MATH 402 Homework 2

Due Friday September 15, 2017

Exercise 1. This exercise will lead you through the proof that the Exterior Angle Theorem implies that the sum of the interior angles of a triangle is always less than $180°$. (Notice that you proved in Worksheet 2 that in Euclidean geometry, the sum of the interior angles is always *equal* to $180°$. But the proof you will give here is independent of the parallel postulate, so it holds in non-Euclidean geometry too.)

a. *[5 pts]* First show that the Exterior Angle Theorem implies that the sum of any two interior angles is less than $180°$.

b. *[5 pts]* Now suppose that you have a triangle ΔABC. Label the angle at B by β. Let M be the midpoint of $\overline{AC}$, and draw a segment $\overline{BE}$ through the points B and M so that M is also the midpoint of $\overline{BE}$. After you draw this picture, prove that ΔBCE is a triangle whose angle sum is equal to that of ΔABC, but which has an angle (either the angle at B or the angle at E) which is less than (or equal to) half of the angle β.

c. **Bonus (up to 5 extra points):** Repeating the above construction, show that for any natural number n you can find a triangle ΔXYZ whose angle sum is equal to that of ΔABC, but which has an angle which is less than $\frac{1}{2^n}\beta$. Use this to show that if ΔABC has angle sum strictly greater than $180°$, you can construct a triangle which contradicts the result in part a.

Exercise 2. *[10 pts]* Show that Playfair's postulate is equivalent to the following statement:

> **(*):** *If a line intersects but is not coincident with one of two parallel lines, then it must also intersect the other line.*

(To prove that they are equivalent, you should prove that Playfair's postulate implies **(*)**, and also that **(*)** implies Playfair's postulate.)

Exercise 3. This question is about the notion of points being on the *same side* or on *opposite sides* of a line.

a. *[2 pts]* State Pasch's Axiom.

b. *[3 pts]* Let A, B, C be three points, none of which is on the line ℓ. State what it means for A and B to be on opposite sides of ℓ. State what is means for A and B to be on the same side of ℓ.

c. *[10 pts]* Assume that A, B, C are not collinear. (The results are also true if they are collinear, but it would be an extra case to check.) Prove that if A and B are on the same side of ℓ, and B and C are on the same side of ℓ, then A and C are also on the same side of ℓ. Now prove that if A and B are on opposite sides of ℓ, and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

Exercise 4.

a. *[5 pts]* Solve exercise 2.2.10 from the book. (SAS congruence $\Rightarrow$ ASA congruence.)

b. *[5 pts]* Solve exercise 2.5.3 from the book. (SSS similarity.)

c. *[5 pts]* Prove Euclid's Proposition 10: Given a line segment $\overline{AB}$, we can construct its midpoint C (using straightedge and compass). Where in your proof do you use the Principle of Circle Continuity?

Hint: for all three of these problems, you may wish to use SAS and/or SSS congruence.

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.