MATH 402 Review for October 15–19

Topics: Rotation, translations, and glide reflections. (5.3, 5.4, 5.6.)
These were covered in lecture. This material will also appear in Homework 7.

1. **Recall from last week:** We took two approaches towards classifying all isometries.

   (1) We proved that the fixed point set S of an isometry f has one of four forms: everything, a line, a point, or the empty set.

   (2) We proved that every isometry can be written as a composition of at most three reflections.

2. **Classification of isometries:** Make sure you understand this table. (You should have it memorized, but it will be much easier to memorize if you understand.)

# of reflections	Relations between the lines	Name of isometry	Fixed point set
0		identity	everything
1: r_ℓ		reflection	ℓ
2: $r_\ell \circ r_m$	$\ell = m$	identity	everything
2: $r_\ell \circ r_m$	ℓ and m are parallel	(non-identity) translation	$\varnothing$
2: $r_\ell \circ r_m$	$\ell \cap m = \{O\}$	(non-identity) rotation	$\{O\}$
3: $r_\ell \circ r_m \circ r_n$	exactly two of the lines intersect at a single point P	glide reflection	$\varnothing$
3: $r_\ell \circ r_m \circ r_n$	any situation other than the previous line	reflection	a single line (we have to work to see which one)

3. **Other things to know about translations:**

   (a) A translation is *defined* to be a composition of two reflections with lines of reflection which are either parallel or coincident. (The identity is the special case where the lines are coincident. This is considered to be a translation.)

   (b) It is often convenient to work in coordinates when dealing with translations. A translation always has the form

$$T(x, y) = (x, y) + (v_1, v_2);$$

   (v_1, v_2) is called the *displacement vector* of T. Conversely, any function of this form is a translation (i.e. can be written as a composition of two reflections...)

   (c) Composition of translations yields another translation; the displacement vectors are added. Translations form a group.

   (d) A line ℓ is invariant under the translation T if and only if it is parallel to the displacement vector.

4. **Other things to know about rotations**

   (a) A rotation is *defined* to be a composition of two reflections where the lines of reflection are not parallel. (The identity is the special case where the lines are equal. This is considered to be a rotation.)

   (b) An isometry is a rotation if and only if it has a unique fixed point. This point is called the *centre of rotation.*

(c) Given a rotation R about a point O, there is an angle ϕ such that for any point $A \neq O$, the angle $\angle AOR(A)$ has measure ϕ. This is called the *angle of rotation.*

5. **Other things to know about glide reflections**

   (a) A glide reflection $G_{\ell,AB}$ is defined to be the composition of a translation T_{AB} with a reflection r_ℓ, where ℓ is parallel to the displacement vector $\vec{AB}$.

   - It doesn't matter whether we write $T_{AB} \circ r_\ell$ or $r_\ell \circ T_{AB}$: we can show that these are equal. Use whichever form is more convenient for you!

   (b) The inverse of $G_{\ell,AB}$ is again a glide reflection, equal to $G_{\ell,BA}$ (same line of reflection, negative of the original displacement vector).

Practice Questions

Tips for working with isometries: We often want to prove that an isometry f is equal to an isometry g. We have several strategies.

(a) Recall that it's enough to show that f and g agree on three non-collinear points, or equivalently that $f^{-1} \circ g$ or $g \circ f^{-1}$ fixes three non-collinear points. So we set out to look for these points. (Let's assume that we're trying to show $g \circ f^{-1}$ fixes three non-collinear points.

(b) We might start by sticking in points where we know how f^{-1} behaves (e.g. if we know fixed points of f^{-1}).

(c) Or, suppose there's a point P where we understand $g(P)$. We maybe don't know anything about $f^{-1}(P)$, but if we plug $f(P)$ into our composition, we get $g(P)$ in the end, so we're happy.

(d) Sometimes we can only find two fixed points of $g \circ f^{-1}$, say they're called P and Q. Let ℓ be the line through P and Q. Now, based on our classification of fixed point sets of isometries, we see that $g \circ f^{-1}$ is either the identity or r_ℓ. If we suppose towards a contradiction that $g \circ f^{-1} = r_\ell$, then we should mess around with this formula to see if we can pick out contradictory behaviour.

(e) For example, we might rewrite this equation as $g = r_\ell \circ f$, and we might be able to identify the fixed point sets of each side. If they don't match, we've found our contradiction!

(f) We introduced the terminology "orientation-preserving" and "orientation-reversing" this week. We'll discuss it in more depth next week, but once we are good at working with these concepts, we'll be able to skip steps (d) and (e): once we know that $g \circ f^{-1}$ is either the identity or a reflection, we just need to figure out whether it is orientation-preserving (then it must be the identity) or orientation-reversing (then it must be a reflection).

Look through proofs we've done in the lectures, and identify places where we've used each of these tips.