega||BW||_V^p + \big\langle A(u + BW(t)), BW(t)\big\rangle_{V^*,V}\Big).$$

So condition (1.13) is satisfied.

Finally, we want to check that condition (1.14) of Theorem 1.3 is satisfied. From our assumptions, we can say that for all $t \in [0, T]$,

$$\begin{aligned}||A(t)u||_{V^*} &= ||A(u + BW(t))||_{V^*} \\ &\leq C_2(1 + ||u + BW||_V^{p-1}).\end{aligned}$$

Using the triangle inequality, we have

$$||A(t)u||_{V^*} \leq C_2(1 + ||u||_V^{p-1} + ||BW||_V^{p-1}).$$

Then we have

$$||A(t)u||_{V^*} \leq \begin{cases} 2C_2\left(||u||_V^{p-1} + 1\right) & \text{if } \beta := \sup_{t\in[0,T]} ||BW(t)||_V \leq 1 \\ C_2\beta\left(||u||_V^{p-1} + 1\right) & \text{if } \beta > 1. \end{cases}$$

So condition (1.14) is satisfied.

So we can apply Theorem 1.3 to find a unique solution $u = u(t)$ for problem (1.10), and then

$$X(t) = u(t) + BW(t)$$

satisfies problem (1.3) for $\mathbb{P}$-a.e. $\omega \in \Omega$ as desired.

We now want to show that this solution $X(t)$ satisfies conditions (1.4) and (1.5). From Theorem 1.3, we know that our solution $u(t)$ to problem (1.10) solves the more general problem (1.15) with a nonzero $f(t)$ term. Since we know that $\frac{du}{dt} \in L^q([0,T];V^*)$, we can say that

$$\left\langle \frac{du}{dt}, v \right\rangle_{V^*,V} + \langle A(t)u(t), v\rangle_{V^*,V} = \langle f(t), v\rangle_{V^*,V}$$

for some constant $v \in V$. Integrating, we have that

$$\int_0^t \left\langle \frac{du}{ds}, v \right\rangle_{V^*,V} ds + \int_0^t \langle A(s)u(s), v\rangle_{V^*,V}\, ds = \int_0^t \langle f(s), v\rangle_{V^*,V}\, ds$$

and then integrating by parts, we see

$$\langle u(t) - u_0, v\rangle_{V^*,V} + \int_0^t \langle A(s)u(s), v\rangle_{V^*,V}\, ds = \int_0^t \langle f(s), v\rangle_{V^*,V}\, ds \tag{3.8}$$

for any $t > 0$. We recall that $u(t) = X(t) - BW(t)$ and $A(t)u(t) = AX(t)$. This gives

$$\langle u(t) - u_0, v\rangle_{V^*,V} = \langle X(t) - BW(t) - u_0, v\rangle_{V^*,V} \tag{3.9}$$

and

$$\int_0^t \langle A(s)u(s), v, \rangle_{V^*,V}\, ds = \left\langle \int_0^t AX(s)ds, v, \right\rangle_{V^*,V} \tag{3.10}$$

where we have used the continuity and bi-linearity of the bracket operation as well as

$$\int_0^t g(t)ds = \lim_{N\to\infty}\left(\sum_{i=1}^{N} g(\xi_i)(s_i - s_{i-1})\right)$$

to interchange the integral and duality brackets.

So then, using equations (3.8), (3.9), and (3.10), we have that

$$\langle X(t) - BW(t) - u_0, v\rangle_{V^*,V} + \left\langle \int_0^t AX(s)ds, v\right\rangle_{V^*,V} = \left\langle \int_0^t f(s)ds, v\right\rangle_{V^*,V}$$

and rearranging

$$\left\langle X(t) - BW(t) - u_0 + \int_0^t AX(s)ds - \int_0^t f(s)ds, v\right\rangle_{V^*,V} = 0$$

for all constant $v \in V$. As $BW(t) = \int_0^t BdW(s)$, we can conclude that

$$X(t) = u_0 - \int_0^t AX(s)ds + \int_0^t f(s)ds + \int_0^t BdW(s)$$

or, in the case of problem (1.10)

$$X(t) = u_0 - \int_0^t AX(s)ds + \int_0^t BdW(s)$$

as given in condition (1.4).

