Universal $\mathcal{D}$-modules, and factorisation structures on Hilbert schemes of points

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To my father, Gerald Cliff,
with admiration, gratitude, and love.

When I grow up, I'm going to be a mathematician,
just like you!

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Abstract

This thesis concerns the study of chiral algebras over schemes of arbitrary dimension n.

In Chapter I, we construct a chiral algebra over each smooth variety X of dimension n. We do this via the Hilbert scheme of points of X, which we use to build a factorisation space over X. Linearising this space produces a factorisation algebra over X, and hence, by Koszul duality, the desired chiral algebra. We begin the chapter with an overview of the theory of factorisation and chiral algebras, before introducing our main constructions. We compute the chiral homology of our factorisation algebra, and show that the $\mathcal{D}$-modules underlying the corresponding chiral algebras form a universal $\mathcal{D}$-module of dimension n.

In Chapter II, we discuss the theory of universal $\mathcal{D}$-modules and $\mathcal{O}$-modules more generally. We show that universal modules are equivalent to sheaves on certain stacks of étale germs of n-dimensional varieties. Furthermore, we identify these stacks with the classifying stacks of groups of automorphisms of the n-dimensional disc, and hence obtain an equivalence between the categories of universal modules and the representation categories of these groups. We also define categories of *convergent* universal modules and study them from the perspectives of the stacks of étale germs and the representation theory of the automorphism groups.

Contents

Introduction

Vertex algebras and vertex operator algebras have been studied and applied fruitfully in a number of areas, ranging from physical disciplines such as conformal field theory and string theory to finite group theory and the geometric Langlands correspondence. Beilinson and Drinfeld [4] reformulated the axioms of a vertex algebra in geometric language in terms of chiral algebras, and showed that the latter are equivalent to factorisation algebras—both geometric objects which take the form of $\mathcal{D}$-modules over a complex curve, equipped with additional structure. The sophisticated machinery of factorisation and chiral algebras elegantly captures the data of vertex algebras in an often more intuitive way.

In the one-dimensional setting, Frenkel and Ben-Zvi [10] explain the relationship between vertex algebras and chiral algebras over curves. To make this relationship precise, we need adjectives on both sides. First, we require our vertex algebras to be *quasi-conformal*, or equipped with a one-dimensional infinitesimal translation. On the other hand, the chiral algebras we obtain are *universal*: they are defined over all smooth families of curves, and are compatible with pullback by étale morphisms of these families. Roughly, the infinitesimal translation allows us to spread the vector space underlying the vertex algebra canonically along any complex curve C. In this way, we obtain a $\mathcal{D}$-module on C which will have the structure of a chiral algebra. The fact that this procedure works for any smooth curve C means that we obtain a universal chiral algebra.

Francis and Gaitsgory [9] showed that Beilinson–Drinfeld's definitions can be extended to higher dimensions. They identify chiral and factorisation algebras as specific Lie algebra and cocommutative coalgebra objects in a certain monoidal category, and then show that the equivalence of chiral and factorisation algebras amounts to a particular case of Koszul duality. We expect Frenkel–Ben-Zvi's result to extend to this setting: that is, a higher-dimensional analogue of a vertex algebra should correspond to a higher-dimensional universal chiral algebra. However, although some attempt has been made to define such higher vertex algebras (see for example Borcherds [5]), these definitions have not been compared with the algebras of Francis–Gaitsgory.

In this thesis, we approach the study of higher-dimensional chiral algebras from two perspectives. In the first chapter, we use the Hilbert scheme of points to construct concrete examples of chiral algebras of arbitrary dimension. On the other hand, in the second chapter, we study the theory of universal $\mathcal{D}$-modules, which is a structure weaker than that of universal chiral algebras. We develop a thorough understanding of the relationship between the category of universal $\mathcal{D}$-modules and the category of representations of the group of automorphisms of the formal disc; the functors that we provide between these two categories agree with the functors exhibiting the equivalence between universal chiral algebras and quasi-conformal vertex algebras. This understanding should shed light on the extension of the definition of a vertex algebra to higher dimensions.

The Hilbert scheme of points is a natural starting point when attempting to find examples of chiral algebras. One reason for this is the well-known result of Nakajima [32] and Grojnowski [21], that the cohomology of the Hilbert scheme of points on a smooth projective surface X has a canonical structure of a vertex algebra, the *Heisenberg vertex algebra* modelled on the integral cohomology of the surface X. By the above, this means that for the fixed surface X and for any curve C, there is a chiral algebra on C coming from the cohomology of the Hilbert scheme of X.

However, we are interested in constructing chiral algebras over higher-dimensional varieties, not just curves. In fact, in this thesis we take a different approach to Nakajima and Grojnowski, and use the Hilbert scheme of points to define a chiral algebra over X, for X a smooth variety of any dimension, not just for a surface. We do this via the category of factorisation algebras: an important advantage of working in the setting of factorisation algebras rather than that of chiral algebras or vertex algebras is that the definition of a factorisation algebra extends in a straightforward way to non-linear settings, leading to the notions of factorisation spaces and factorisation categories. In particular, once one has constructed a factorisation space living over the variety X one can linearise it in several natural ways to obtain factorisation algebras over X. Our strategy is to exploit the geometry of the Hilbert scheme to construct a factorisation space over X; we linearise this space to produce a factorisation algebra over X, and hence a chiral algebra. We study these objects in Chapter I.

In Chapter II we study categories of universal $\mathcal{D}$-modules and universal $\mathcal{O}$-modules of dimension n. Motivated by a claim of Beilinson and Drinfeld [4] we relate these categories to categories of representations of groups of automorphisms of the n-dimensional formal disc. A universal $\mathcal{D}$-module is a rule assigning to each smooth n-dimensional variety a $\mathcal{D}$-module in a way compatible with pullback by étale morphisms between the varieties. A key observation is that all of this data is equivalent to the data of a single sheaf on a stack parametrising étale germs of n-dimensional varieties.

A second critical observation is that, using a generalisation of Artin's approximation theorem to the relative setting, we can relate this stack to the classifying stack of the group G of automorphisms of the n-dimensional formal disc. More precisely, our stack is equivalent to the classifying stack of the group $G^{\text{ét}}$ of automorphisms *of étale type*, which is a dense subgroup of G. It follows that a universal $\mathcal{D}$-module is equivalent to a representation of $G^{\text{ét}}$; furthermore, any representation of G restricts to give a representation of $G^{\text{ét}}$ and hence a universal $\mathcal{D}$-module.

A natural question to ask is whether we can characterise those universal $\mathcal{D}$-modules which come from representations of G rather than representations of just the subgroup $G^{\text{ét}}$. We give two characterisations of these universal $\mathcal{D}$-modules, which we call *convergent universal $\mathcal{D}$-modules*. However, at the time of writing, we do not know whether all universal $\mathcal{D}$-modules are actually convergent. This question is equivalent to the question of whether all representations of $G^{\text{ét}}$ extend uniquely to representations of G. We are able to show that if an extension exists, it is unique. Furthermore, any finite-dimensional representation of $G^{\text{ét}}$ extends to a representation of G, and in fact this is true of any representation of $G^{\text{ét}}$ satisfying a weaker finiteness condition which we call being *$K^{\text{ét}}$-locally-finite*. All representations of G satisfy this condition, and so the question is reduced to the existence of representations of $G^{\text{ét}}$ which are not $K^{\text{ét}}$-locally-finite. If such a representation exists, it will give rise to a universal $\mathcal{D}$-module which is not convergent; on the other hand, it seems that such an object would be unlikely to arise in ordinary applications of the theory and would be unpleasant to work with. In other words, we are only interested in working with $\mathcal{D}$-modules satisfying the properties implied by convergence, and we suggest that the category of convergent universal $\mathcal{D}$-modules is the correct category in which to work.

An intended application of this theory is the following. As in Francis–Gaitsgory [9], we know that chiral algebras over a variety X are certain Lie algebra objects in the category of $\mathcal{D}$-modules on the Ran space of X, which is equipped with a monoidal structure called the *chiral* monoidal structure. In the universal setting, we can introduce Ran versions of the stack of étale germs and the automorphism groups

G and $G^{\text{ét}}$, and define chiral monoidal structures on the associated categories of quasi-coherent sheaves. We should then interpret the monoidal structure in terms of the classifying stack in order to obtain higher-dimensional analogues of vertex algebras as Lie algebra objects in the representation category with this chiral monoidal structure. This will be explored in future work.

Chapter I

Factorisation structures on the Hilbert scheme of points

In this chapter, our goal is to use the Hilbert scheme of points of a variety X of arbitrary dimension n to define a factorisation space and factorisation algebra over X. We will begin in section 1 by establishing the necessary definitions and results regarding factorisation spaces and factorisation algebras. In section 2 we define and and study our factorisation space and the resulting factorisation algebra.

0.1 Conventions

We fix an algebraically closed field k of characteristic zero. We work in the categories Sch of schemes over k and $\mathrm{Sch}_{\mathrm{f.t.}}$ of schemes of finite type over k. Let us emphasise that these are classical rather than DG schemes.

The "spaces" we consider will all be in particular *prestacks, locally of finite type*, that is, functors

$$\mathrm{Sch}_{\mathrm{f.t.}}^{\mathrm{op}} \to \infty\text{-Grpd}.$$

We denote this category by $\mathrm{PreStk}_{\mathrm{l.f.t.}}$, and view $\mathrm{Sch}_{\mathrm{f.t.}}$ as a full subcategory under the Yoneda embedding. See Appendix A.1 for some basic definitions and properties.

We work with the DG-categories of $\mathcal{D}$-modules on our prestacks. See Appendix A.2.3 for an overview of the theory, or [18] and III.4 of [20] for a more complete account; the key idea to keep in mind is that for any prestack $\mathcal{Y}$ which is locally of finite type, $\mathcal{D}(\mathcal{Y})$ is defined as

$$\lim_{(S\to\mathcal{Y})\in\left((\mathrm{Sch}_{\mathrm{f.t.}}^{\mathrm{Aff}})_{/\mathcal{Y}}\right)^{\mathrm{op}}} \mathcal{D}(S),$$

where the limit is taken with respect to the functor sending $f : S \to S'$ to $f^! : \mathcal{D}(S') \to \mathcal{D}(S)$.

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The factorisation algebra $\mathcal{A}_{\operatorname{Ran} X}$ was defined via a different construction by Vladimir Kotov [26] in the case that X is a surface.

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1 Preliminaries on factorisation

In this section, we introduce the theory of factorisation spaces and factorisation algebras. We begin in 1.1 by defining the Ran space of a separated scheme X, and its variant, the Ran space of marked points. We show, following Beilinson–Drinfeld [4], that if X is connected, both of these spaces are homologically contractible. In 1.2 we define factorisation spaces and algebras, and show how we can produce examples of factorisation algebras from factorisation spaces by linearisation. In 1.3 we define the notion of a chiral algebra, and focus in particular on commutative chiral algebras. In 1.4 we discuss the equivalence between the categories of chiral algebras and factorisation algebras.

Finally, we conclude by introducing the notion of a vertex algebra in 1.5, and by discussing the equivalence between quasi-conformal vertex algebras and chiral algebras over curves. The material of 1.5 is not necessary for the second section of the chapter, but it is the motivation for the study of universal $\mathcal{D}$-modules in Chapter II.

1.1 The Ran space

Fix a separated scheme X of finite type over k. In this section we introduce the Ran space of the surface X, as well as its variant, the Ran space of marked points, and prove that, as long as X is connected, they are homologically contractible.

Definition 1.1.1. Let fSet denote the category of finite non-empty sets I and surjections $\alpha : I \twoheadrightarrow J$.

Key construction 1.1.2. Given any finite non-empty set I, let X^I denote the I-fold fibre product of X over $\operatorname{Spec} k$. For any surjective map $\alpha : I \twoheadrightarrow J$ there is a natural map $X^J \to X^I$ sending the point $(x_j)_{j\in J} \in X^J$ to the I-tuple $(y_i)_{i\in I}$ in X^I which has coordinates $y_i = x_{\alpha(i)}$ for each $i \in I$. We denote this map by $\Delta(\alpha) : X^J \to X^I$. It is easy to see that we obtain a functor

$$\begin{aligned} X^{\mathrm{fSet}} : \mathrm{fSet}^{\mathrm{op}} &\to \mathrm{Sch} \\ I &\mapsto X^I \\ (\alpha : I \twoheadrightarrow J) &\mapsto (\Delta(\alpha) : X^J \to X^I), \end{aligned}$$

and that the maps $\Delta(\alpha)$ are closed embeddings of schemes. It follows that the colimit (in the category $\mathrm{PreStk}_{\mathrm{l.f.t.}}$)

$$\operatorname*{colim}_{I\in \mathrm{fSet}^{\mathrm{op}}} X^I$$

is a pseudo-indscheme.

Definition 1.1.3. We denote the above pseudo-indscheme by $\operatorname{Ran} X$, and call it the *Ran space* of X.

We are interested in the category of $\mathcal{D}$-modules on $\operatorname{Ran} X$,

$$\mathcal{D}(\operatorname{Ran} X) \simeq \lim_{I\in \mathrm{fSet}} \mathcal{D}^!(X^I) \simeq \operatorname*{colim}_{I\in \mathrm{fSet}^{\mathrm{op}}} \mathcal{D}_!(X^I).$$

(See Appendix A2.3.2 for a discussion of $\mathcal{D}$-modules on prestacks.) For any $I \in \mathrm{fSet}$, we denote by $((\Delta^I)_!, (\Delta^I)^!)$ the adjoint pair of functors $\mathcal{D}(X^I) \rightleftarrows \mathcal{D}(\operatorname{Ran} X)$ given by the tautological functors from the descriptions of $\mathcal{D}(\operatorname{Ran} X)$ as a colimit and limit, respectively.

It follows from the discussion in Appendix A2.3.3 that the de Rham cohomology of $\operatorname{Ran} X$

$$\mathrm{H}_\bullet(\operatorname{Ran} X) := (p_{\operatorname{Ran} X})_! \circ (p_{\operatorname{Ran} X})^!(k) \in \mathrm{Vect} = \mathcal{D}(\mathrm{pt})$$

is defined, and that there is a canonical map

$$\mathrm{Tr}_{\operatorname{Ran} X} : \mathrm{H}_\bullet(\operatorname{Ran} X) \to \mathrm{H}_\bullet(\mathrm{pt}) = k.$$

The following important fact is a theorem of Beilinson and Drinfeld when X is a curve (see the proposition in 4.3.3, [4]), and is proved by the same methods for X of higher dimension in section 6 of [16].

Proposition 1.1.4. *For X a connected separated scheme of finite type over k, the trace map*

$$\mathrm{Tr}_{\mathrm{Ran}\,X} : \mathrm{H}_\bullet(\mathrm{Ran}\,X) \to k$$

is an isomorphism of DG-vector spaces over k. In other words, $\mathrm{Ran}\,X$ *is homologically contractible.*

Beilinson and Drinfeld's proof of Proposition 1.1.4 is easily generalised to the following setting:

Lemma 1.1.5. *Let $\mathcal{Y}$ be prestack expressible as a colimit of schemes*

$$\mathcal{Y} \simeq \operatorname*{colim}_{I \in \mathcal{S}} Z(I),$$

and satisfying the following properties:

1. *For each $I \in \mathcal{S}$, $Z(I)$ is either empty or connected, and for at least some $I \in \mathcal{S}$, $Z(I)$ is non-empty.*
2. *There is a map $m : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ which is associative and commutative, and such that its composition with the diagonal map is the identity:*

$$\mathcal{Y} \xrightarrow{\Delta_{\mathcal{Y}}} \mathcal{Y} \times \mathcal{Y} \xrightarrow{m} \mathcal{Y}, \qquad m \circ \Delta_{\mathcal{Y}} = \mathrm{id}_{\mathcal{Y}}.$$

Then $\mathcal{Y}$ is homologically contractible.

In the case of the Ran space, the map m corresponds to taking the union of two finite sets:

$$\begin{aligned} \mathrm{union} : \mathrm{Ran}\,X \times \mathrm{Ran}\,X &\to \mathrm{Ran}\,X \\ (S,T) &\mapsto S \cup T. \end{aligned}$$

It is easy to see that this satisfies the conditions of (2).

