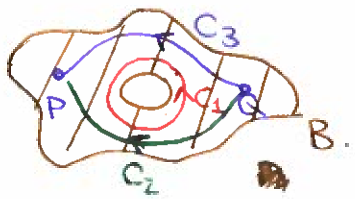
MATH 241 - LECTURE 32 - §16.5 Divergence & Curl. Weds. April 10, 2019

<u>Last time:</u> <u>Green's Theorem.</u> Let $D \subset \mathbb{R}^2$ such that ∂D is made up of one or more simple closed curves.

Let $\vec{F} = \langle P, Q \rangle$, where P, Q have continuous first order partial derivatives on D.

Then $\iint_D Q_x - P_y \, dA = \oint_C F \cdot d\vec{r}$

Consider the following:



Let $\vec{F} = \langle P, Q \rangle$ and assume $P_y = Q_x$ on B.

Suppose $\int_{C_1} \vec{F} \cdot d\vec{r} = 3$

$\int_{C_2} \vec{F} \cdot d\vec{r} = 1.$ what is $\int_{C_3} \vec{F} \cdot d\vec{r}$?

[see slides]

<u>Notation:</u>

- $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\partial_z = \frac{\partial}{\partial z}$

- $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$ "a vector"

  ↳ $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$ for $f : \mathbb{R}^3 \to \mathbb{R}$

  $= \langle f_x, f_y, f_z \rangle$ (gradient).

<u>Definition</u> for a vector field $\vec{F}$ on $B \subset \mathbb{R}^3$, the curl of $\vec{F}$ is

$$\operatorname{curl} \vec{F} = \text{"}\nabla \times \vec{F}\text{"} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} \quad (\vec{F} = \langle P, Q, R \rangle)$$

$$= \langle R_y - Q_z, P_z - R_x, \underbrace{Q_x - P_y}_{\text{Same term in Green's theorem}} \rangle$$

<u>Example:</u> Find $\operatorname{curl} \langle xz, xyz, -y^2 \rangle$

$\operatorname{curl} \langle xz, xyz, -y^2 \rangle = \langle -2y - xy, x - 0, yz - 0 \rangle$

$= \langle -2y - xy, x, 0 \rangle.$

<u>Example:</u> Find $\operatorname{curl} \langle P(x,y), Q(x,y), 0 \rangle$ [see slides]

<u>Example:</u> Find $\operatorname{curl} \langle e^x, z\cos y, \sin y \rangle$

Recall: • if $\vec{F} = \langle P, Q \rangle$ (2d vector field) is conservative,

$Q_x - P_y$ ("2d curl") $= 0$.

• the converse holds if the set D is simply connected.

Theorem: Let f be a function on an open set $D \subset \mathbb{R}^3$ with continuous second order partial derivatives

Then $\text{curl}(\nabla f) = \vec{0}$

proof: $\nabla f = \langle f_x, f_y, f_z \rangle$.

$$\text{curl}(\nabla f) = \langle (f_z)_y - (f_y)_z, (f_x)_z - (f_z)_x, (f_y)_x - (f_x)_y \rangle$$

$$= \langle 0, 0, 0 \rangle \text{ by Clairaut's Theorem.} \quad \square$$

↳ so if $\vec{F}$ is a vector field with $\text{curl}(\vec{F}) \neq 0$, then $\vec{F}$ is not conservative.

The converse is true if $D = \mathbb{R}^3$.

Theorem: Let $\vec{F}$ be a vector field on $\mathbb{R}^3$ with continuous first order partial derivatives.

If $\text{curl}(\vec{F}) = \vec{0}$, then $\vec{F}$ is conservative.

[Not ready to prove yet - need something stronger than Green's Theorem]

1. Is $\vec{F} = \langle xz, xyz, -y^2 \rangle$ conservative?

Is $\vec{F} = \langle e^x, z\cos y, \sin y \rangle$ conservative?

[See slides]

Definition: The divergence of $\vec{F} = \langle P, Q, R \rangle$ is

$$\text{div } \vec{F} = \text{"}\nabla \cdot \vec{F}\text{"} = \langle \partial_x, \partial_y, \partial_z \rangle \cdot \langle P, Q, R \rangle$$

$$= P_x + Q_y + R_z.$$

* It's a function of three variables

Theorem: Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field on an open set $D \subset \mathbb{R}^3$ with continuous second order derivatives.

Then
$$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0.$$

<u>proof:</u> $\mathrm{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$\Rightarrow \mathrm{div}(\mathrm{curl}\,\vec{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$
$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}.$$
$$= 0 \quad \text{by Clairaut's theorem. } \square.$$

<u>Example:</u> Does there exist $\vec{G}$ with $\mathrm{curl}(\vec{G}) = \langle x, y, z \rangle$?

Note: $\mathrm{div}\langle x, y, z \rangle = 1 + 1 + 1 = 3 \neq 0$

and if $\langle x, y, z \rangle = \mathrm{curl}(\vec{G})$, we would have

$$\mathrm{div}\langle x, y, z \rangle = \mathrm{div}(\mathrm{curl}(\vec{G})) = 0$$

Contradiction. $\square$.

<u>The big picture:</u>

$$\begin{tikzcd}
\{\text{constants}\} \arrow[r, squiggly] \arrow[rr, bend right, "0"'] & \{\text{functions}\} \arrow[r, squiggly, "\nabla"] \arrow[rr, bend left, "0"] & \{\text{vector fields}\} \arrow[r, squiggly, "\text{curl}"] \arrow[rr, bend right, "0"'] & \{\text{vector fields}\} \arrow[r, squiggly, "\text{div}"] & \{\text{functions}\}.
\end{tikzcd}$$

- Any two arrows in a row give zero.
- The other combinations of two arrows either
  - don't make sense (e.g. ~~div...~~ $\mathrm{curl}(\underbrace{\mathrm{div}\,\vec{F}}_{\text{function}})$ not defined)
  - or don't necessarily give 0:

$$\mathrm{div}(\nabla f) \quad \text{makes sense}$$
$$\parallel$$
$$\nabla \cdot (\nabla f) = \mathrm{div}\langle f_x, f_y, f_z \rangle$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

This is called the Laplacian of f.

we write $\nabla^2 f = \mathrm{div}(\nabla f)$.