We now want to show that this solution satisfies condition (1.5). Again, we start with the solution $u(t)$ to problem (1.10), which satisfies equation (1.20) by Remark 1.1. Given that in this case, $f(t) \equiv 0$, we know

$$\sup_{t\in[0,T]}\left[||u(t)||_H^2 + \frac{2\omega}{q}\int_0^t ||u(s)||_V^p ds\right] \leq ||u_0||_H^2 \tag{3.11}$$

and so we can deduce that

$$\sup_{t\in[0,T]} ||u(t)||_H^2 \leq ||u_0||_H^2 \tag{3.12}$$

and

$$\sup_{t\in[0,T]}\left[\frac{2\omega}{q}\int_0^t ||u(s)||_V^p ds\right] = \frac{2\omega}{q}\int_0^T ||u(t)||_V^p dt \leq ||u_0||_H^2 \tag{3.13}$$

must hold true.

Since we have that for $\mathbb{P}$-a.e.$\omega$, $X(\omega,t) = u(t) + BW(\omega,t)$, and then by hypothesis we know $BW \in L^p([0,T];V) \cap C([0,T];H)$ $\mathbb{P}$-a.s., one has for $X(t) := X(\omega,t)$

$$\begin{aligned} \sup_{t\in[0,T]} ||X(t)||_H &\le \sup_{t\in[0,T]} ||u(t)||_H + \sup_{t\in[0,T]} ||BW(t)||_H \\ &\le ||u_0||_H + \sup_{t\in[0,T]} ||BW(t)||_H \end{aligned} \tag{3.14}$$

by equation (3.12). Using the inequality referenced in [4], one also has

$$\begin{aligned} ||X(\omega,\cdot)||^p_{L^p([0,T];V)} &\le \left(||u||_{L^p([0,T];V)} + ||BW(\omega,\cdot)||_{L^p([0,T];V)}\right)^p \\ &\le 2^{p-1}||u||^p_{L^p([0,T];V)} + 2^{p-1}||BW(\omega,\cdot)||^p_{L^p([0,T];V)} \\ &\le \frac{2^{p-2}q}{\omega}||u_0||^2_H + 2^{p-1}||BW(\omega,\cdot)||^p_{L^p([0,T];V)} \end{aligned} \tag{3.15}$$

by equation (3.13).

So then, from equations (3.14) and (3.15) we get

$$\begin{aligned} \mathbb{E}\left[\int_0^T ||X(t)||^p_V + ||X(t)||^2_H dt\right] &\le \mathbb{E}\left[\int_0^T 2^{p-1}||BW(t)||^p_V + 2||BW||^2_H dt\right] \\ &\quad + 2\left(\frac{2^{p-3}q}{\omega} + 1\right)||u_0||^2_H \end{aligned} \tag{3.16}$$

which is finite by the continuity of $BW$ over the domain $[0,T]$. This satisfies condition (1.5) for $\alpha = p-1$.

Finally, we want to demonstrate that the solution is continuously dependent on initial conditions. Consider solutions $X(t)$ and $\hat{X}(t)$ which satisfy problem (1.3) with initial conditions $u_0$ and $\hat{u}_0$ respectively. From equation (3.7), we know that $u(t)$ is stable under changes in initial conditions, and so we have

$$\begin{aligned} ||X(t) - \hat{X}(t)||_H &\le ||u(t) + BW(t) - \hat{u}(t) - BW(t)||_H \\ &\le ||u(t) - \hat{u}(t)||_H \\ &\le ||u_0 - \hat{u}_0||_H \end{aligned} \tag{3.17}$$

which demonstrates that $X(t)$ is continuously dependent on the initial data. This concludes the proof. □

It should be noted that the existence of solutions is not restricted to the case where $p \ge 2$. We can prove Theorem 1.3 for $1 < p < 2$ by similar method, defining the problem on a modified Gelfand Triple, and by extension we can also prove Theorem 1.2 for the full range $p > 1$.

# 4 Application for SPDE driven by the $p$-Laplace operator

We now return to our problem (1.2), and use the tools established in Section 3 to prove Theorem 1.1 and so find a unique solution for problem (1.1).

***Proof of Theorem 1.1***: We want to show that Theorem 1.2 can be applied to problem (1.2). Given that $p \geq 2$, from Brezis [3, Section 9.4], we have that

$$W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,q}(\Omega) \tag{4.1}$$

and this is the Gelfand Triple on which we can consider our abstract Cauchy problem (1.2).

We need to show that the autonomous operator $-\Delta_p^D$ satisfies the assumptions in Theorem 1.2. Then we can invoke the theorem to conclude that a unique solution $X = X(t)$ exists for any $T > 0$, and the real-valued function $X(t)(x)$ solves problem (1.1).