We will now consider another example of a prestack which is homologically contractible by this same lemma. We have the following variant of the Ran space, introduced by Gaitsgory:

Definition 1.1.6 (2.5.2, [16])**.** Fix a finite set A, and consider the category fSet_A whose objects are finite sets I equipped with any map $a_I : A \to I$, and whose morphisms from $(A \to I)$ to $(A \to J)$ are commutative triangles:

$$\begin{array}{ccc} & A & \\ {}^{a_I}\swarrow & & \searrow^{a_J} \\ I & \xrightarrow[\alpha]{} & J. \end{array}$$

Now fix a k-point $y^A \in X^A$ and for any $(a_I : A \to I) \in \mathrm{fSet}_A$ let $X_A^{a_I} := \left\{y^A\right\} \times_{X^A} X^I$.

Notice that for any $\alpha : I \twoheadrightarrow J$ giving a map $(a_I : A \to I) \to (a_J : A \to J)$ in fSet_A, the closed embedding $\Delta(\alpha) : X^J \hookrightarrow X^I$ induces a closed embedding $X_A^{a_J} \hookrightarrow X_A^{a_I}$. Therefore we can consider the colimit

$$\operatorname{Ran} X_A := \operatorname*{colim}_{(I,a_I)\in \mathrm{fSet}_A^{\mathrm{op}}} X_A^{\alpha}.$$

It is a pseudo-indscheme, and we will call it the *Ran space of X with marked points.*

Intuitively, $\operatorname{Ran} X_A$ parametrises all finite non-empty subsets of X containing y^a for each $a \in A$.

It follows from Proposition 1.1.4 above and Corollary 2.5.10 of [16] that $\operatorname{Ran} X_A$ is homologically contractible; however, it is easy enough to see it directly from Lemma 1.1.5. We explain this now.

Proposition 1.1.7. *If X is connected, then the trace map*

$$\mathrm{Tr}_{\operatorname{Ran} X_A} : \mathrm{H}_\bullet(\operatorname{Ran} X_A) \to k$$

is an equivalence.

Proof. Let us first check condition (1) of Lemma 1.1.5. Given any $(A \xrightarrow{a_I} I) \in \mathrm{fSet}_A$, we can decompose I as $I = I' \sqcup I''$, where $I' := \mathrm{im}(a_I)$. Then we have

$$X_A^I := \{y^A\} \times_{X^A} X^I \simeq \left(\{y^A\} \times_{X^A} X^{I'}\right) \times X^{I''}.$$

Since the map $X^{I'} \to X^A$ is a closed embedding, the scheme $\{y^A\} \times_{X^A} X^{I'}$ is either empty or a single point (and in the case that a_I is injective, it is a point), and so the scheme X_A^I is either empty or isomorphic to $X^{I''}$, which is a connected scheme.

So to complete the proof, it remains to define a suitable map

$$m : \operatorname{Ran} X_A \times \operatorname{Ran} X_A \to \operatorname{Ran} X_A.$$

Intuitively, the map we will use is just the map sending two finite subsets of X containing $\left\{y^A\right\}$ to their union; let us now define this rigorously.

Recall from the universal property of colimits that to define a map $m : \mathrm{Ran}\, X_A \times \mathrm{Ran}\, X_A \to \mathrm{Ran}\, X_A$ it suffices to give compatible maps $m_{I,J} : X_A^I \times X_A^J \to \mathrm{Ran}\, X_A$ for any pair of objects $(A \xrightarrow{a_I} I)$ and $(A \xrightarrow{a_J} J)$ of fSet_A.

Given such a pair, we define $A \xrightarrow{a_{I \sqcup_A J}} I \sqcup_A J$ to be the pushout $I \sqcup_A J$ of I and J along the maps from A, together with the natural map from A to $I \sqcup_A J$. (Note that this is an object of fSet_A, but it is not the coproduct of $(A \xrightarrow{a_I} I)$ and $(A \xrightarrow{a_J} J)$.) Then $X^{I \sqcup_A J}$ is isomorphic to the fibre product $X^I \times_{X^A} X^J$, and so we can see that

$$X_A^{I \sqcup_A J} \simeq \{y^A, y^A\} \times_{X^A \times X^A} (X^I \times X^J).$$

We need to define a map $m_{I,J}$ from $X_A^I \times X_A^J$ into $\mathrm{Ran}\, X_A$, and since $X_A^I \times X_A^J \simeq \{y^A, y^A\} \times_{X^A \times X^A} (X^I \times X^J)$, we can simply take $m_{I,J}$ to be the natural map

$$X_A^{I \sqcup_A J} \to \operatorname*{colim}_{K \in \mathrm{fSet}_A^{\mathrm{op}}} X_A^K = \mathrm{Ran}\, X_A.$$

This gives a compatible family of maps into $\mathrm{Ran}\, X_A$, and hence defines a map $m : \mathrm{Ran}\, X_A \times \mathrm{Ran}\, X_A \to \mathrm{Ran}\, X_A$. The associativity and commutativity of this map m follow from the corresponding properties of the coproduct of finite sets. It is also easy to see that m is left inverse to the diagonal map $\mathrm{Ran}\, X_A \to \mathrm{Ran}\, X_A \times \mathrm{Ran}\, X_A$, and so m satisfies all the required properties. Hence, by Lemma 1.1.5, the proof is complete. □

1.2 Factorisation spaces and factorisation algebras

Let us again fix X a separated scheme of finite type over k.

Definition 1.2.1. A *factorisation space* over X consists of the following data:

1. A prestack $\mathcal{Y}_{\mathrm{Ran}\, X}$ expressible as a colimit over $\mathrm{fSet}^{\mathrm{op}}$:

$$\mathcal{Y}_{\mathrm{Ran}\, X} \simeq \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{op}}} Y_{X^I},$$

   where each Y_{X^I} is an indscheme and for any $\alpha : I \twoheadrightarrow J$, the map $Y(\alpha) : Y_{X^J} \to Y_{X^I}$ is ind-proper.

2. For each finite set I a map

$$f^I : Y_{X^I} \to X^I,$$

   natural in fSet, i.e. a natural transformation $f : Y_{X^{\mathrm{fSet}}} \Longrightarrow X^{\mathrm{fSet}}$.

3. *Ran's condition:* For any surjection $\alpha : I \twoheadrightarrow J$, there is a natural map $\nu_\alpha : Y_{X^J} \to X^J \times_{X^I} Y_{X^I}$ given by

$$\begin{array}{ccccc} Y_{X^J} & & \xrightarrow{Y(\alpha)} & & \\ & \overset{\nu_\alpha}{\searrow} & & & \\ & & X^J \times_{X^I} Y_{X^I} & \longrightarrow & Y_{X^I} \\ & {\scriptstyle f^J} & \downarrow & & \downarrow {\scriptstyle f^I} \\ & & X^J & \xrightarrow[\Delta(\alpha)]{} & X^I. \end{array}$$

We require that ν_α be an equivalence of indschemes, and that ν be associative in the obvious sense.

4. *Factorisation:* Given $\alpha : I \twoheadrightarrow J$ as above, we obtain a partition of I as $\bigsqcup_{j\in J} I_j$, where $I_j = \{i \in I \mid \alpha(i) = j\}$, and we consider the following open subscheme of X^I:

$$U = U(\alpha) := \left\{(x_i)_{i\in I} \in X^I \mid x_{i_1} \neq x_{i_2} \text{ unless } \alpha(i_1) = \alpha(i_2)\right\}.$$

We let $j = j(\alpha)$ denote the open embedding $U \hookrightarrow X^I \cong \prod_{j\in J} X^{I_j}$, and consider the following two pullback diagrams:

$$\begin{array}{ccc} U \times_{X^I} Y_{X^I} & \xrightarrow{j'} & Y_{X^I} \\ {\scriptstyle (f^I)'} \downarrow & & \downarrow {\scriptstyle f^I} \\ U & \underset{j}{\hookrightarrow} & X^I \end{array} \qquad \begin{array}{ccc} U \times_{X^I} \left(\prod_{j\in J} Y_{X^{I_j}}\right) & \xrightarrow{j''} & \prod_{j\in J} Y_{X^{I_j}} \\ {\scriptstyle \left(\prod_{j\in J} f^{I_j}\right)''} \downarrow & & \downarrow {\scriptstyle \prod_{j\in J} f^{I_j}} \\ U & \underset{j}{\hookrightarrow} & \prod_{j\in J} X^{I_j}. \end{array}$$

We require an equivalence

$$d_\alpha : U \times_{X^I} \left(\prod_{j\in J} Y_{X^{I_j}}\right) \xrightarrow{\sim} U \times_{X^I} Y_{X^I}$$

of indschemes over U. Moreover, these equivalences d_α should be associative and compatible with the other structure maps ν_α.

Let us describe explicitly a compatibility condition between the maps ν and d. Suppose that we have three finite sets I, J, K with surjections:

$$I \overset{\alpha}{\twoheadrightarrow} J \overset{\beta}{\twoheadrightarrow} K.$$

Write α_k for the restriction of α to the pre-image I_k of k under the composition $\beta \circ \alpha$, so that we have

$$\alpha_k : I_k = \bigsqcup_{j \in J_k} I_j \twoheadrightarrow J_k.$$

Notice that $U(\beta) = U(\beta \circ \alpha) \times_{X^I} X^J$. Then we require that the following diagram be commutative.

$$\begin{array}{ccc}
U(\beta) \times_{X^J} \left(\prod_{k \in K} Y_{X^{J_k}} \right) & \xrightarrow[d_\beta]{\sim} & U(\beta) \times_{X^J} Y_{X^J} \\
\downarrow \wr \, (\nu_{\alpha_k})_{k \in K} & & \downarrow \wr \, \nu_\alpha \\
U(\beta) \times_{X^J} \left(\prod_{k \in K} \left(X^{J_k} \times_{X^{I_k}} Y_{X^{I_k}} \right) \right) & & U(\beta) \times_{X^J} \left(X^J \times_{X^I} Y_{X^I} \right) \\
\downarrow \wr & & \downarrow \wr \\
X^J \times_{X^I} \left(U(\beta \circ \alpha) \times_{X^I} \left(\prod_{k \in K} Y_{X^{I_k}} \right) \right) & \xrightarrow[d_{\beta \circ \alpha}]{\sim} & X^J \times_{X^I} \left(U(\beta \circ \alpha) \times_{X^I} Y_{X^I} \right).
\end{array}$$

Remark 1.2.2. Note that, by the universal property of colimits, the maps f^I in (2) give rise to a map $f : \mathcal{Y}_{\operatorname{Ran} X} \to \operatorname{Ran} X$.

An important example of a factorisation space is due to Beilinson and Drinfeld [3]:

Example 1.2.3. Let G be an algebraic group and let $X = C$ be a curve. For $I \in \mathrm{fSet}$ define Gr_{G,C^I} to be the prestack that sends a test-scheme S to the groupoid

$$\left\{ (c^I, \mathcal{P}, \alpha) \;\middle|\; \begin{array}{l} c^I : S \to C^I \\ \mathcal{P} \to S \times C \text{ a principal } G\text{-bundle} \\ \alpha : S \times C \setminus \{c^I\} \to \mathcal{P} \text{ a trivialisation} \end{array} \right\}.$$

Let

$$\mathrm{Gr}_{G,\operatorname{Ran} C} := \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{op}}} \mathrm{Gr}_{G,C^I}.$$

This is a factorisation space, known as the *Beilinson-Drinfeld Grassmannian*. (Note that it is not very interesting for X with $\dim X \geq 2$.)

Convention 1.2.4. Whenever we have $x^I : S \to X^I$, or equivalently a collection of maps $x_i : S \to X$ indexed by I, we write $\{x^I\}$ to mean the closed subset of $S \times X$ given by the union of the graphs of the functions x_i.

We are interested in factorisation spaces because they are non-linear analogues of factorisation algebras. As we will see in 1.2.8, we can use factorisation spaces to construct examples of factorisation algebras.

Definition 1.2.5. A *factorisation algebra* over X is a $\mathcal{D}$-module $\mathcal{A}_{\operatorname{Ran} X}$ over $\operatorname{Ran} X$ together with the data of *factorisation isomorphisms*. Recall that, as a $\mathcal{D}$-module over $\operatorname{Ran} X$, $\mathcal{A}_{\operatorname{Ran} X}$ is given by a collection $(\mathcal{A}^I \in \mathcal{D}(X^I))_{I \in \mathrm{fSet}}$ together with compatible isomorphisms

$$\nu_\alpha : \Delta(\alpha)^!(\mathcal{A}^I) \simeq \mathcal{A}^J$$

for any $\alpha : I \twoheadrightarrow J$ (c.f. Ran's condition for factorisation spaces). Then we demand factorisation isomorphisms

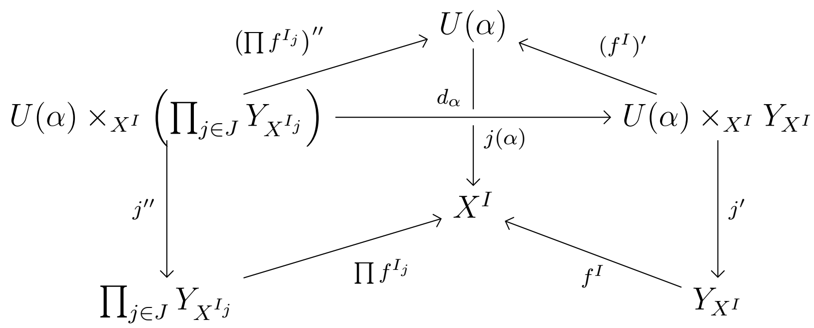
$$c_\alpha : j(\alpha)^* \mathcal{A}^I \xrightarrow{\sim} j(\alpha)^* (\boxtimes_{j \in J} \mathcal{A}^{I_j})$$

for any $\alpha : I \twoheadrightarrow J$. These isomorphisms must be compatible with composition of α and with the structure isomorphisms ν_α (c.f. the factorisation condition for factorisation spaces).

Remark 1.2.6. In fact, instead of beginning with a $\mathcal{D}$-module on $\operatorname{Ran} X$, we could start with a quasi-coherent sheaf. Assume in addition that there are no non-zero local sections supported at the diagonal $X \subset \operatorname{Ran} X$. Then it follows from the factorisation structure that this sheaf has a canonical structure of crystal on $\operatorname{Ran} X$, i.e. that it is a $\mathcal{D}$-module. (See 3.4.7 [4] for the construction of the canonical connection when working with a curve X and in the abelian setting.)

Convention 1.2.7. From now on, if we write $U \subset X^2$ without specifying the map α, we mean $U = U(\alpha)$ for $\alpha = \mathrm{id} : \{1,2\} \to \{1,2\}$. On the other hand, if we write Δ we always mean $\Delta = \Delta(\beta) : X \to X^2$ for $\beta : \{1,2\} \to \{\mathrm{pt}\}$. More generally, for any finite set I, $j(I) : U(I) \to X^I$ comes from to the map $\mathrm{id}_I : I \to I$, while $\Delta(I) : X \to X^I$ comes from the projection $I \to \{\mathrm{pt}\}$.

Key construction 1.2.8. Let $\mathcal{Y}_{\operatorname{Ran} X} = \operatorname{colim} Y_{X^I}$ be a factorisation space, and assume that the structure maps $f^I : Y_{X^I} \to X^I$ are ind-proper. Assume also that we have a $\mathcal{D}$-module $\mathscr{F}$ on $\mathcal{Y}_{\operatorname{Ran} X}$, i.e. a compatible family of $\mathcal{D}$-modules $(\mathscr{F}^I \in \mathcal{D}(Y_{X^I}))$, and suppose that $\mathscr{F}$ is compatible with the factorisation structure on $\mathcal{Y}_{\operatorname{Ran} X}$. Let us explain what we mean by this. Recall that for each $\alpha : I \twoheadrightarrow J$ we have a commutative diagram as follows:



With this notation, $\mathscr{F}$ is *compatible* with the factorisation structure on $\mathcal{Y}_{\operatorname{Ran} X}$ if we have isomorphisms

$$\widetilde{c}_\alpha : (j')^*(\mathscr{F}^I) \xrightarrow{\sim} (d_\alpha)_*(j'')^* \left(\boxtimes_{j\in J} \mathscr{F}^{I_j} \right)$$

which are themselves compatible with respect to composition of α and the isomorphisms $Y(\alpha)^!(\mathscr{F}^I) \simeq \mathscr{F}^J$.

