Last time: oriented surfaces and their boundaries

- Point your head in the direction of the positive unit normal vector **n**.
- Orient ∂S so that S is to your left as you walk along ∂S.

Example: Consider the surface of the unit cube $[0,1] \times [0,1] \times [0,1]$, oriented outwards.

Let S_1 be the bottom and sides of the cube, and let S_2 be the top of the cube, so ∂S_1 and ∂S_2 are oriented curves.

(a) $\partial S_1 = \partial S_2$

(b) $\partial S_1 = -\partial S_2$

(c) Neither is true.

(d) I don't know.

Announcements

- Deadline to request a regrade for midterm 3 is this Thursday.
- Final exam is next Friday. (!) I will organize some kind of review session next Wednesday/Thursday/Friday. Fill out the survey on the course webpage indicating your availability if you're interested.

More on Stokes' Theorem and Curl

Recall:

- We assume we have a vector field $\mathbf{F}$ defined on some open region $D \subset \mathbb{R}^3$, with continuous first order partial derivatives on D.
- S is an oriented surface contained in D. We assume S is "nice":
  - S is piecewise smooth.
  - ∂S consists of one or more simple closed paths.

Theorem (Stokes' Theorem)

$$\iint_S curl\, \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

More on Stokes' Theorem and Curl

Let $\mathbf{F}$ be the velocity field of a fluid flow in $\mathbb{R}^3$. Choose a point P in $\mathbb{R}^3$, and choose any vector unit vector $\mathbf{n}$ at P.

Let D be a small disk with centre P and unit normal $\mathbf{n}$, and place a tiny paddle wheel at P with its axis of rotation in direction $\mathbf{n}$.

The counterclockwise force on the wheel is related to the **circulation** of $\mathbf{F}$ around ∂D:

$$\sim \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

But by Stokes' theorem, this is

$$\iint_D \text{curl}\mathbf{F} \cdot \mathbf{n} \; dA.$$

The counterclockwise force on the wheel is related to the **circulation** of $\mathbf{F}$ around ∂D:

$$\sim \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}\mathbf{F} \cdot \mathbf{n} \; dA.$$

We approximate the function $\text{curl}\mathbf{F} \cdot \mathbf{n}$ over the small disk D by its value at the centre point P.

- The wheel rotates counterclockwise if $\text{curl}\mathbf{F} \cdot \mathbf{n} > 0$ at P.
- It rotates clockwise if $\text{curl}\mathbf{F} \cdot \mathbf{n} < 0$ at P.
- It doesn't rotate at all if $\text{curl}\mathbf{F} \cdot \mathbf{n} = 0$.

The speed of rotation is related to $|\text{curl} \cdot \mathbf{n}|$.

If we want to place a tiny wheel at P oriented so that it will spin as quickly as possible, we should choose the angle/direction $\mathbf{n}$ so that $|\text{curl} \cdot \mathbf{n}|$ is as large as possible.

i.e. we should choose $\mathbf{n}$ to be pointing in the same direction $(\pm)$ as curl.

Analogy

If f is a function, the gradient $\nabla f(P)$ points in the direction we should face if we want to increase as quickly as possible.

If $\mathbf{F}$ is a vector field, the curl $\nabla \times \mathbf{F}(P)$ points in the direction we should stand if we want to be spun around as quickly as possible.

Practice with Stokes' theorem: computing a hard surface integral by changing it into an easy surface integral

Let S be the blob drawn on the board, oriented outward, with boundary edges of the square $[0,1] \times [0,1] \times \{1\}$.
Let $\mathbf{F}$ be as before.
What is $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$?

(a) -1

(b) 0

(c) 1

(d) Not enough information.

(e) I don't know.

Practice with Stokes' theorem

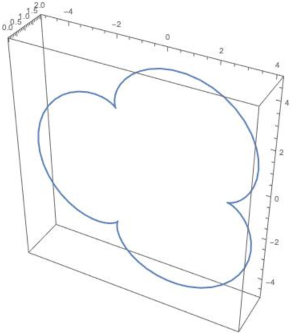
$$\mathbf{F} = \langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, e^{z^2} \rangle.$$

This is defined everywhere except the z-axis, $\{x = y = 0\}$.
Claim:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & e^{z^2} \end{vmatrix} = \mathbf{0}.$$

ST: Converting a hard line integral to an easy surface integral

Let C_1 be the curve parametrized by
$\mathbf{r}_1(\theta) = \langle 4\cos\theta - \cos 4\theta, 1, 4\sin\theta - \sin 4\theta \rangle$, $0 \le \theta \le 2\pi$.



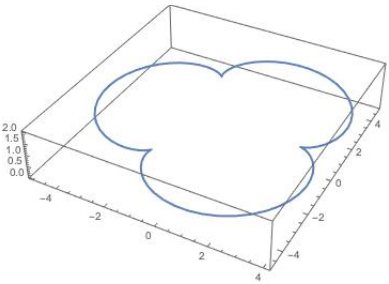
Let S be the surface we get by filling in the curve in the $y = 1$ plane. Observe that S doesn't intersect the z-axis, so $\mathbf{F}$ is defined on all of S.

Orient S so that $\partial S = C_1$. Then Stokes' Theorem says:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = 0.$$

More practice with Stokes' Theorem

Let $\mathbf{F}$ be as before, but now let C_2 be the curve parametrized by $\mathbf{r}_2(\theta) = \langle 4\cos\theta - \cos 4\theta, 4\sin\theta - \sin 4\theta, 1\rangle$, $0 \leq \theta \leq 2\pi$.



Does the previous argument work to show that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$? Why or why not?

(a) No.

(b) Yes.

No: any surface S with boundary C_2 passes through the z-axis, so $\mathbf{F}$ is not defined on all of S.