MATH 595 Tuesday, 20 February

More practice with Ext and Čech cohomology

(1) **Chapter III, Exercise 6.5.**

Let X be a noetherian scheme such that $\mathrm{Coh}(X)$ has enough locally frees. For a sheaf $\mathscr{F} \in \mathrm{Coh}(X)$, the *homological dimension* $\mathrm{hd}(\mathscr{F})$ is defined to be the least length of a locally free resolution of $\mathscr{F}$.

(a) Prove that $\mathscr{F}$ is locally free if and only if $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for every $\mathcal{O}_X$-module $\mathscr{G}$.

(b) Now use induction on n to prove that $\mathrm{hd}(\mathscr{F}) \leq n$ if and only if $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for every $\mathcal{O}_X$-module $\mathscr{G}$ and every $i > n$.

(c) Finally, show that $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$.

(2) **Chapter III, Exercise 6.6.**

Let A be a regular local ring, M a finitely generated A-module.

(a) Prove that M is projective if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > 0$.
(Hints for 'if': use descending induction to show that $\mathrm{Ext}^i(M, N) = 0$ for all $i > 0$ and any finitely generated module N. Use this to show that M is a direct summand of a free module.)

(b) Use (a) to show that for any n, $\mathrm{pd}(M) \leq n$ if and only if $\mathrm{Ext}^i(M, A) = 0$ for all $i > n$.

(3) **Chapter III, Exercise 4.7** Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation of degree d: $f(x_0, x_1, x_2) = 0$. Assume that $(1, 0, 0)$ is not a point of X.

(a) Find a suitable open affine cover of X with two pieces.

(b) Use Čech cohomology to compute that $H^0(X, \mathcal{O}_X)$ has dimension 1, and $H^1(X, \mathcal{O}_X)$ has dimension $\frac{1}{2}(d-1)(d-2)$.

(4) Let $X = \mathbb{A}^2$, and let $Y = X \times X \setminus \Delta$. What can you say about the cohomology of Y. What if you replace X by $\mathbb{A}^n$?