In such a setting, we obtain a factorisation algebra $\mathcal{A}_{\operatorname{Ran} X}$ by setting $\mathcal{A}^I := f^I_*(\mathscr{F}^I)$.

Indeed, we have isomorphisms

$$\Delta(\alpha)^! f^I_*(\mathscr{F}^I) \simeq f^J_* Y(\alpha)^!(\mathscr{F}^I) \xrightarrow{\sim} f^J_*(\mathscr{F}^J),$$

and

$$\begin{aligned} j(\alpha)^* f^I_*(\mathscr{F}^I) &\simeq (f^I)'_*(j')^*(\mathscr{F}^I) \\ &\xrightarrow{\sim} (f^I)'_*(d_\alpha)_*(j'')^* \left(\boxtimes_{j\in J} \mathscr{F}^{I_j} \right) \\ &\simeq \left(\prod f^{I_j} \right)''_* (j'')^* \left(\boxtimes_{j\in J} \mathscr{F}^{I_j} \right) \\ &\simeq j(\alpha)^* \left(\prod f^{I_j} \right)_* \left(\boxtimes_{j\in J} \mathscr{F}^{I_j} \right) \\ &\simeq j(\alpha)^* \left(\boxtimes_{j\in J} f^{I_j}_*(\mathscr{F}^{I_j}) \right). \end{aligned}$$

Example 1.2.9. Let $\mathcal{Y}_{\operatorname{Ran} X} = \operatorname{colim}_{I\in \mathrm{fSet}^{\mathrm{op}}} Y_{X^I}$ be a factorisation space such that each f^I is ind-proper, and consider the $\mathcal{D}$-module $\omega_{\mathcal{Y}_{\operatorname{Ran} X}}$ on $\mathcal{Y}_{\operatorname{Ran} X}$. It is not hard to check that this is compatible with the factorisation structure, and so

$$f_*(\omega_{\mathcal{Y}_{\operatorname{Ran} X}}) = \left\{ f^I_*(\omega_{Y_{X^I}}) \right\}_{I\in \mathrm{fSet}}$$

is a factorisation algebra.

1.3 Chiral algebras

In this section, we introduce the notion of a chiral algebra, following [9]. We work in the category $\mathcal{D}(\operatorname{Ran} X)$ of $\mathcal{D}$-modules on the Ran space, and begin by defining two natural monoidal structures on $\mathcal{D}(\operatorname{Ran} X)$.

To describe a monoidal structure, we need to give compatible maps

$$\mathcal{D}(\operatorname{Ran} X)^{\otimes J} \to \mathcal{D}(\operatorname{Ran} X)$$

for all $J \in \mathrm{fSet}$. Using the presentation of $\mathcal{D}(\operatorname{Ran} X)$ as a colimit, it suffices to define compatible maps

$$\bigotimes_{j \in J} \mathcal{D}(X^{I_j}) \to \mathcal{D}(\operatorname{Ran} X)$$

for any collection $\{I_j\}$ of finite sets parametrised by another finite set J. Such a family of maps can be defined from the following data: for each $J \in \mathrm{fSet}$, a functor

$$m_J : \mathrm{fSet}^{\mathrm{op}} \times \cdots \times \mathrm{fSet}^{\mathrm{op}} \to \mathrm{fSet}^{\mathrm{op}},$$

together with a natural transformation between the resulting two functors

$$\left(F_1 : (I_j)_{j \in J} \mapsto \otimes_{j \in J} \mathcal{D}(X^{I_j})\right) \Rightarrow \left(F_2 : (I_j)_{j \in J} \mapsto \mathcal{D}(X^{m_J(I_j)})\right).$$

This data should be compatible with the operation $\sqcup$ on fSet.

Definition 1.3.1. The $\star$ *symmetric monoidal structure* on $\mathcal{D}(\operatorname{Ran} X)$ is the symmetric monoidal structure coming from the map $m_J : (I_j) \mapsto I := \sqcup_{j \in J} I_j$, together with the natural transformation $\tau : F_1 \Rightarrow F_2$ defined for each $(I_j)_{j \in J}$ by the external tensor product of D-modules:

$$\begin{aligned} \tau_{(I_j)} : \bigotimes_{j \in J} \mathcal{D}(X^{I_j}) &\to \mathcal{D}(X^I) \\ (M^{I_j} \in \mathcal{D}(X^{I_j})) &\mapsto \boxtimes_{j \in J} (M^{I_j}). \end{aligned}$$

We denote this monoidal product by $\otimes^\star$.

Given $M_1, M_2 \in \mathcal{D}(\operatorname{Ran} X)$, the fibre of $M_1 \otimes^\star M_2$ at a point $(S \subset X) \in \operatorname{Ran} X$ is

$$(M_1 \otimes^\star M_2)_{[S]} \simeq \bigoplus_{\{(S_1, S_2) | S = S_1 \cup S_2, S_i \neq \emptyset\}} (M_1)_{S_1} \otimes (M_2)_{S_2}.$$

Definition 1.3.2. The *chiral monoidal structure* on $\mathcal{D}(\operatorname{Ran} X)$ is the symmetric monoidal structure coming from the maps $m_J : \mathrm{fSet} \times \ldots \mathrm{fSet} \to \mathrm{fSet}$ as in definition 1.3.1 and the natural transformations $\tau' : F_1 \Rightarrow F_2$ given for $\{I_j\}_{j\in J}$ by

$$\tau'_{(I_j)} : \bigotimes_{j\in J} \mathcal{D}(X^{I_j}) \to \mathcal{D}(X^I)$$
$$\big(M_j \in \mathcal{D}(X^{I_j})\big) \mapsto j(\alpha)_* \circ j(\alpha)^*(\boxtimes_{j\in J} M^{I_j}).$$

(Here we denote by α the obvious surjection $I = \sqcup_{j\in J} I_j \twoheadrightarrow J$.) We denote this monoidal operation by $\otimes^{\mathrm{ch}}$.

Given $M_1, M_2 \in \mathcal{D}(\operatorname{Ran} X)$, the fibre of $M_1 \otimes^{\mathrm{ch}} M_2$ at a point $(S \subset X) \in \operatorname{Ran} X$ is

$$(M_1 \otimes^{\mathrm{ch}} M_2)_S \simeq \bigoplus_{\{(S_1,S_2)|S=S_1\sqcup S_2, S_i\neq\emptyset\}} (M_1)_{S_1} \otimes (M_2)_{S_2},$$

where now the direct sum is over decompositions of S by disjoint sets S_i.

Suppose we have a surjection $\alpha : I \twoheadrightarrow J$ in fSet, with $I_j := \alpha^{-1}(j)$, and for each $j \in J$ let $M_j \in \mathcal{D}(\operatorname{Ran} X)$. By definition of $\otimes^{\mathrm{ch}}$, we have

$$(\Delta^I)_! \left(j(\alpha)_* \circ j(\alpha)^*(\boxtimes_J (\Delta^{I_j})^! M_j)\right) \simeq \otimes_J^{\mathrm{ch}} \left((\Delta^{I_j})_! \circ (\Delta^{I_j})^! M_j\right)$$

and hence we obtain a map

$$j(\alpha)_* \circ j(\alpha)^*(\boxtimes_J (\Delta^{I_j})^! M_j) \to (\Delta^I)^! \left(\otimes_J^{\mathrm{ch}} M_j\right).$$

Lemma 1.3.3 (Lemma 2.3.4, [9])**.** *For I, J and $M_j \in \mathcal{D}(\operatorname{Ran} X)$ as above, the resulting map*

$$\bigoplus_{\alpha: I\twoheadrightarrow J} \left(j(\alpha)_* \circ j(\alpha)^*(\boxtimes_J (\Delta^{I_j})^! M_j)\right) \to (\Delta^I)^! \left(\otimes_J^{ch} M_j\right).$$

is a homotopy equivalence (where the direct sum is taken over all surjections $\alpha : I \to J$, i.e. all partitions of I into $|J|$ non-empty subsets).

This result allows us to understand the structure of a J-fold chiral tensor product by breaking it down into exterior tensor products.

Given the two symmetric monoidal structures on $\mathcal{D}(\operatorname{Ran} X)$, we can consider algebra and coalgebra objects in $\mathcal{D}(\operatorname{Ran} X)$. In particular, we will use the Lie operad and the co-commutative cooperad. (See e.g. Loday and Vallette [27] or Markl, Shnider and Stasheff [29] for an introduction to operads, and Chapter 2 of Lurie [28] for the theory of ∞-operads.)

Definition 1.3.4. A *chiral algebra* $\mathcal{C}$ over X is a Lie algebra object in the symmetric monoidal category $(\mathcal{D}(\operatorname{Ran} X), \otimes^{\mathrm{ch}})$ which is supported on $X \subset \operatorname{Ran} X$.

That is, we have $\mathcal{B} \in \mathcal{D}(X)$ such that for any finite set I, $\mathcal{C}^I \simeq \Delta(I)_! \mathcal{B} \in \mathcal{D}(X^I)$. Moreover, for any surjection $\alpha : I \twoheadrightarrow J$ we have a *chiral operation*

$$\mu_{\mathcal{B}} : j(\alpha)_* j(\alpha)^* \left(\boxtimes_{j \in J} \Delta(I_j)_! \mathcal{B} \right) \to \Delta(I)_! \mathcal{B}$$

corresponding to the chiral Lie bracket

$$\mu_{\mathcal{C}} : \bigotimes_J^{\mathrm{ch}} \mathcal{C} \to \mathcal{C}.$$

The chiral operations satisfy relations coming from the anti-symmetry, Leibniz rule, and Jacobi identity of the chiral Lie bracket.

We denote the category of chiral algebras by $\text{Lie-alg}^{\mathrm{ch}}(X)$. We will often refer to the underlying $\mathcal{D}_X$-module $\mathcal{B}$ as the chiral algebra rather than $\mathcal{C}$.

Example 1.3.5. Let $\mathcal{B} = \omega_X \in \mathcal{D}(X)$. Then we have a canonical exact sequence

$$0 \to \omega_X \boxtimes \omega_X \to j_* j^* (\omega_X \boxtimes \omega_X) \xrightarrow{\mu_{\omega_X}} \Delta_!(\omega_X) \to 0,$$

and the map μ_{ω_X} gives a chiral bracket on ω_X.

Definition 1.3.6. Similarly, a *Lie $\star$ algebra* on X is a Lie algebra object in the symmetric monoidal category $(\mathcal{D}(\operatorname{Ran} X), \otimes^{\star})$ which is supported on X.

If $\mathcal{S}$ is a Lie $\star$ algebra on X, we will denote the Lie bracket by $[\ ]_{\mathcal{S}}$. We denote the category of Lie $\star$ algebras on X by $\text{Lie-alg}^{\star}(X)$.

For any $\mathcal{C} \in \mathcal{D}(\operatorname{Ran} X)$, there are natural maps

$$\bigotimes_J^{\star} \mathcal{C} \to \bigotimes_J^{\mathrm{ch}} \mathcal{C},$$

meaning that every chiral algebra is in particular a Lie $\star$ algebra. This gives rise to a forgetful functor

$$F : \text{Lie-alg}^{\mathrm{ch}}(X) \to \text{Lie-alg}^{\star}(X)$$

Definition 1.3.7. If $\mathcal{C}$ is a chiral algebra such that the Lie bracket $[\ ]_{F\mathcal{C}}$ on the underlying Lie $\star$ algebra vanishes, then we say that $\mathcal{C}$ is a *commutative* chiral algebra.

Suppose $M \in \mathcal{D}(X)$ is a commutative algebra object with respect to the $\otimes^!$ tensor structure on the category $\mathcal{D}(X)$ of right $\mathcal{D}$-modules. That is, we have a morphism

$$m : M \otimes^! M \to M$$

in $\mathcal{D}(X)$ which is associative and commutative. We want to use m to define a chiral algebra structure on M, i.e. a map $j_*j^*(M \boxtimes M) \to \Delta_! M$ (and analogues for more general $\alpha : I \twoheadrightarrow J$). By definition of the tensor structure on $\mathcal{D}(X) = \mathrm{IndCoh}\,(X_{\mathrm{dR}})$, we have

$$M \otimes^! M = \Delta^!(M \boxtimes M).$$

Hence we define the chiral operation μ_M to be the composition

$$j_*j^*(M \boxtimes M) \to \Delta_!\Delta^!(M \boxtimes M) \simeq \Delta_!(M \otimes^! M) \xrightarrow{\Delta_!(m)} \Delta_! M.$$

Here the first map is the canonical map from the Cousin complex for $M \boxtimes M$:

$$M \boxtimes M \to j_*j^*(M \boxtimes M) \to \Delta_!\Delta^!(M \boxtimes M).$$

It has the following alternate description: recall that if $M_1, M_2 \in \mathcal{D}(X)$, then $M_1 \otimes^! M_2 \simeq M_1^\ell \otimes_{\mathcal{O}_X} M_2$, where $M_1^\ell := M_1 \otimes \omega_X^{-1}$ is the left $\mathcal{D}$-module associated to M_1. (See for example Appendix A.2.3.15.) Note also that we have

$$\begin{aligned} j_*j^*(M \boxtimes M) &\simeq j_*j^*\left((M^\ell \boxtimes M^\ell) \otimes (\omega_X \boxtimes \omega_X)\right) \\ &\simeq j_*\left(j^*(M^\ell \boxtimes M^\ell) \otimes j^*(\omega_X \boxtimes \omega_X)\right) \\ &\simeq (M^\ell \boxtimes M^\ell) \otimes j_*j^*(\omega_X \boxtimes \omega_X), \end{aligned}$$

using the projection formula (see Lemma A.2.2.11 (1)).

Now we apply the canonical map

$$\mu_{\omega_X} : j_*j^*(\omega_X \boxtimes \omega_X) \to \Delta_!(\omega_X) \tag{I.1}$$

to obtain a map into $(M^\ell \boxtimes M^\ell) \otimes \Delta_!(\omega_X)$. Again using the projection formula, we have

$$\begin{aligned} (M^\ell \boxtimes M^\ell) \otimes \Delta_!(\omega_X) &\simeq \Delta_!\left(\Delta^*(M^\ell \boxtimes M^\ell) \otimes \omega_X\right) \\ &\simeq \Delta_!\left(M^\ell \otimes M^\ell \otimes \omega_X\right) \\ &\simeq \Delta_!\left(M \otimes^! M\right), \end{aligned}$$

as required.

The skew-symmetry and Jacobi identity satisfied by the map (I.1) together with the symmetry and associativity of m ensure that the composition

$$\mu_M : j_*j^*(M \boxtimes M) \to \Delta_! M$$

is indeed a chiral bracket. Moreover, this gives a commutative chiral algebra, because of the exactness of the canonical sequence

$$0 \to \omega_X \boxtimes \omega_X \to j_*j^* (\omega_X \boxtimes \omega_X) \to \Delta_!(\omega_X) \to 0.$$

In fact, a chiral algebra $\mathcal{B}$ is commutative precisely if it comes from a commutative $\otimes^!$ algebra in this way. That is, we have the following equivalence:

Proposition 1.3.8. *The above construction gives an equivalence of categories*

$$\text{Com-alg}(\mathcal{D}(X), \otimes^!) \xrightarrow{\sim} \text{Lie-alg}^{ch}(X)_{com}.$$

1.4 Koszul duality

In this section, we will see that the category of chiral algebras is equivalent to the category of factorisation algebras.

Let us begin by considering again the symmetric monoidal category

$$(\mathcal{D}(\operatorname{Ran} X), \otimes^{\mathrm{ch}}).$$

A cocommutative coalgebra object in this category is a $\mathcal{D}$-module $\mathcal{M} = (\mathcal{M}^I)$ together with comultiplication maps

$$\delta^I_{\mathcal{M}} : \mathcal{M} \to \bigotimes_I^{\mathrm{ch}} \mathcal{M}$$

which are coassociative and cocommutative.

