Last time: curl and div

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on $D \subset \mathbb{R}^3$.

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle;$$
$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z.$$

Let $\mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle$. Compute curl$\mathbf{F}$ and find the function curl$\mathbf{F} \cdot \mathbf{k}$, where $\mathbf{k}$ is the vector $\langle 0, 0, 1 \rangle$.

(a) $P_x - Q_y$

(b) $Q_x - P_y$

(c) $P_x + Q_y$

(d) $P_y + Q_x$

(e) I don't know how.

Physical interpretation of curl

Let **F** be a vector field on $D \subset \mathbb{R}^3$, representing the velocity of a fluid flowing through the region D.

For a point $P \in D$, we consider the vector $\text{curl}(\mathbf{F})(P)$.

- The line through P in the direction of $\text{curl}(\mathbf{F})(P)$ is the axis of rotation of a tiny ball at point P.
- The direction of $\text{curl}(\mathbf{F})(P)$ is related to the direction of rotation by the right-hand rule.
- The magnitude $|\text{curl}(\mathbf{F})(P)|$ is proportional to the speed of rotation.

In particular, when $\text{curl}(\mathbf{F}) = 0$, the little ball doesn't rotate at all; we say that **F** is irrotational at P.

Note: the ball can still be moving! It's floating along the current, it's just not spinning as it moves past the point P.

Practice with curl

Let $\mathbf{F}(x, y, z) = \langle y, 0, 0 \rangle$. By imagining a tiny ball placed at different locations in the vector field, decide whether curl($\mathbf{F}$) points up, points down, or is zero.

(a) It always points up.

(b) It always points down.

(c) It's always zero.

(d) It depends what point we look at.

(e) I don't know.

If you're done, calculate curl($\mathbf{F}$) from the definition and see if it matches your prediction.

Physical interpretation of div

- If div**F** is positive, fluid flows out of B, a small ball around the point.
- If div**F** is negative, fluid flows in to B.
- If div**F** is zero, there is no net change: the volume of fluid coming in is equal to the volume of fluid going out. In that case, we say that **F** is **incompressible**.

Practice with div

Let $\mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle$. Imagine a small region around a point. Is fluid leaving the region more quickly than it is entering it?
Use your observation to decide whether div$\mathbf{F}$ is

(a) always positive.

(b) always negative.

(c) always zero.

(d) It depends on the point.

(e) I don't know.

If you're done, calculate div$\mathbf{F}$ from the definition, and see if your prediction is correct.