The Dirichlet-to-Neumann Map on the Half Space

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Motivation (not for me...for the topic!)

Suppose we have

- Some medium Ω in $\mathbb{R}^3$ that conducts electricity.

When we apply

- A voltage φ to the boundary/surface of Ω.

This induces

- A potential u that satisfies Ohm's Law in the domain Ω and
- The gradient ∇u of u describes an electric field through the medium.
- The normal component $\nabla u \cdot \nu =: \frac{\partial u}{\partial \nu}$ of that electric field ∇u at the boundary of Ω describes the current flux density through the surface.

For a specified domain Ω, the Dirichlet-to-Neumann Map is an operator that sends the voltage φ to the normal component $\frac{\partial u}{\partial \nu}$ of the induced field ∇u at the boundary. Comparing abstractly computed expected values with actual measured values at the boundary gives us information about the properties of the medium Ω.

Our Classical Friends, Dirichlet...

Suppose Ω is an open set in $\mathbb{R}^d$.
The Dirichlet Problem. Let φ be a function on the boundary $\partial\Omega$. Does there exist a unique twice differentiable function u on Ω such that

$$\begin{cases} -\Delta u = 0 \text{ on } \Omega \text{ (Laplace's Equation)} \\ u = \varphi \text{ on } \partial\Omega. \end{cases}$$

Here $\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$ and is called the Laplacian.

...and Neumann, Introduce Us to DtN

The Neumann Problem. Let ψ be a function on the boundary $\partial\Omega$. Does there exist a unique twice differentiable function u on Ω such that

$$\begin{cases} -\Delta u = 0 \text{ on } \Omega \text{ (Laplace again!)} \\ \frac{\partial u}{\partial \nu} = \psi \text{ on } \partial\Omega. \end{cases}$$

Here $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ is the normal derivative of u at the boundary with respect to a given normal ν.

The Dirichlet-to-Neumann Map. As the name suggests, the Dirichlet-to-Neumann (DtN) Map sends boundary value data to normal derivative data via a solution.

$$\Lambda : \varphi \mapsto u \mapsto \frac{\partial u}{\partial \nu},$$

where u is the solution to the Dirichlet Problem.

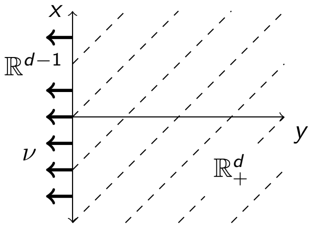
Is the DtN Map on the Halfspace an Old Friend?

What does the Dirichlet-to-Neumann Map actually look like? In particular, we want to investigate the DtN Map on the Halfspace

$$\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0, \infty) = \{(x, y) : x \in \mathbb{R}^{d-1}, y > 0\}$$

with boundary and normal vector

$$\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\} = \mathbb{R}^{d-1} \text{ and } \nu = (0, ..., 0, -1).$$



Is the DtN Map on the Halfspace an Old Friend? (cont.)

Let us consider the DtN Map in the smooth case. Suppose that we have $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d-1})$, with $u \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^d})$ a solution to

$$\begin{cases} -\Delta u = 0 \text{ in } \mathbb{R}_+^d \\ u(x,0) = \varphi \text{ (on } \mathbb{R}^{d-1}). \end{cases}$$

Then the normal derivative of u is given by

$$\frac{\partial u}{\partial \nu} = \nabla u|_{\mathbb{R}^{d-1}} \cdot \nu = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_d})|_{\mathbb{R}^{d-1}} \cdot (0, ..., -1) = -\frac{\partial u}{\partial x_d}(x,0).$$

In terms of the DtN map,

$$\Lambda \varphi = -\frac{\partial u}{\partial x_d}(x,0) \in \mathcal{C}_c^\infty(\mathbb{R}^{d-1}).$$

We can then apply the DtN map once more, to get $\Lambda^2 \varphi$.

Is the DtN Map on the Halfspace an Old Friend? (cont.)

We want the solution to the Dirichlet problem for the boundary function $-\frac{\partial u}{\partial x_d}(x,0)$, that is, v such that

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^d_+ \\ v(x,0) = -\frac{\partial u}{\partial x_d}(x,0) \text{ (on } \mathbb{R}^{d-1}). \end{cases}$$

However, by Schwarz' Theorem, we have that

$$-\Delta \frac{\partial u}{\partial x_d} = -\frac{\partial}{\partial x_d}\Delta u = 0 \text{ in } \mathbb{R}^d_+.$$

And trivially,

$$-\frac{\partial u}{\partial x_d}(x,0) = -\frac{\partial u}{\partial x_d}(x,0) \text{ on } \mathbb{R}^{d-1}.$$

It follows that $v = -\frac{\partial u}{\partial x_d}$ is the solution of the particular Dirichlet problem.

Is the DtN Map on the Halfspace an Old Friend? (cont.)

Using the same normal, we have

$$\frac{\partial}{\partial \nu}\left(-\frac{\partial u}{\partial x_d}\right) = -\frac{\partial}{\partial x_d}\left(-\frac{\partial u}{\partial x_d}\Big|_{\mathbb{R}^{d-1}}\right) = \frac{\partial^2 u}{\partial x_d^2}(x,0).$$

Hence,

$$\Lambda^2 \varphi = \Lambda(\Lambda\varphi) = \Lambda\left(-\frac{\partial u}{\partial x_d}(x,0)\right) = \frac{\partial^2 u}{\partial x_d^2}(x,0).$$

But we know that

$$\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \Delta u = 0,$$

so it follows that

$$\Lambda^2 \varphi = \frac{\partial^2 u}{\partial x_d^2}(x,0) = -\sum_{i=1}^{d-1} \frac{\partial^2 u}{\partial x_i^2}(x,0) = -\Delta_{d-1}\varphi.$$

The $d-1$ Laplacian! The Project.

We get the identity for smooth functions,

$$\Lambda^2 = -\Delta_{d-1}.$$

This identity is the main focus of the research project. We sought to:

- Generalise the classical Dirichlet/Neumann Problems to weaker nonclassical conditions - Sobolev spaces, weak derivatives, Lebesgue spaces etc - and investigate the Well-Posedness of these problems.
- Generalise the Dirichlet-to-Neumann Map with respect to the above weaker conditions.
- Investigate how, on appropriately constructed generalised spaces, as operators,

$$\Lambda = (-\Delta_{d-1})^{1/2}.$$

- Prove the above well-known identity using new and previously undiscovered methods.

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