Given such an object $(\mathcal{M}, \delta_{\mathcal{M}})$ and a surjection $\alpha : I \twoheadrightarrow J$, we use Lemma 1.3.3 and the $(j(\alpha)^*, j(\alpha)_*)$ adjunction to obtain morphisms

$$j(\alpha)^*(\boxtimes_J \mathcal{M}^{I_j}) \to j(\alpha)^*(\mathcal{M}^I);$$

if these morphisms are all isomorphisms, then $\mathcal{M}$ is a factorisation algebra on X. Conversely, given a factorisation algebra, we can apply the adjuction isomorphisms to the factorisation isomorphisms, and we obtain the structure maps of a cocommutative coalgebra in $(\mathcal{D}(\operatorname{Ran} X), \otimes^{\mathrm{ch}})$. That is, we have the following result:

Lemma 1.4.1. *Let $\mathcal{M}$ be a $\mathcal{D}$-module on $\operatorname{Ran} X$. The structure of a factorisation algebra on $\mathcal{M}$ is equivalent to the structure of a chiral cocommutative coalgebra structure on $\mathcal{M}$ such that the induced maps*

$$j(\alpha)^*(\boxtimes_J \mathcal{M}^{I_j}) \to j(\alpha)^*(\mathcal{M}^I);$$

are all isomorphisms.

As described by Francis and Gaitsgory in [9], there is a Koszul duality between the Lie operad and the cocommutative coalgebra cooperad, which is particularly well-behaved in our setting of $(\mathcal{D}(\operatorname{Ran} X), \otimes^{\mathrm{ch}})$:

Theorem 1.4.2 (Theorem 5.1.1, [9])**.** *The adjoint functors*

$$C^{ch} : \text{Lie-alg}^{ch}(\operatorname{Ran} X) \rightleftarrows \text{Com-coalg}^{ch}(\operatorname{Ran} X) : \operatorname{Prim}^{ch}[-1]$$

provided by Koszul duality are in fact mutually inverse equivalences.

Furthermore, we have the full subcategories

$$\text{Lie-alg}^{\mathrm{ch}}(X) \subset \text{Lie-alg}^{\mathrm{ch}}(\operatorname{Ran} X) \text{ and } \quad \operatorname{Fact}(X) \subset \text{Com-coalg}^{\mathrm{ch}}(\operatorname{Ran} X),$$

and the Koszul duality functors also respect these:

Theorem 1.4.3 (Theorem 5.2.1, [9])**.** *The above equivalences C^{ch} and $\operatorname{Prim}^{ch}[-1]$ restrict to give mutually inverse equivalences*

$$C^{ch} : \text{Lie-alg}^{ch}(X) \rightleftarrows \operatorname{Fact}(X) : \operatorname{Prim}^{ch}[-1].$$

In other words, a cocommutative coalgebra $(\mathcal{M}, \delta_{\mathcal{M}})$ is a factorisation algebra if and only if the corresponding Lie algebra object is supported on the diagonal $X \subset \operatorname{Ran} X$.

In sections 3.4.11–12 of [4] there is an explicit description of $C^{\mathrm{ch}}(\mathcal{B})$ as a left $\mathcal{D}$-module on $\operatorname{Ran} X$, in the case that $X = C$ is a curve and $\mathcal{B} \in \mathcal{D}(C)$ is concentrated in degree zero. Namely, we define the *Chevalley-Cousin complex* $\operatorname{Chev}(\mathcal{B})$ as follows: for $I \in \mathrm{fSet}$ we set $\operatorname{Chev}(\mathcal{B})_{C^I}$ to be the complex given in degree n by

$$\operatorname{Chev}(\mathcal{B})^n_{C^I} := \bigoplus_{\alpha: I \twoheadrightarrow T} \Delta(\alpha)_! j(T)_* j(T)^* \left(\mathcal{B}[1]\right)^{\boxtimes T}$$

where we take the sum over all surjections α from I to a set T of cardinality $|T| = -n$. We define the differential d by specifying its components

$$d_{T,T'} : \Delta(\alpha)_! j(T)_* j(T)^* \left(\mathcal{B}[1]\right)^{\boxtimes T} \to \Delta(\alpha')_! j(T')_* j(T')^* \left(\mathcal{B}[1]\right)^{\boxtimes T'}$$

for two sets T, T' with $T = T'' \sqcup \{t_1, t_2\}$, $T = T'' \sqcup \{t_0\}$ and surjections α, α' making the following diagram commute:

$$\begin{array}{ccc} & I & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\alpha'} \\ T & \xrightarrow[t_1, t_2 \mapsto t_0]{} & T'. \end{array}$$

Then $d_{T,T'}$ comes from the chiral bracket $\mu_{\mathcal{B}} : j_* j^*(\mathcal{B}_{t_1} \boxtimes \mathcal{B}_{t_2}) \to \Delta_! \mathcal{B}_{t_0}$. The Jacobi identity ensures that the resulting map d satisfies $d^2 = 0$.

This complex $\mathrm{Chev}(\mathcal{B})$ satisfies Ran's condition and factorisation, and we have

$$C^{\mathrm{ch}}(\mathcal{B})^I = \left(H^{-|I|} \, \mathrm{Chev}(\mathcal{B})_{C^I}\right)^{\ell}.$$

In particular, for $I = \mathrm{pt}$ we have that $\mathrm{Chev}(\mathcal{B})_{C^{\mathrm{pt}}} = \mathcal{B}[1]$, and so

$$C^{\mathrm{ch}}(\mathcal{B})^{\mathrm{pt}} = \mathcal{B}^{\ell}.$$

A natural question to ask is the following: what property of a factorisation algebra will ensure that the corresponding chiral algebra is commutative? We have the following characterisation of these so-called *commutative factorisation algebras*:

Lemma 1.4.4 (Proposition 3.4.20, [4])**.** *A factorisation algebra $\mathcal{A}_{\mathrm{Ran}\,X} = (\mathcal{A}^I)$ is commutative if and only if for every $\alpha : I \twoheadrightarrow J$ the isomorphism*

$$j(\alpha)^* \left(\boxtimes_{j \in J} \mathcal{A}^{I_j} \right) \xrightarrow{\sim} j(\alpha)^* \left(\mathcal{A}^I \right)$$

extends to a morphism

$$\boxtimes_{j \in J} \mathcal{A}^{I_j} \to \mathcal{A}^I.$$

1.5 Vertex algebras

In this section, we give the definition of a vertex algebra, and discuss the relationship between vertex algebras and chiral algebras. See for example [10] or [12] for more details.

Definition 1.5.1 (Definition 1.3.1, [10])**.** A *vertex algebra*

$$V = (V, |0\rangle, T, Y(\cdot, z))$$

consists of the following data:

- The *space of states*: a graded complex vector space
$$V = \bigoplus_{i \geq 0} V_i.$$
- The *vacuum vector*: $|0\rangle \in V_0$.
- The *translation operator*: $T : V \to V$ a linear map of degree 1.
- The *vertex operators*: $Y(\cdot, z) : V \to \operatorname{End} V[\![z, z^{-1}]\!]$ a linear map such that if we have $A \in V_i$ and write
$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1},$$
then the $(-n-1)$th coefficient $A_{(n)} \in \operatorname{End} V$ is of degree $-n + i - 1$.

These data are subject to the following conditions:

- *The vacuum axiom:*
$$Y(|0\rangle, z) = \mathrm{id}_V .$$
Furthermore, for any $A \in V$, $A_{(n)}|0\rangle = 0$ for $n \geq 0$, and $A_{(-1)}|0\rangle = A$.
- *The translation axiom:*
$$[T, Y(A, z)] = \partial_z Y(A, z) \qquad \forall A \in V,$$
$$T|0\rangle = 0.$$
- *The locality axiom:* For any $A, B \in V$ there exists $N \in \mathbb{N}$ such that
$$(z - w)^N [Y(A, z), Y(B, w)] = 0 \in \operatorname{End} V[\![z^{\pm 1}, w^{\pm 1}]\!].$$

It is also possible to modify the definition slightly to work in the super-setting, as in remark 1.3.2 of [10].

Definition 1.5.2 (Definition 2.5.8, [10])**.** A vertex algebra
$$V = (V, |0\rangle, T, Y(\cdot, z))$$
is said to be *conformal* of central charge $c \in \mathbb{C}$ if we have a *conformal vector* $\omega \in V_2$ with the following property. We introduce the notation
$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{n \in \mathbb{Z}} L^V_{n-1} z^{-n-1};$$

we require that the operators $L_n^V \in \operatorname{End} V[\![z, z^{-1}]\!]$ satisfy the *Virasoro relations*:

$$\begin{aligned}
[L_n^V, L_m^V] &= (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,-m}c, \\
L_{-1}^V &= T, \\
L_0^V|_{V_i} &= i\,\mathrm{id}_{V_i}\,.
\end{aligned}$$

Definition 1.5.3 (Definition 6.3.4, [10])**.** A vertex algebra is called *quasi-conformal* if it carries an action of the Lie algebra $\operatorname{Der}\mathcal{O} = \mathbb{C}[\![z]\!]\partial_z$ satisfying the following properties: if we let L_m denote the linear operator on V defined by the action of $-z^{m+1}\partial_z$, $m \geq -1$, then:

1. $L_{-1} = T$.

2. L_0 acts semi-simply with integral eigenvalues.

3. Given a vector field

$$v(z) = \sum_{n \geq -1} v_n z^{n+1}\partial_z \in \operatorname{Der}\mathcal{O}$$

defining the linear operator

$$\mathbf{v} = -\sum_{n \geq -1} v_n L_n,$$

we require that

$$[\mathbf{v}, Y(A,w)] = -\sum_{m \geq -1} \frac{1}{(m+1)!}\left(\partial_w^{m+1} v(w)\right) Y(L_m A, w).$$

4. $\operatorname{Der}_+\mathcal{O} = z^2\mathbb{C}[\![z]\!]\partial_z$ acts locally nilpotently.

Example 1.5.4. A conformal vertex algebra is in particular quasi-conformal. If $(V, |0\rangle, T, Y(\cdot, z))$ is a vertex algebra with conformal vector $\omega \in V_2$, then we can define an action of $\operatorname{Der}\mathcal{O}$ on V by letting $z^{m+1}\partial_z$ act on V by $-L_m^V$ (i.e. the coefficient of z^{-m-2} in $Y(\omega, z)$). This makes V into a quasi-conformal vertex algebra.

We are interested in quasi-conformal vertex algebras because of their close relation to chiral algebras over curves. This was made precise by Frenkel–Ben-Zvi, but we follow here the exposition of section 6.2 of [25]:

Key construction 1.5.5. Let $(V, |0\rangle, T, Y(\cdot, z))$ be a quasi-conformal vertex algebra, and let $D = \operatorname{Spf} k[\![t]\!]$ be the formal one-dimensional disc. Then we can construct a chiral algebra $\mathcal{V}$ over any curve C as follows.

We give $V[\![t]\!]$ the unique vertex algebra structure such that:

1. The translation operator $\widetilde{T}$ is the sum of the translation operator T of V and the differential ∂_t.
2. The vacuum vector is equal to the vacuum vector $|0\rangle$ of V.
3. $\widetilde{Y}(at^n, z) = (t+z)^n Y(a, z)$.

Let $\mathcal{B}_V$ be the sheaf on D associated to the $k[\![t]\!]$-module $V[\![t]\!] \cdot dt$. Then $\mathcal{B}_V$ is a chiral algebra, with chiral operation induced by

$$\begin{aligned} V \otimes V[\![t_1, t_2]\!][(t_1 - t_2)^{-1}] &\to V \otimes V[\![t_1, t_2]\!][(t_1 - t_2)^{-1}]/V \otimes V[\![t_1, t_2]\!] \\ f(t_1, t_2) A \otimes B &\mapsto f(t_1, t_2) Y(A, t_1 - t_2)(B) \mod V \otimes V[\![t_1, t_2]\!] \end{aligned}$$

for $f(t_1, t_2) \in k(t_1 - t_2), A, B \in V$.

So far we have not used the quasi-conformal structure on V, but we will use it now: the action of $\operatorname{Der} \mathcal{O}$ can be exponentiated to give an action of $K = \operatorname{Aut}(\mathcal{O})$, so that V is a $(\operatorname{Der} \mathcal{O}, \operatorname{Aut}(\mathcal{O}))$-module, or equivalently an object of $\operatorname{Rep}(G)$, where $G = \underline{\operatorname{Aut}}(\mathcal{O})$ (see Chapter II, section 2 for the precise definitions of K and G). It follows from the results of Chapter II that V corresponds to a universal $\mathcal{D}$-module $\mathcal{V}$ of dimension one, such that for any c in C the restriction of $\mathcal{V}(C)$ to the disc D_c around c is isomorphic to $\mathcal{B}_V$. (The isomorphism depends only on a choice of formal coordinate at c.)

In other words, we obtain a *universal chiral algebra* of dimension one.

Conversely, any universal chiral algebra of dimension one is in particular a universal D-module, and the corresponding $\underline{\operatorname{Aut}}(\mathcal{O})$-module has a uniquely determined structure of quasi-conformal vertex algebra.

2 The main constructions

With the basic theory established, we now restrict our attention to X a smooth variety over k. We will use the Hilbert scheme Hilb_X to construct a factorisation space $\mathscr{H}ilb_{\operatorname{Ran} X}$ over the variety X; by linearising we produce a factorisation algebra $\mathcal{A}_{\operatorname{Ran} X}$.

In 2.1 we define the prestack $\mathscr{H}ilb_{\operatorname{Ran} X}$, and prove that it is in fact an ind-proper factorisation space over X. In 2.2 we also introduce a slight variant, $\widetilde{\mathscr{H}ilb}_{\operatorname{Ran} X}$, and compare it to our original factorisation space. In 2.3 we consider the natural map

$$\mathscr{H}ilb_{\operatorname{Ran} X} \to \operatorname{Hilb}_X,$$

and show that its fibres are contractible. This allows us to compute the chiral homology of the factorisation algebra $\mathcal{A}_{\operatorname{Ran} X}$ defined by linearising $\mathscr{H}ilb_{\operatorname{Ran} X}$. We do this in 2.4.

Finally, in 2.5 we allow the base variety X to vary, and consider the corresponding factorisation spaces $\mathscr{H}ilb_{\operatorname{Ran} X}$, and especially the indscheme $\mathscr{H}ilb_X$ living over a single copy of the variety X. We show that the factorisation space can be defined over families $X \to S$ of smooth varieties. We also show that the assignment

$$X/S \to \mathscr{H}ilb_{X/S}$$

is compatible with pullback by étale maps between smooth families. It follows that the assignment

$$X/S \to \mathcal{A}_{X/S} \in \mathcal{D}(X/S)$$

defined by linearising this space is also compatible with pullback by étale maps. We will see in the next chapter that this means that $\{\mathcal{A}_{X/S}\}$ gives a universal $\mathcal{D}$-module.

2.1 The factorisation space

We begin by recalling some preliminaries about the Hilbert scheme of points, before defining the factorisation space.

Recall the following definition:

Definition 2.1.1. For $n \in \mathbb{Z}_{\geq 0}$, the *Hilbert scheme of n points in X* is the scheme Hilb_X^n representing the functor

$$\begin{aligned} &\operatorname{Sch}^{\mathrm{op}} \to \operatorname{Set} \\ &S \mapsto \left\{ \xi \subset S \times X \;\middle|\; \begin{array}{c} \xi \text{ is flat over } S \text{ with zero-dimensional support} \\ \text{on fibres over } S \text{ of length } n \end{array} \right\}. \end{aligned}$$

See for example [33]. It is a theorem of Grothendieck [22] that this functor is indeed representable for any smooth projective scheme X. It is also representable for any affine scheme $\operatorname{Spec} A$ (see for example [23]), and for any smooth variety. It follows from the valuative criterion of properness that whenever it exists, the Hilbert scheme is in fact proper.

Note that $\mathrm{Hilb}_X^0 \simeq \mathrm{pt}$ and $\mathrm{Hilb}_X^1 \simeq X$.