We can show that $-\Delta_p^D$ is demicontinuous (and indeed, continuous) for all $u \in W_0^{1,p}(\Omega)$ by proving that the functional

$$\phi(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p$$

is $C^1$ for all $u \in W_0^{1,p}(\Omega)$. We refer the interested reader to Hauer [6, Theorem 1] for details.

Second, we need to show that $-\Delta_p^D$ is monotone. Now we have

$$\big\langle -\Delta_p^D u, w \big\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla w(x) dx,$$

for any $u, v \in W_0^{1,p}(\Omega)$. So we can see that

$$\big\langle -\Delta_p^D u + \Delta_p^D v, u - v \big\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} = \int_\Omega \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) (\nabla u - \nabla v) \, dx,$$

which will always non-negative for any $u, v \in W_0^{1,p}(\Omega)$, meaning that $-\Delta_p^D$ is monotone.

Next, we need to show that $-\Delta_p^D$ satisfies

$$\big\langle -\Delta_p^D u, u \big\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} \geq \omega ||u||^p_{W_0^{1,p}(\Omega)} \tag{4.2}$$

for some $\omega > 0, p > 1$, and for all $u \in W_0^{1,p}(\Omega)$. To prove this, note that

$$\begin{aligned} \big\langle -\Delta_p^D u, u \big\rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} &= \int_\Omega |\nabla u(x)|^p dx \\ &= |||\nabla u(x)|||^p_{L^p(\Omega)} \end{aligned}$$

for $u \in W_0^{1,p}$. Then by Poincaré's inequality (refer to Brezis [3, Corollary 9.19]), we have

$$\big\langle -\Delta_p^D u, u\big\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)} \geq \omega ||u||^p_{W_0^{1,p}(\Omega)}$$

for some constant $\omega$ which depends on $p$ and $\Omega$, and this satisfies the inequality (4.2).

Finally, we need to show that $-\Delta_p^D$ satisfies

$$|| -\Delta_p^D u||_{W^{-1,q}(\Omega)} \leq C_2(1 + ||u||^{p-1}_{W_0^{1,p}(\Omega)}) \tag{4.3}$$

for all $u \in W_0^{1,p}(\Omega)$. For any $u, v \in W_p^{1,p}(\Omega)$, we have

$$|| -\Delta_p^D u||_{W^{-1,q}(\Omega)} = \sup_{||v||\leq 1} \big|\big\langle -\Delta_p^D u, v\big\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}\big|.$$

Then by the triangle equality and using Cauchy-Schwartz inequality, we have

$$\begin{aligned}\big|\big\langle -\Delta_p^D u, v\big\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}\big| &\leq \int_\Omega \big||\nabla u|^{p-2}\nabla u \nabla v\big| dx \\ &\leq \int_\Omega |\nabla u|^{p-1}|\nabla v| dx.\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}\big|\big\langle -\Delta_p^D u, v\big\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}\big| &\leq \left(\int_\Omega |\nabla u|^{(p-1)q} dx\right)^{\frac{1}{q}} ||\nabla v||_p \\ &\leq ||\nabla u||_p^{p-1}||\nabla v||_p,\end{aligned}$$

and using Poincaré's inequality again,

$$\begin{aligned}\big|\big\langle -\Delta_p^D u, v\big\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}\big| &\leq C_2||u||^{p-1}_{W_0^{1,p}(\Omega)}||v||_{W_0^{1,p}(\Omega)} \\ &\leq C_2||u||^{p-1}_{W_0^{1,p}(\Omega)}\end{aligned}$$

as $||v||_{W_0^{1,p}(\Omega)} \leq 1$, and so inequality (4.3) is satisfied.

So the assumptions of Theorem 1.2 are satisfied, and we can apply Theorem 1.2 to problem (1.2) to conclude that there exists a unique solution $X = X(t)$ $\mathbb{P}$-a.s which is continuously dependent on initial conditions.

We now need to show that we have a unique solution for problem (1.1). Recall that $X(t)$ is our notation for the function $x \mapsto X(x,t)$ for $x \in \Omega$. We can reconsider $X$ to be a function of space and time for $x \in \Omega, t \in [0,T]$, and then $X(t)(x)$ is a solution for problem (1.1). The function $X(t)(x)$ is the compilation of unique solutions for all $t \geq 0$, and from Theorem 1.2 we have that it is unique. Moreover, from equation (3.17) we see that the solution is continuously dependent on initial data. □

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