We let Hilb_X be the disjoint union of Hilb_X^n for $n \geq 0$. The scheme Hilb_X is a scheme of infinite type, which we prefer to think of as an indscheme. Indeed, Hilb_X is the colimit over $N \in \mathbb{Z}_{\geq 0}$ of the schemes

$$\bigsqcup_{n=0}^{N} \mathrm{Hilb}_X^n,$$

which are each of finite type.

Notice that the data of a closed subscheme ξ of $S \times X$ is equivalent to the data of a quasi-coherent sheaf $\mathscr{F} \in \mathrm{QCoh}\left(\mathcal{O}_{S\times X}\right)$ together with a surjection $\phi : \mathcal{O}_{S\times X} \twoheadrightarrow \mathscr{F}$ (flat over S). Setting $\mathscr{E}_\xi := \ker \phi$, we obtain a torsion-free sheaf with trivial first Chern class and second Chern class equal to n. In fact this gives an equivalent description of Hilb_X^n as the moduli space of rank one torsion-free sheaves on X with trivial first Chern class and $\mathrm{ch}_2 = n$. It will sometimes be useful to keep this alternative description in mind.

The Hilbert scheme is closely related to the symmetric product $\mathrm{Sym}_X^n := X^n/S_n$. Indeed, we have the *Hilbert-Chow morphism*

$$\begin{aligned} \pi : \mathrm{Hilb}_X^n &\to S^n(X) \\ \xi &\mapsto \sum_{x\in X} \mathrm{length}(\xi_x)[x]. \end{aligned}$$

If we restrict to the open loci of Hilb_X^n and Sym_X^n where all n points are distinct, then π becomes an isomorphism. When X is a smooth surface over k, Fogarty [8] showed that Hilb_X^n is particularly well-behaved:

Theorem 2.1.2. *If X is a smooth surface, then Hilb_X^n is non-singular of dimension $2n$, and the Hilbert-Chow morphism $\pi : \mathrm{Hilb}_X^n \to \mathrm{Sym}_X^n$ is a resolution of singularities.*

With these definitions in mind, we can introduce our main construction:

Definition 2.1.3. Given $I \in \mathrm{fSet}$, let $\mathscr{H}ilb_{X^I}$ be the prestack which sends a test scheme S to the set

$$\left\{ (\xi, x^I) \;\middle|\; \begin{array}{l} x^I : S \to X^I; \\ \xi \in \mathrm{Hilb}_X(S); \\ \mathrm{Supp}(\xi) \subset_\star \{x^I\} \end{array} \right\}.$$

Recall from example 1.2.3 that $\{x^I\}$ is the subset given by the union of the graphs of the functions $x^i : S \to X^I \xrightarrow{\mathrm{pr}_i} X$. The condition on the support of ξ is not to be interpreted simply set-theoretically, but instead in the following way: $\xi : S \to \mathrm{Hilb}_X^n$ induces a map $S \to \mathrm{Sym}_X^n$ via composition with the Hilbert-Chow morphism. In this way we obtain a family of n (unordered) maps $\xi_j : S \to X$. We require that for each $j = 1, \ldots, n$, there exists some $i \in I$ such that $\xi_j = x^i$.

Remark 2.1.4. We can define a factorisation space with the set-theoretic containment condition instead, but it will not be quite as well-behaved, as we will see in Definition 2.2.1 and Lemma 2.2.3 below.

We can write $\mathscr{H}ilb_{X^I}$ as the disjoint union of subfunctors $\mathscr{H}ilb_{X^I}^n$, defined in the obvious way.

Lemma 2.1.5. *For any $I \in fSet$ and any $n \geq 0$, $\mathscr{H}ilb_{X^I}^n$ is representable by a closed subscheme of $X^I \times \mathrm{Hilb}_X^n$; in particular it is proper over X^I.*

Proof. We will identify $\mathscr{H}ilb_{X^I}^n$ as the pullback of a closed subscheme of $X^I \times \mathrm{Sym}_X^n$ along the map $\mathrm{id}_{X^I} \times \pi$ induced by the Hilbert-Chow morphism.

Consider the *incidence scheme*, the closed reduced subscheme of $X \times X^n$ given by

$$\Gamma_{X^n} := \{(x, (x_1, \ldots, x_n)) |\ x = x_i \text{ for some } i\} \subset X \times X^n.$$

For any $i \in I$, we let $\Gamma_{X^n}^i$ be its pullback along the ith projection map $X^I \times X^n \to X \times X^n$. This is closed in $X^I \times X^n$, and hence so is the finite union $\Gamma_{X^n}^I$ of the $\Gamma_{X^n}^i$ over all $i \in I$. Now, we let $\widetilde{\Gamma}_{X^n}^I$ be the image of $\Gamma_{X^n}^I$ in the quotient $X \times \mathrm{Sym}_X^n$; this is also closed by definition of the quotient topology, and we give it the reduced scheme structure.

We claim that $\mathscr{H}ilb_{X^I}^n$ is the pullback

$$\widetilde{\Gamma}_{X^n}^I \times_{X^I \times \mathrm{Sym}_X^n} (X^I \times \mathrm{Hilb}_X^n).$$

Indeed, under the Hilbert-Chow morphism, a point (x^I, ξ) of $\mathscr{H}ilb_{X^I}^n(S)$ gives rise to an unordered collection of n maps ξ_j from $S \to X$. The resulting map $S \to X^I \times \mathrm{Sym}_X^n$ factors through $\widetilde{\Gamma}_{X^n}^I$ exactly when for each $j = 1, \ldots, n$ there is an element $i \in I$ such that $\xi_j = x^i$, or equivalently such that the graphs of ξ_j and x^i are *scheme-theoretically* equal in $S \times X$. In turn, this corresponds to the statement that the support of ξ is contained in $\{x^I\}$, as required.

It is clear that the natural map from $\mathscr{H}ilb_{X^I}^n$ to X^I given by $(x^I, \xi) \mapsto x^I$ is compatible with the projection from $X^I \times \mathrm{Hilb}_X^n$, which is proper. Hence it is a composition of a closed embedding with a proper map, and is again proper, as claimed. □

The following is immediate:

Corollary 2.1.6. *For any $I \in fSet$, $\mathscr{H}ilb_{X^I}$ is an ind-closed sub-indscheme of $X^I \times \mathrm{Hilb}_X$, ind-proper over X^I.*

Proposition 2.1.7. *The assignment*

$$I \mapsto \mathscr{H}ilb_{X^I}$$

extends to a functor

$$fSet^{op} \to IndSch,$$

and defines a factorisation space $\mathscr{H}ilb_{\mathrm{Ran}\,X} := \mathrm{colim}_{I \in fSet^{op}} \mathscr{H}ilb_{X^I}$ over X.

Proof. This is a matter of routine checking: we will proceed through the defining axioms of a factorisation space, and show that each one is satisfied.

Step 1 Given a map $\alpha : I \twoheadrightarrow J \in \mathrm{fSet}$, we need to define $Y(\alpha) : \mathscr{H}ilb_{X^J} \to \mathscr{H}ilb_{X^I} \in \mathrm{IndSch}$ and prove that it is ind-proper. On a connected test scheme S, an element of $\mathscr{H}ilb_{X^J}$ is a pair $(x^J, \xi) \in X^J(S) \times \mathrm{Hilb}^n_X(S)$ (for some $n \in \mathbb{Z}^{\geq 0}$) such that $\mathrm{Supp}(\xi) \subset \{x^J\}$ in $S \times X$. We define $Y(\alpha)_S(x^J, \xi)$ to be $(\Delta(\alpha) \circ x^J, \xi)$; it is immediate that this lies in $\mathscr{H}ilb_{X^I}(S)$, and that this assignment is natural in S, giving a map of prestacks $Y(\alpha) : \mathscr{H}ilb_{X^J} \to \mathscr{H}ilb_{X^I}$.

To show that the resulting map is ind-proper, we must show that for any scheme S lying over $\mathscr{H}ilb_{X^I}$, the pullback $S \times_{\mathscr{H}ilb_{X^I}} \mathscr{H}ilb_{X^J}$ (which is automatically an indscheme over $\mathscr{H}ilb_{X^J}$) is ind-proper over S. In fact, we will show that $S \times_{\mathscr{H}ilb_{X^I}} \mathscr{H}ilb_{X^J} \to S$ is a closed embedding of schemes.

It is enough to check for connected schemes S, but in that case a morphism $S \to \mathscr{H}ilb_{X^I}$ corresponds to a pair (x^I, ξ), where $x^I : S \to X^I$, and $\xi \in \mathrm{Hilb}^n_X(S)$ for some $n \geq 0$. Then a morphism from a connected test scheme T to $S \times_{\mathscr{H}ilb_{X^I}} \mathscr{H}ilb_{X^J}$ corresponds to a morphism $\beta : T \to S$ such that $x^I \circ \beta = \Delta(\alpha) \circ y^J$ for some $y^J : T \to X^J$. That is, $S \times_{\mathscr{H}ilb_{X^I}} \mathscr{H}ilb_{X^J}$ can be identified with the pullback $S \times_{X^I} X^J$, and the projection $S \times_{\mathscr{H}ilb_{X^I}} \mathscr{H}ilb_{X^J}$ is just the closed embedding into S.

Step 2 We need to define a natural transformation $f : \mathscr{H}ilb_{X^{\mathrm{fSet}}} \Longrightarrow X^{\mathrm{fSet}}$. Given $I \in \mathrm{fSet}$, $f^I : \mathscr{H}ilb_{X^I} \to X^I$ is just the obvious forgetful functor, sending a pair (x^I, ξ) to x^I. It is easy to see that this is natural in I.

Step 3 Given $\alpha : I \twoheadrightarrow J$, we wish to show that $X^J \times_{X^I} \mathscr{H}ilb_{X^I} \simeq \mathscr{H}ilb_{X^J}$. Again, this is easy to see at the level of connected test schemes: the set $(X^J \times_{X^I} \mathscr{H}ilb_{X^I})(S)$ is equal to

$$\left\{ \left(x^J, (x^I, \xi)\right) \;\middle|\; \begin{array}{l} x^J : S \to X^J; \\ (x^I, \xi) \in \mathscr{H}ilb^n_{X^I}(S), n \in \mathbb{Z}^{\geq 0}; \\ x^I = \Delta(\alpha) \circ x^J \end{array} \right\}.$$

This is naturally in bijection with the following set:

$$\left\{ (x^J, \xi) \;\middle|\; \begin{array}{l} x^J : S \to X^J; \\ \xi : S \to \mathrm{Hilb}^n_X(S), n \in \mathbb{Z}^{\geq 0}; \\ (\Delta(\alpha) \circ x^J, \xi) \in \mathscr{H}ilb^n_{X^I}(S) \end{array} \right\},$$

which can in turn be rewritten as

$$\left\{ (x^J, \xi) \in \mathscr{H}ilb^n_{X^J}(S) \mid n \in \mathbb{Z}^{\geq 0} \right\} = \mathscr{H}ilb_{X^J}(S),$$

using the fact that $\left\{x^J\right\} = \left\{\Delta(\alpha) \circ x^J\right\}$ because α is surjective.

Step 4 Finally, given $\alpha : I \twoheadrightarrow J$, we wish to show there is a natural equivalence of indschemes

$$j^* \left(\mathscr{H}ilb_{X^I}\right) \simeq j^* \left(\prod_{j \in J} \mathscr{H}ilb_{X^{I_j}} \right).$$

Given a connected test scheme S and a map $x^I : S \to X^I$, the image of x^I lies in $U \subset X^I$ precisely when x^I can be written as a product of maps $x^{I_j} : S \to X^{I_j}$ such that the sets $\left\{x^{I_j}\right\} \subset S \times X$ are pairwise disjoint as j varies.

Then for $\xi \in \mathrm{Hilb}^n_X(S)$, $n \in \mathbb{Z}^{\geq 0}$, the condition $\mathrm{Supp}(\xi) \subset_\star \left\{x^I\right\}$ is satisfied if and only if ξ can be written as a disjoint union of closed subschemes $\xi_j \subset S \times X$ such that for each j, $\mathrm{Supp}(\xi_j) \subset_\star \left\{x^{I_j}\right\}$. In that case, each $(x^{I_j}, \mathscr{H}ilb_{X^{I_j}}) \in \mathscr{H}ilb^{n_j}_{X^{I_j}}(S)$, where n_j are some non-negative integers such that $\sum_{j \in J} n_j = n$.

This gives a natural assignment

$$j^*(\mathscr{H}ilb_{X^I})(S) \to j^* \left(\prod_{j \in J} \mathscr{H}ilb_{X^{I_j}} \right)(S)$$

$$(x^I, \xi) \mapsto (x^{I_j}, \xi_j)_{j \in J}$$

which defines a morphism of indschemes

$$j^*(\mathscr{H}ilb_{X^I}) \to j^*\left(\prod_{j\in J}\mathscr{H}ilb_{X^{I_j}}\right).$$

It follows from the above discussion that this morphism is an equivalence. It is clear from the definitions that the structure morphisms are compatible with composition and with each other.

□

2.2 A variation on the factorisation space

As indicated in Remark 2.1.4 above, we can consider a slightly different factorisation space by letting the condition on the support of schemes be strictly set-theoretic. We introduce this space now.

Definition 2.2.1. Given $I \in \mathrm{fSet}$, let $\widetilde{\mathscr{H}ilb}_{X^I}$ be the prestack sending a test scheme S to the set

$$\left\{ (\xi, x^I) \;\middle|\; \begin{array}{l} x^I : S \to X^I; \\ \xi \in \mathrm{Hilb}_X(S); \\ \mathrm{Supp}(\xi) \subset \left\{x^I\right\} \text{ set-theoretically} \end{array} \right\}.$$

Remark 2.2.2. The condition on the support of $\xi \in \mathrm{Hilb}^n_X(S)$ can be interpreted as follows: recall that ξ gives rise to n morphisms $\xi_j : S \to X$; then for each j the graph of ξ_j must be equal set-theoretically to the graph of some $x^i : S \to X$. Equivalently, we must have $\xi_j \circ \iota_S = x^i \circ \iota_S : S_{red} \to X$. (Here, $\iota_S : S_{red} \hookrightarrow S$ is the canonical inclusion.)

It follows that the two prestacks $\mathscr{H}ilb_{X^I}$ and $\widetilde{\mathscr{H}ilb}_{X^I}$ agree when evaluated on reduced schemes; in particular they have the same k-points.

We can see that $(x^I, (\xi_j)_{j=1}^n) : S \to X^I \times \mathrm{Sym}^n_X$ factors through the formal neighbourhood $\widehat{\Gamma}^I_{X^n}$ of the incidence scheme $\widetilde{\Gamma}^I_{X^n}$ in $X^I \times \mathrm{Sym}^n_X$ precisely when the original pair $(x^I, \xi) : S \to X^I \times \mathrm{Hilb}^n_X$ factors through $\widetilde{\mathscr{H}ilb}^n_{X^I}$. That is, $\widetilde{\mathscr{H}ilb}^n_{X^I}$ is the fibre-product $\widehat{\Gamma}^I_{X^n} \times_{X^I \times \mathrm{Sym}^n_X} (X^I \times \mathrm{Hilb}^n_X)$. We have proved:

Lemma 2.2.3. *For any $I \in fSet$ and any $n \geq 0$, $\widetilde{\mathscr{H}ilb}^n_{X^I}$ is representable by an ind-closed sub-indscheme of $X^I \times \mathrm{Hilb}^n_X$; in particular it is ind-proper over X^I.*

Proof. Indeed, fibre products commute with filtered colimits. Since the formal completion $\widehat{\Gamma}^I_{X^n}$ is the colimit over $k \in \mathbb{N}$ of the kth infinitesimal neighbourhoods of $\widehat{\Gamma}^I_{X^n}$, which are closed subschemes of $X^I \times \mathrm{Sym}^n_X$, $\widetilde{\mathscr{H}ilb}^n_{X^I}$ is the colimit of the fibre product of these closed subschemes with $X^I \times \mathrm{Hilb}^n_X$, and in particular is a colimit of closed subschemes as required. ☐

It is also clear that we have analogues of Corollary 2.1.6 and Proposition 2.1.7 for $\widetilde{\mathscr{H}ilb}_{X^I}$.

Example 2.2.4. Let $X = \operatorname{Spec} k[t_1, t_2]$, $S = \operatorname{Spec} k[\epsilon]/\epsilon^2$, and $n = 2$. We will construct an S-point of $\widetilde{\mathscr{H}ilb}^n_X$ which is not a point of $\mathscr{H}ilb^n_X$. Let $x_1 : S \to X$ be given by

$$k[t_1, t_2] \to k[\epsilon]/\epsilon^2$$
$$t_1, t_2 \mapsto 0,$$

and let $x_2 : S \to X$ be given by

$$t_1 \mapsto 0; \qquad t_2 \mapsto \epsilon.$$

Now let $\xi_1, \xi_2 \subset S \times X = \operatorname{Spec} k[\epsilon, t_1, t_2]/\epsilon^2$ be the closed subschemes cut out by the ideals $I_1 = (t_1, t_2)$ and $I_2 = (t_1, t_2 - \epsilon)$, respectively. Then the point $\xi \in \mathrm{Hilb}^2_X$ corresponding to the union of ξ_1 and ξ_2 is the closed subscheme cut out by the ideal $I = (t_1, t_2(t_2 - \epsilon))$. We consider the pair $(x_1, \xi) : S \to X \times \mathrm{Hilb}^2_X$.

The graph of x_1 is given by $\operatorname{Spec} k[\epsilon, t_1, t_2]/(\epsilon^2, t_1, t_2) \subset S \times X$: on the level of reduced schemes, it is given by $\operatorname{Spec} k[t_1, t_2]/(t_1, t_2) \subset \mathrm{pt} \times X$. This is the same as the support of ξ_1 and ξ_2 on the level of reduced schemes, and hence $(x_1, \xi) \in \widetilde{\mathscr{H}ilb}^2_X(S)$. On the other hand, it is not a point of $\mathscr{H}ilb^2_X(S)$ because the image of ξ under the Hilbert-Chow morphism is given by $x_1 + x_2$, and $x_2 \neq x_1 : S \to X$.

Conversely, if we take any S-point ξ' of the Hilbert scheme which is in the preimage of $2 \cdot x_1 \in \mathrm{Sym}^2_X(S)$ under the Hilbert-Chow morphism, then (x_1, ξ') will live in $\mathscr{H}ilb^2_X(S)$ (as well as $\widetilde{\mathscr{H}ilb}^2_X(S)$), even though ξ' will not be scheme-theoretically contained in the graph of x_1. For example, we could take ξ' to be cut out by the ideal $I' = (t_1, t_2^2)$.

Remark 2.2.5. Many definitions of a factorisation space $\{Y_{X^I}\}_I$ require that each space Y_{X^I} be equipped with a connection over X^I. That is, they require that the space Y_{X^I} descends to some space $Y^{\mathrm{dR}}_{X^I}$ over X^I_{dR}; then the collection of spaces $\left\{Y^{\mathrm{dR}}_{X^I}\right\}$ satisfies suitable analogues of Ran's condition and the factorisation condition. We do not need this property for most of our constructions, and so we have not included it in our definition of a factorisation space.

However, if we *do* wish to work with this stronger definition, we must work with $\widetilde{\mathscr{H}ilb_{X^I}}$ rather than $\mathscr{H}ilb_{X^I}$. It is clear that the former descends to the de Rham stack X^I_{dR}, while the latter does not. On the other hand, the two spaces themselves have the same de Rham stack, and since our factorisation algebra constructions in 2.4 begin with a $\mathcal{D}$-module on the factorisation space, it does not matter which space we work with.

2.3 The fibre of $\mathscr{H}ilb_{\mathrm{Ran}\,X}$ over Hilb_X

There is a natural forgetful map $\rho : \mathscr{H}ilb_{\mathrm{Ran}\,X} \to \mathrm{Hilb}_X$; we shall now study its fibre over a given point $\xi \in \mathrm{Hilb}_X$. The goal is to show that the fibre is homologically contractible, and more generally that $\rho^! : \mathcal{D}(\mathrm{Hilb}_X) \to \mathcal{D}(\mathscr{H}ilb_{\mathrm{Ran}\,X})$ is fully faithful.

We introduce some general lemmas that will allow us to prove our result.

First, the argument in the first step of the proof of Theorem 4.1.6 [16] generalises to give the following:

Lemma 2.3.1. *Let $\rho : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. Given an affine scheme S and a map $f : S \to \mathcal{Y}_2$, we form the Cartesian diagram:*

$$\begin{array}{ccc} S \times_{\mathcal{Y}_2} \mathcal{Y}_1 & \xrightarrow{f'} & \mathcal{Y}_1 \\ {\scriptstyle \rho_S}\downarrow & & \downarrow{\scriptstyle \rho} \\ S & \xrightarrow[f]{} & \mathcal{Y}_2. \end{array}$$

Suppose that for all S and f as above, the functor $\rho^!_S$ is fully faithful. Then $\rho^!$ is fully faithful as well.

Proof. (We follow the argument of Gaitsgory from the proof of Theorem 4.1.6, [16].) We wish to show that for any $\mathscr{F}, \mathscr{G} \in \mathcal{D}(\mathcal{Y}_2)$, the map

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{Y}_2)}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathcal{D}(\mathcal{Y}_1)}(\rho^!\mathscr{F}, \rho^!\mathscr{G})$$

is an isomorphism. We know that

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{Y}_2)}(\mathscr{F}, \mathscr{G}) \simeq \lim_{\substack{(S \xrightarrow{g} \mathcal{Y}_2) \\ \in \mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}_2}}} \mathrm{Hom}_{\mathcal{D}(S)}(g^!\mathscr{F}, g^!\mathscr{G}).$$

On the other hand

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}(\mathcal{Y}_1)}(\rho^!\mathscr{F}, \rho^!\mathscr{G}) &\simeq \lim_{\substack{(S \xrightarrow{g} \mathcal{Y}_2) \\ \in \operatorname{Sch}^{\operatorname{Aff}}_{/\mathcal{Y}_2}}} \operatorname{Hom}_{\mathcal{D}(S\times_{\mathcal{Y}_2}\mathcal{Y}_1)}((g')^!\rho^!\mathscr{F}, (g')^!\rho^!\mathscr{G}) \\ &\simeq \lim_{\substack{(S \xrightarrow{g} \mathcal{Y}_2) \\ \in \operatorname{Sch}^{\operatorname{Aff}}_{/\mathcal{Y}_2}}} \operatorname{Hom}_{\mathcal{D}(S\times_{\mathcal{Y}_2}\mathcal{Y}_1)}(\rho_S^! g^!\mathscr{F}, \rho_S^! g^!\mathscr{G}).\end{aligned}$$

So it suffices to show that for any $(S \xrightarrow{g} \mathcal{Y}_2)$, the map

$$\operatorname{Hom}_{\mathcal{D}(S)}(g^!\mathscr{F}, g^!\mathscr{G}) \to \operatorname{Hom}_{\mathcal{D}(S\times_{\mathcal{Y}_2}\mathcal{Y}_1)}(\rho_S^! g^!\mathscr{F}, \rho_S^! g^!\mathscr{G})$$

is an isomorphism. This is immediate from the assumption. □

In fact, in special situations, we can prove something even stronger:

Lemma 2.3.2. *Let $\rho : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks and suppose that ρ has a section $u : \mathcal{Y}_2 \to \mathcal{Y}_1$. Then to show that $\rho^!$ is fully faithful, it suffices to show that $\rho^!_{pt}$ is fully faithful for any map $f : pt \to \mathcal{Y}_2$, or equivalently that $pt \times_{\mathcal{Y}_2} \mathcal{Y}_1$ is homologically contractible.*

Proof. Since $u! \circ \rho^! = \operatorname{id}_{\mathcal{D}(\mathcal{Y}_2)}$, it is clear that $\rho^!$ is faithful, so it remains to check that it is full. That is, given $\mathscr{F}, \mathscr{G} \in \mathcal{D}(\mathcal{Y}_2)$ and some $\phi : \rho^!\mathscr{F} \to \rho^!\mathscr{G} \in \mathcal{D}(\mathcal{Y}_1)$, we need to find $\psi : \mathscr{F} \to \mathscr{G}$ such that $\phi \simeq \rho^!\psi$. Thanks to the section u, we have an obvious candidate for ψ, namely $u^!\phi$.

Therefore, to prove the claim it suffices to show that $\phi \simeq \rho^! u^! \phi$. In fact, it suffices to show that these two morphisms agree when pulled back to the fibre $\operatorname{pt} \times_{\mathcal{Y}_2} \mathcal{Y}_1$ over any k-point of $\mathcal{Y}_2$. We have the following diagram:

$$\begin{array}{ccc} \operatorname{pt} \times_{\mathcal{Y}_2} \mathcal{Y}_1 & \overset{f'}{\hookrightarrow} & \mathcal{Y}_1 \\ u_{\operatorname{pt}} \uparrow\downarrow \rho_{\operatorname{pt}} & & u \uparrow\downarrow \rho \\ \operatorname{pt} & \underset{f}{\hookrightarrow} & \mathcal{Y}_2. \end{array}$$

We wish to show that $(f')^!\rho^! u^! \phi \simeq (f')^!\phi$. From the commutativity of the diagram, we have that

$$\begin{aligned}(f')^!\rho^! u^! \phi &\simeq \rho^!_{\operatorname{pt}} f^! u^! \mathscr{F} \\ &\simeq \rho^!_{\operatorname{pt}} (u_{\operatorname{pt}})^! (f')^! \phi.\end{aligned}$$

By assumption $\rho^!_{\operatorname{pt}}$ is fully faithful, and so $\rho^!_{\operatorname{pt}}(u_{\operatorname{pt}})^!(f')^!\phi \simeq (f')^!\phi$ as required. □

With these general results established, let us now turn our attention to the map

$$\rho : \mathscr{H}ilb_{\operatorname{Ran} X} \to \operatorname{Hilb}_X .$$

We wish to show that $\rho^!$ is fully faithful. Note that we can construct a section $u : \operatorname{Hilb}_X \to \mathscr{H}ilb_{\operatorname{Ran} X}$ as follows. We map an S-point ξ of Hilb_X to the pair (x^I, ξ), where $x^I : S \to \operatorname{Ran} X$ is given by taking the support of ξ (i.e. a representative in $X^n(S)$ of $\pi(\xi) \in \operatorname{Sym}^n_X(S)$). There are different choices of representative on the level of maps $S \to X^I$, but all representatives are identified in the colimit $\operatorname{Ran} X$, so the section u is well-defined.

Therefore, by Lemma 2.3.2, it is enough to show that

$$\rho^{-1}(\xi) := \operatorname{pt} \times_{\operatorname{Hilb}_X} \mathscr{H}ilb_{\operatorname{Ran} X}$$

is fully faithful for any k-point $\xi : \operatorname{pt} \to \operatorname{Hilb}_X$.

We have

$$\rho^{-1}(\xi) \simeq \operatorname*{colim}_{I \in \mathrm{fSet}^{\mathrm{op}}} \rho_I^{-1}(\xi),$$

where

$$\rho_I : \mathscr{H}ilb_{X^I} \to \operatorname{Hilb}_X$$

is given at the level of a test scheme S by

$$(x^I, \eta) \mapsto \eta.$$

It is easy to see that for such an S, we have that

$$\begin{aligned} \rho_I^{-1}(\xi)(S) &= \left\{ (x^I, \eta) \in \mathscr{H}ilb_{X^I}(S) \mid \eta = S \times \xi \right\} \\ &\simeq \left\{ x^I \mid \operatorname{Supp}(S \times \xi) \subset_\star \left\{ x^I \right\} \right\}. \end{aligned}$$

In other words, $\rho_I^{-1}(\xi)(S)$ consists of S points of X^I such that for each point $s \in S$, the set $\{x^I(s)\} \subset X$ contains the support of ξ. Motivated by this, we formulate the following proposition:

Proposition 2.3.3. *For our fixed $\xi \in \operatorname{Hilb}_X$, let $\{y_1, \ldots, y_n\}$ be a complete and repetition-free list of the points in X at which ξ is supported. Let $A := \{1, \ldots, n\}$ and let $y^A \in X^A = X^n$ be the point corresponding to this list. Then we have that*

$$\rho^{-1}(\xi) \simeq \operatorname{Ran} X_A$$

as prestacks.

Proof. We begin by defining a map

$$F : \operatorname{Ran} X_A \to \rho^{-1}(\xi)$$

in terms of a compatible family of maps given at the level of test schemes S by

$$\begin{aligned} F_{(I,a_I)} : X_A^{(I,a_I)}(S) &\to \rho^{-1}(\xi)(S) \\ (y^A, x^I) &\mapsto \lambda_I(x^I, S \times \xi). \end{aligned}$$

Here $\lambda_I : \mathscr{H}ilb_{X^I} \to \mathscr{H}ilb_{\operatorname{Ran} X}$ are the structure maps and y^A is viewed as a constant map from S. This definition makes sense because the condition $(y^A, x^I) \in X_A^{(I,a_I)}(S)$ implies that $\operatorname{Supp}(S \times \xi) = \{y^A\} \subset_\star \{x^I\}$, so that $(x^I, S \times \xi)$ is indeed in $\mathscr{H}ilb_{X^I}(S)$; moreover it is clear that $(x^I, S \times \xi)$ lies in $\rho_I^{-1}(\xi)(S)$ and that the maps $F_{(I,a_I)}$ are compatible over $(I, a_I) \in \operatorname{fSet}_A^{\mathrm{op}}$.

Now we will define an inverse $G : \rho^{-1}(\xi) \to \operatorname{Ran} X_A$ to F, by giving a compatible family of morphisms

$$G_I : \rho_I^{-1}(\xi) \to \operatorname{Ran} X_A.$$

Recall that an S-point of $\rho_I^{-1}(\xi)$ consists of a pair $(x^I, \eta) \in \mathscr{H}ilb_{X^I}$ such that $\eta = S \times \xi$. Since the support of ξ is given by the set $\{y^A\} \subset X$, the support of $S \times \xi$ is given by the union $\{c_{y^a}\}$ of the graphs of the constant functions $c_{y^a} : S \to X$. Therefore $\{c_{y^a}\} \subset_* \{x^I\}$, and so for each $a \in A$ we can choose some $i \in I$, which we'll call $a_I(a)$, such that $c_{y^a} = x^i$.

These choices define a map $a_I : A \to I$, and x^I gives an S-point of $X_A^{a_I}$. We define $G_{I,S}(x^I, \eta) := \mu_{a_I}(x^I)$, where μ_{a_I} is the structure map $X_A^{a_I} \to \operatorname{Ran} X_A$.

We need to check first of all that this gives a well-defined map of sets

$$G_{I,S} : \rho_I^{-1}(\xi)(S) \to \operatorname{Ran} X_A(S);$$

i.e. that $\mu_{a_I}(x^I)$ is independent of the choice of map $a_I : A \to I$. To see this, let us impose an equivalence relation on I by setting

$$i \sim j \text{ if there exists } a \in A \text{ such that } x^i = c_{y^a} \circ \phi = x^j.$$

(In fact this is already an equivalence relation because we chose our list $\{y^A\}$ to be repetition-free; if we had not done this, then we would now need to take the equivalence relation generated by $\sim$.) Then the map

$$\begin{aligned} q \circ a_I : A &\to I/\sim \\ a &\mapsto [a_I(a)] \end{aligned}$$

is independent of the choice of a_I, and makes $(A \to I/\sim)$ into an object of fSet^A. By construction of $I/\sim$, $(x^{[i]})$ gives an S-point of $X_A^{(I/\sim, q\circ a_I)}$, whose image in $\left(X_A^{(I,a_I)}\right)(S)$ is (x^I).

We have the following commutative diagram

$$\begin{array}{ccc} & \operatorname{Ran} X_A & \\ {\scriptstyle \mu_{q\circ a_I}} \nearrow & & \nwarrow {\scriptstyle \mu_{a_I}} \\ X_A^{(I/\sim, q\circ a_I)} & \longrightarrow & X_A^{(I,a_I)} \end{array}$$

and so $\mu_{a_I}(x^I) = \mu_{q\circ a_I}(x^{[i]})$ is independent of the choice of a_I. This shows that $G_{I,S}$ is well-defined.

It is also straightforward to check that $G_{I,\bullet}$ is natural in S, and hence gives a map of prestacks

$$G_I : \rho_I^{-1}(\xi) \to \operatorname{Ran} X_A.$$

The last thing to check is that the maps G_I are compatible under surjections $\alpha : I \twoheadrightarrow J$; this is straightforward as well. Given a point $(x^J, \eta) \in \rho_J^{-1}(\xi)(S)$, we have $(\Delta(\alpha)(x^J), \eta) \in \rho_I^{-1}(\xi)(S)$. We choose any $a_I : A \to I$ as above, such that for each $a \in A$ we have $c_{y^a} = (\Delta(\alpha)(x^J))^{a_I(a)} = x^{\alpha\circ a_I(a)}$. Then we take $a_J : A \to J$ to be given by $\alpha \circ a_I$, and we see from the definitions that $G_J(x^I, \eta) = G_I(\Delta(\alpha)(x^J), \eta)$, which is the required compatibility.

Therefore, we obtain a map $G : \rho^{-1}(\xi) \to \operatorname{Ran} X_A$.

It is clear that G and F are mutually inverse, and so exhibit an equivalence of prestacks

$$\rho^{-1}(\xi) \simeq \operatorname{Ran} X_A.$$

□

Combining Lemma 2.3.2 with Propositions 1.1.7 and 2.3.3 we obtain the desired result:

Theorem 2.3.4. *Assuming that X is connected, the map*

$$\rho^! : \mathcal{D}(\mathrm{Hilb}_X) \to \mathcal{D}(\mathscr{H}ilb_{\operatorname{Ran} X})$$

is fully faithful.

Remark 2.3.5. An alternate proof of this fact is the following: we can generalise the proof of Proposition 2.3.3 to show that for any $S \to \mathrm{Hilb}_X$ the fibre-product $S \times_{\mathrm{Hilb}_X} \mathscr{H}ilb_{\mathrm{Ran}\,X}$ is isomorphic to $\mathrm{Ran}\,X_{A,S}$, the so-called *relative Ran space with marked points*. (See 2.5.12 [16] for the definition.) It is easy to check that for any $s \in S$, the fibre of $\mathrm{Ran}\,X_{A,S}$ is the ordinary Ran space with marked points; it is also easy to define a section $S \to \mathrm{Ran}\,X_{A,S}$. Then the Theorem follows from Lemma 2.3.2 and Proposition 1.1.7.

2.4 A factorisation algebra over the variety X

In this section, we consider a factorisation algebra produced by linearising the factorisation space $\mathscr{H}ilb_{\mathrm{Ran}\,X}$.

Definition 2.4.1. Set

$$\mathcal{A}^I := f^I_* \omega_{\mathscr{H}ilb_{X^I}} \in \mathcal{D}\left(X^I\right);$$
$$\mathcal{A}_{\mathrm{Ran}\,X} := f_* \omega_{\mathscr{H}ilb_{\mathrm{Ran}\,X}} \in \mathcal{D}\left(\mathrm{Ran}\,X\right).$$

It is a factorisation algebra on X. We denote the chiral algebra corresponding to $\mathcal{A}$ by $\mathcal{B}$.

Let us make some remarks on this definition:

Remark 2.4.2.

1. Given a smooth surface S, Kotov [26] defines a factorisation algebra $\mathcal{B}$ over S. This factorisation algebra agrees with our factorisation algebra for $X = S$. Kotov claims that for S simply connected, $\mathcal{B}$ is commutative (Theorem 4, [26]). We do not yet know an algebro-geometric proof of this fact, but we expect that such a proof exists and should generalise to higher dimensions. We expect to make use of the results of II.7.4.
2. Since we are working with the de Rham cohomology of the factorisation space $\mathscr{H}ilb_{\mathrm{Ran}\,X}$, we could equally well have used the variant $\widetilde{\mathscr{H}ilb}_{\mathrm{Ran}\,X}$ for this definition.

We wish to compute the chiral homology of $\mathcal{B}$, which by definition is given by

$$\int_X \mathcal{B} := (p_{\mathrm{Ran}\,X})_! \mathcal{A}_{\mathrm{Ran}\,X}.$$

Remark 2.4.3. The chiral homology of a chiral algebra, defined by Beilinson–Drinfeld [4], is a generalisation of the notion of the space of conformal blocks associated to a vertex algebra. More specifically, in the case that V is a quasi-conformal vertex algebra and $\mathcal{B}_V$ is the corresponding chiral algebra on $\mathbb{A}^1$, the degree zero piece of the chiral homology of $\mathcal{B}$ is isomorphic to the space of conformal blocks of V.

Proposition 2.4.4. *The chiral homology of $\mathcal{B}$ is given by the de Rham cohomology of the Hilbert scheme:*

$$\int_X \mathcal{B} \simeq \mathrm{H}_\bullet(\mathrm{Hilb}_X).$$

Proof. We have the following diagram, which is trivially commutative:

$$\begin{array}{ccccc} & & \mathscr{H}\!ilb_{\operatorname{Ran} X} & & \\ & \swarrow^{\rho} & & \searrow^{f} & \\ \mathrm{Hilb}_X & & & & \operatorname{Ran} X \\ & \searrow_{p_{\mathrm{Hilb}_X}} & & \swarrow_{p_{\operatorname{Ran} X}} & \\ & & \mathrm{pt}. & & \end{array}$$

Then we can see that

$$\int_X \mathcal{B} := (p_{\operatorname{Ran} X})_! f_* \omega_{\mathscr{H}\!ilb_{\operatorname{Ran} X}} \simeq (p_{\mathrm{Hilb}_X})_! \rho_! \rho^! \omega_{\mathrm{Hilb}_X}.$$

By Theorem 2.3.4, we have an isomorphism

$$(p_{\mathrm{Hilb}_X})_! \rho_! \rho^! \omega_{\mathrm{Hilb}_X} \xrightarrow{\sim} (p_{\mathrm{Hilb}_X})_! \omega_{\mathrm{Hilb}_X} := \mathrm{H}_\bullet(\mathrm{Hilb}_X),$$

so the theorem is proved. □

2.5 Universality of $\mathscr{H}\!ilb_\bullet$

In this section we consider how the spaces $\mathscr{H}\!ilb_{X^I}$ change as we change the base variety X.

We begin by fixing an arbitrary scheme S of finite type together with

$$\pi : X \to S$$

a smooth morphism of dimension n. Then we can consider a relative version of the Hilbert scheme:

Definition 2.5.1. The *relative Hilbert scheme of n points on X/S* is the scheme $\mathrm{Hilb}^n_{X/S}$ representing the functor

$$\begin{aligned}&\left(\mathrm{Sch}^{\mathrm{Aff}}_{/S}\right)^{\mathrm{op}} \to \infty\text{-Grpd}\\ &(T/S) \mapsto \left\{ \xi \;\middle|\; \begin{array}{c}\xi \subset T \times_S X \text{ is a closed subscheme, flat over T with}\\ \text{zero dimensional fibres of finite length } n.\end{array} \right\}.\end{aligned}$$

As in the non-relative case, Grothendieck [22] proved that this functor is representable; it is also representable for any affine base scheme S with X affine over S (see again [23]), and hence by gluing for any smooth family $X \to S$. It is proper over S, again by the valuative criterion of properness.

We let $\mathrm{Hilb}_{X/S}$ be the disjoint union of $\mathrm{Hilb}^n_{X/S}$ for $n \geq 0$. It is an indscheme, ind-proper over S.

Our goal is to use this definition to construct a relative version of our factorisation space. First let us generalise the definition of a factorisation space to the relative setting. The Ith component of a factorisation space will be an l.f.t. prestack over S, that is, a functor

$$\left(\left(\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}}\right)_{/S}\right)^{\mathrm{op}} \to \infty\text{-Grpd}.$$

For any finite set I let $(X/S)^I = X \times_S X \times_S \ldots \times_S X \simeq X^I \times_{S^I} S$ denote the I-fold fibre product of X over S. For any $\alpha : I \twoheadrightarrow J$, let $U(\alpha)_S \subset (X/S)^I$ be given by the fibre product

$$U(\alpha) \times_{X^I} (X/S)^I.$$

Definition 2.5.2. A *relative factorisation space* over X/S is given by the following data:

1. For each $I \in \mathrm{fSet}$ we have a prestack $Y_{(X/S)^I} \in \mathrm{PreStk}_{/S}$ representable by an indscheme, and equipped with a map

   $$f^I : Y_{(X/S)^I} \to (X/S)^I$$

   over S.

2. For any $\alpha : I \twoheadrightarrow J$ in fSet, an identification

   $$Y_{(X/S)^J} \xrightarrow{\sim} X^J \times_{X^I} Y_{(X/S)^J}$$

   of indschemes over $(X/S)^J$. In particular, we have an ind-closed embedding

   $$Y(\alpha) : Y_{(X/S)^J} \to Y_{(X/S)^I}.$$

3. For any $\alpha : I \twoheadrightarrow J$ in fSet, an identification

$$U(\alpha) \times_{X^I} \left(\left(\prod_{j \in I} Y_{(X/S)^{I_j}} \right) \times_{S^J} S \right) \xrightarrow{\sim} U(\alpha) \times_{X^I} Y_{(X/S)^I}.$$

of indschemes over $U(\alpha)_S$.

We require that these identifications be compatible with each other and with composition.

Remark 2.5.3. Note that a factorisation space over X/S is not necessarily a factorisation space over the total space X: the condition in (2) is the same for both definitions, but the condition in (3) is weaker than the requirement for a factorisation space over X.

On the other hand, suppose that $\left\{ Y_{X^I} \to X^I \right\}_{I \in \mathrm{fSet}^{\mathrm{op}}}$ is a factorisation space over X. Define $\widetilde{Y}_{(X/S)^I} := Y_{X^I} \times_{S^I} S$ for each $I \in$ fSet. Then

$$\left\{ \widetilde{Y}_{(X/S)^I} \to (X/S)^I \right\}_{I \in \mathrm{fSet}}$$

gives a factorisation space over X/S.

Definition 2.5.4. We define the $\mathscr{H}ilb_{(X/S)^I} \in \mathrm{PreStk}_{/S}$ to be the functor which sends a scheme $T \to S$ to the set of pairs (x^I, ξ), where $x : T \to X$ is a morphism of schemes over S and $\xi : T \to \mathrm{Hilb}_{(X/S)}$ with $\mathrm{Supp}(\xi) \subset_\star \{x^I\}$.

Here the inclusion $\subset_\star$ is to be interpreted in the same sense as in the definition 2.1.3. That is, via the relative version of the Hilbert-Chow morphism, ξ determines an unordered finite collection of morphisms $\xi_j : T \to X$ over S, and the condition is that each of these ξ_j is equal to the ith projection

$$x^i : T \xrightarrow{x^I} (X/S)^I \xrightarrow{\mathrm{pr}_i} X$$

of x^I for some $i = i(j) \in I$.

Let $f^I_{X/S} : \mathscr{H}ilb_{(X/S)^I} \to (X/S)^I$ denote the natural projection.

Proposition 2.5.5. *The collection*

$$I \mapsto \mathscr{H}ilb_{(X/S)^I}$$

defines a relative factorisation space over X/S.

Proof. The proof uses the same ideas as in Lemma 2.1.5 and Proposition 2.1.7. □

We have seen that linearising a factorisation space produces a factorisation algebra; similarly, linearising a relative factorisation space produces a relative version of a factorisation algebra. Recall from remark 1.2.6 that, in the non-relative setting, rather than assuming that a factorisation algebra is a $\mathcal{D}$-module, we can define a factorisation structure on a $\mathcal{O}$-module and then obtain the $\mathcal{D}$-module structure canonically from the axioms. However, the factorisation condition for relative factorisation algebras is weaker than for ordinary factorisation algebras: the factorisation isomorphism is given only over $U(\alpha)_S$ rather than all of $U(\alpha)$.

This means that the construction used to produce the connection on the quasi-coherent sheaf underlying a factorisation algebra no longer produces a connection in the relative setting. Instead, we obtain only a connection along the fibres of the morphism $X \to S$. That is, we should expect a relative factorisation algebra to be in particular a relative $\mathcal{D}$-module, i.e. a family of objects $\mathrm{IndCoh}\left(X^I_{\mathrm{dR}} \times_{S^I_{\mathrm{dR}}} S\right)$.

Apart from this subtlety, the definition of a relative factorisation algebra is exactly the linear analogue of a relative factorisation space. We omit the details.

For the remainder of the section, let us restrict our attention to the case $I = \mathrm{pt}$. Consider the dualising sheaf in $\mathrm{IndCoh}\big((\mathscr{H}ilb_{X/S})_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S\big)$; by abuse of notation we will denote it by $\omega_{\mathscr{H}ilb_{X/S}}$. Let $\mathcal{A}_{X/S}$ denote the pushforward of $\omega_{\mathscr{H}ilb_{X/S}}$ under the map

$$g_{X/S} : (\mathscr{H}ilb_{X/S})_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \to (X/S)_{\mathrm{dR}} := X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S.$$

(This map is proper, so the $*$ and ! pushforwards coincide.) This is the $I = \{\mathrm{pt}\}$ component of a relative factorisation algebra, and is in particular a relative $\mathcal{D}$-module over X/S.

Proposition 2.5.6. *Suppose that we have a fibrewise étale morphism $\varphi = (\varphi_X, \varphi_S)$ of smooth families*

$$\begin{array}{ccc} X & \xrightarrow{\varphi_X} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ S & \xrightarrow[\varphi_S]{} & S'. \end{array}$$

That is, X/S and X'/S' are smooth families, necessarily of the same dimension n, and (φ_X, φ_S) are compatible maps such that for any point $s \in S$ with $s' := \varphi_S(s) \in S'$, the induced morphism on fibres $(X)_s \to (X')_{s'}$ is étale.

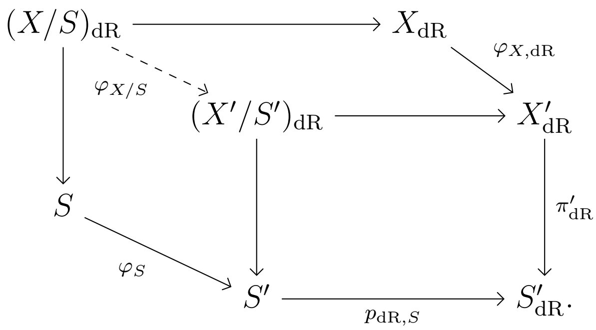
Then there is a natural morphism $\varphi_{X/S} : (X/S)_{dR} \to (X'/S')_{dR}$*, and we have a canonical identification*

$$\mathcal{A}(\varphi) : \mathcal{A}_{X/S} \xrightarrow{\sim} \varphi^!_{X'/S'} \mathcal{A}_{X'/S'}.$$

Moreover, these identifications are compatible with composition of fibrewise étale morphisms

$$X/S \xrightarrow{\varphi} X'/S' \xrightarrow{\psi} X''/S''.$$

Proof. First note that the map $\varphi_{X/S}$ is defined as follows:



With this in mind, the result follows easily from the following claim:

Lemma 2.5.7. *In the above setting, there is a natural map of indschemes*

$$\mathscr{H}(\varphi) : \mathscr{H}ilb_{X/S} \to \mathscr{H}ilb_{X'/S'}$$

such that the diagram

$$\begin{array}{ccc} \mathscr{H}ilb_{X/S} & \xrightarrow{\mathscr{H}(\varphi)} & \mathscr{H}ilb_{X'/S'} \\ {\scriptstyle f_{X/S}}\downarrow & & \downarrow{\scriptstyle f_{X'/S'}} \\ X & \xrightarrow[\varphi_X]{} & X' \end{array}$$

is Cartesian.

Let us assume the lemma for the moment, and show how it implies the statement of the proposition.

The map $\mathscr{H}(\varphi)$ induces a map

$$\mathscr{H}(\varphi)_{X/S} : \left(\mathscr{H}ilb_{X/S}\right)_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \to \left(\mathscr{H}ilb_{X'/S'}\right)_{\mathrm{dR}} \times_{T_{\mathrm{dR}}} T$$

such that the diagram

$$\begin{array}{ccc} \left(\mathscr{H}ilb_{X/S}\right)_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S & \xrightarrow{\mathscr{H}(\varphi)_{X/S}} & \left(\mathscr{H}ilb_{X'/S'}\right)_{\mathrm{dR}} \times_{T_{\mathrm{dR}}} T \\ {\scriptstyle g_{X/S}}\big\downarrow & & \big\downarrow{\scriptstyle g_{X'/S'}} \\ (X/S)_{\mathrm{dR}} & \xrightarrow[\varphi_{X/S}]{\qquad\qquad} & (X'/S')_{\mathrm{dR}} \end{array}$$

is Cartesian.

Then we have by proper base change

$$\begin{aligned} \varphi^!_{X/S}\mathcal{A}_{X'/S'} &= \varphi^!_{X/S}(g_{X'/S'})_* \left(\omega_{\mathscr{H}ilb_{X'/S'}}\right) \\ &\simeq (g_{X/S})_*(\mathscr{H}(\varphi)_{X/S})^! \left(\omega_{\mathscr{H}ilb_{X'/S'}}\right) \\ &\simeq (g_{X/S})_* \left(\omega_{\mathscr{H}ilb_{X/S}}\right) \\ &:= \mathcal{A}_{X/S}. \end{aligned}$$

The compatibility of these identifications with composition comes from the compatibility of the base-change isomorphisms with composition. □

Proof of Lemma 2.5.7. Notice that the fibrewise étale morphism $\varphi = (\varphi_X, \varphi_S)$ can always be factored as

$$\begin{array}{ccccc} X & \xrightarrow{\psi_X} & S \times_{S'} X' & \longrightarrow & X' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ S & = & S & \xrightarrow[\varphi_S]{} & S'. \end{array}$$

where ψ_X is étale and the square on the right is Cartesian. Thus it suffices to prove the claim in two cases: when $\varphi = (\varphi_X, \mathrm{id}_S)$ with φ_X étale, or else when the commutative square formed by (φ_X, φ_S) is a pullback square.

Let us treat first the case that $\varphi = (\varphi_X, \mathrm{id}_S)$. Suppose we have a map $T \to \mathscr{H}ilb_{X/S}$ given by a pair (x, ξ); we wish to show that this is equivalent to a map $T \to X \times_{X'} \mathscr{H}ilb_{X'/S}$. We need to construct compatible maps $T \to X$ and $T \to \mathscr{H}ilb_{X'/S}$ over X'.

The map $T \to X$ is given by x. The map $T \to \mathscr{H}ilb_{X'/S}$ will be given by a pair (x', ξ') where $\mathrm{Supp}(\xi') \subset_\star \{x'\}$. It is clear that we must take $x' = \varphi_X \circ x$. Also if we choose a representative $(\xi'_j) : T \to (X')^n$ of the image of ξ' in $\mathrm{Sym}(X')$ under the Hilbert-Chow morphism, then each ξ'_j must be equal to x'. It remains to give the scheme structure of ξ' in $T \times_S X'$.

By definition of $\mathscr{H}ilb_{X'/S}$, ξ' the closed embedding

$$\xi' \hookrightarrow T \times_S X'$$

must factor through the formal neighbourhood of the graph of x' in $T \times_S X'$. But since φ_X is étale, this is isomorphic to the formal neighbourhood of the graph of x in $T \times_S X$. We have the following:

$$\begin{array}{ccccc} \xi & \hookrightarrow & \widehat{\{x\}} & \hookrightarrow & T \times_S X \\ \| & & \widehat{\varphi} \downarrow \wr & & \downarrow \\ \xi' & \hookrightarrow & \widehat{\{x'\}} & \hookrightarrow & T \times_S X' \end{array}$$

That is, we define ξ' to be the image of ξ under the isomorphism $\widehat{\varphi}$.

It is clear that this gives a bijection between the T-points of $\mathscr{H}ilb_{X/S}$ and those of $X \times_{X'} \mathscr{H}ilb_{X'/S}$. It is also easy to see that this is functorial in T, and hence gives the desired result.

Now let us address the second case: the morphism $\varphi_S : S \to S'$ is arbitrary, but φ_X is its pullback along the smooth map $\pi' : X' \to S'$. Given a T-point (x, ξ) of $\mathscr{H}ilb_{X/S}$, we wish to define a T-point (y, x', ξ') of $\mathscr{H}ilb_{X/S}$. As in the above discussion, it is clear that we must have $x = y$ and $x' = \varphi_X \circ x$, and it remains to specify the subscheme ξ' of $\widehat{\{x\}} \hookrightarrow T \times_S X$.

But in this case we have that $T \times_S X \simeq T \times_{S'} X'$, and that $\widehat{\{x\}}$ and $\widehat{\{x'\}}$ are identified under this isomorphism. So again, we take ξ' to be the subscheme of $T \times_{S'} X'$ corresponding to ξ.

We again obtain a functorial identification of the T-points, and hence an isomorphism of the prestacks

$$\mathscr{H}ilb_{X/S} \simeq X \times_{X'} \mathscr{H}ilb_{X'/S'} .$$

□

Observation 2.5.8. For future reference, we make the following observation about the proof in the case $\varphi = (\varphi_X, \mathrm{id}_S)$. Suppose that our T-point $(x, \xi) : T \to \mathscr{H}ilb^c_{X/S}$ for some $c \in \mathbb{N}$. Then in fact the inclusion of ξ in the formal neighbourhood $\widehat{\{x\}}$ factors through the cth infinitesimal neighbourhood:

$$\begin{array}{ccc} \xi & \hookrightarrow & \widehat{\{x\}}. \\ & \searrow \quad \nearrow & \\ & \{x\}^{(c)} & \end{array}$$

So we could have defined ξ' to be the image of ξ under $\varphi^{(c)} : \{x\}^{(c)} \xrightarrow{\sim} \{x'\}^{(c)}$.

Let us make some further remarks on the result of Proposition 2.5.6.

Remark 2.5.9. 1. The structure given by this compatible family

$$\left\{(X/S) \mapsto \mathcal{A}_{X/S}\right\}$$

is of particular interest, and is the subject of the next part of this thesis: for any fixed dimension n, we have constructed a so-called *universal $\mathcal{D}$-module* of dimension n.

2. We conjecture, further, that the assignment

$$(X/S) \mapsto \left\{\mathcal{A}_{(X/S)^I} \in \mathcal{D}(X^I/S)\right\}_{I \in \mathrm{fSet}}$$

is also *universal* in a similar sense. More specifically, we expect that the chiral algebra structures on the $\mathcal{D}(X/S)$-modules $\mathcal{A}_{X/S}$ are compatible under pull-back by fibrewise étale morphisms. In that case, this data is an example of what should be defined as an n-dimensional vertex algebra.

3. More generally, we expect that there exists some natural condition on an assignment

$$X/S \mapsto Y_{\operatorname{Ran} X/S}$$

of relative factorisation spaces which will ensure that the chiral algebras formed by linearising these spaces give a universal chiral algebra. However, it is not clear to the author what this condition should be.

More precisely, it is certain that the analogue of Lemma 2.5.7 must hold for $I = \mathrm{pt}$, but we do not expect it to hold for $|I| \geq 2$. It already does not hold for $n = 2$ and the factorisation spaces given by $\mathscr{H}ilb_\bullet$ or the affine Grassmannian over curves. In these cases, we do not even have maps

$$\mathscr{H}ilb_{X^2} \to \mathscr{H}ilb_{(X')^2} \text{ or } \mathrm{Gr}_{G,X^2} \to \mathrm{Gr}_{G,(X')^2},$$

unless $\varphi : X \to X'$ is an open embedding.

On the other hand, Kapranov–Vasserot [25] define a factorisation space denoted $\mathcal{L}(X)_{\operatorname{Ran} C}$ over any smooth curve C and claim (for example in the statements of Proposition 3.4.6 and 3.6.2) that for any étale map $\pi : C \to D$ and for any finite set I there *is* an étale map $\mathcal{L}(\pi) : \mathcal{L}(X)_{C^I} \to \mathcal{L}(X)_{D^I}$ having many desirable properties. They do not provide a construction of this map.

Chapter II

Universal $\mathcal{D}$-modules and stacks of étale germs

The goal of this chapter is to understand universal families of $\mathcal{D}$-modules and $\mathcal{O}$-modules as quasi-coherent sheaves on certain stacks. These universal modules are rules assigning to each n-dimensional variety a $\mathcal{D}$-module or an $\mathcal{O}$-module in a way compatible with étale morphisms; we will introduce several stronger versions of the compatibility condition, which will allow us to define the notion of a *convergent* universal module. In particular, we introduce stacks classifying étale germs of n-dimensional varieties, and show that the universal modules are quasi-coherent sheaves on these stacks; we also introduce variations on these stacks corresponding to the stronger compatibility conditions.

Moreover, we show that these stacks are isomorphic to the classifying stacks of certain automorphism groups of the formal n-dimensional disc $\operatorname{Spf} k[\![t_1, \ldots, t_n]\!]$, and hence that the categories of quasi-coherent sheaves on our stacks are the same as the representation categories of these automorphism groups. The difference between $\mathcal{D}$-modules and $\mathcal{O}$-modules amounts to an action by infinitesimal translations, present only in the case of $\mathcal{D}$-modules; in the case of the stacks in this chapter, this difference is manifested in the automorphism groups as follows: the group corresponding to $\mathcal{O}$-modules contains only those automorphisms of the formal disc preserving the origin, while in the case of $\mathcal{D}$-modules infinitesimal translations of the origin are permitted.

0.1 The key players

Let us now introduce the main players of this chapter, which fall into three classes: we have two flavours of stacks—those corresponding to classifying stacks, and those corresponding to stacks of germs of varieties—and in addition, we have the categories of universal modules.

Let G denote the group formal scheme of automorphisms of the formal disc. It has a pro-structure, since it can be viewed as the limit of its quotients $G^{(c)}$, which are the automorphism groups of the cth infinitesimal neighbourhood of a point in an n-dimensional variety. The classifying stacks of interest will be those corresponding to these groups; quasi-coherent sheaves on these classifying stacks correspond to representations of the associated group. There is a subgroup $G^{\text{ét}}$ of G, of automorphisms of *étale type*; it is closely related to the stacks of étale germs that we will define later. We will see that this subgroup is dense in G, so that the representation theory of the two groups is very similar. More specifically, placing a finiteness condition on their representations yields equivalent categories of representations.

The second flavour of stacks are those parametrising étale germs of n-dimensional varieties—that is, we are interested in pointed n-dimensional varieties with morphisms given by roofs of étale morphisms, or *common étale neighbourhoods*:

$$\begin{array}{ccc} & (V, v) & \\ \swarrow & & \searrow \\ (X_1, x_1) & & (X_2, x_2). \end{array}$$

Imposing different equivalence relations on these classes of morphisms allows us to define the different versions of the stack that we will need, corresponding to the various classifying stacks mentioned above. The equivalence relations are defined by identifying common étale neighbourhoods which give rise to the same isomorphisms of the formal completions $\widehat{X_1} \xrightarrow{\sim} \widehat{X_2}$ or of the cth infinitesimal neighbourhoods for $c \in \mathbb{N}$. We denote these stacks by $\mathcal{M}_n$, $\mathcal{M}_n^{(c)}$, and $\mathcal{M}_n^{(\infty)}$.

Finally, we consider the category $\mathscr{U}_n^{\mathcal{D}}$ of universal $\mathcal{D}$-modules, as introduced by Beilinson and Drinfeld [4]. These are families of $\mathcal{D}$-modules on n-dimensional varieties, compatible with pullback along étale morphisms. We impose an additional requirement, that these compatibilities be themselves compatible with identifications of étale morphisms giving rise to the same morphisms of infinitesimal neighbourhoods. This allows us to define subcategories $\mathscr{U}_n^{\mathcal{D},(c)}$, and finally the category $\mathscr{U}_n^{\mathcal{D},\text{conv}}$ of *convergent universal $\mathcal{D}$-modules*, which are the kind of modules arising from vertex algebras.

Extending the finiteness condition on representations alluded to above allows us to define analogous conditions on the categories of quasi-coherent sheaves, and finally to give a characterisation of convergent universal $\mathcal{D}$-modules as those universal $\mathcal{D}$-modules which are of *ind-finite type*.

The relationship between these objects is the main focus of this chapter; the results are summarised in Figure 1.[1]

This diagram corresponds to the setting of universal $\mathcal{D}$-modules; we have an entirely analogous diagram to describe the setting of universal $\mathcal{O}$-modules. The key differences are the following:

1. We replace the group G of automorphisms of the formal disc by its reduced subgroup $K = G_{red}$: this is the group of automorphisms which preserve the origin of the disc. By contrast, G includes automorphisms which may involve an infinitesimal translation of the origin. These infinitesimal translations correspond to the action of the sheaves of differential operators on the corresponding universal $\mathcal{D}$-modules, not present in the case of universal $\mathcal{O}$-modules.

2. We replace the stacks of étale germs of n-dimensional varieties by stacks with the same objects but with fewer isomorphisms: in this case we only allow isomorphisms which fix the distinguished points of our pointed varieties, whereas the original stacks permitted isomorphisms which could shift these points infinitesimally. We will denote these stacks by adding a superscript $\bullet^{\mathrm{pt}}$ to the symbol denoting the corresponding stack for the $\mathcal{D}$-module setting (e.g. $\mathcal{M}_n^{\mathrm{pt}}$).

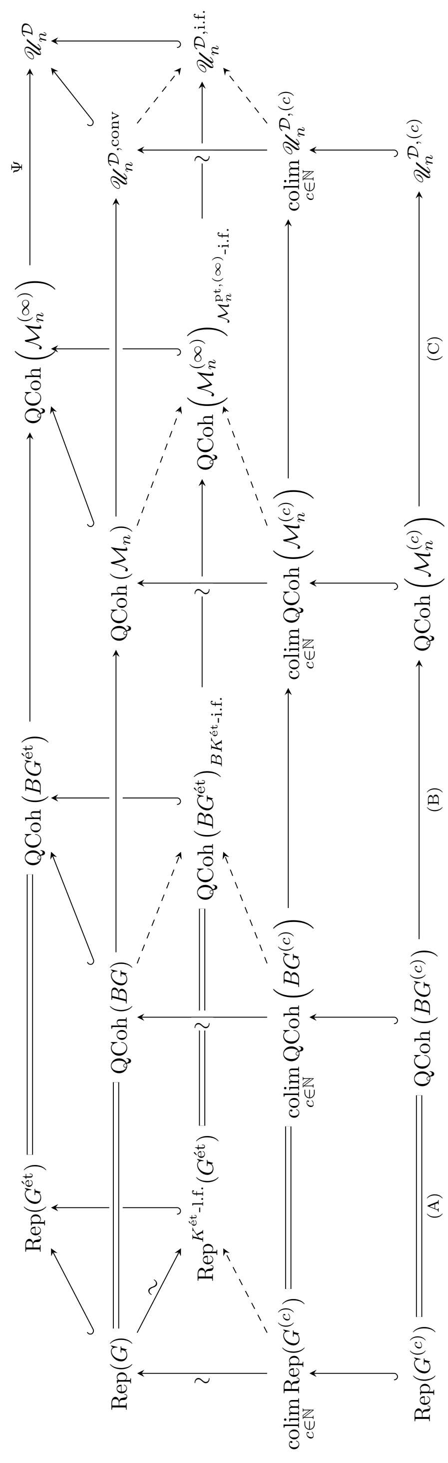
3. Rather than considering universal families of $\mathcal{D}$-modules, we consider families of universal $\mathcal{O}$-modules. We denote these categories by $\mathscr{U}_n^{\mathcal{O}}$, etc.

4. The finiteness conditions in the bottom part of the back row are simpler. In fact, these finiteness conditions are most naturally defined in the $\mathcal{O}$-module setting; the corresponding conditions in the $\mathcal{D}$-module setting are then defined by requiring the objects to be suitably finite when regarded as objects in the $\mathcal{O}$-module setting after applying a forgetful functor.

We will give the full definitions of the stacks and categories for both the $\mathcal{D}$- and $\mathcal{O}$- module settings, but for the proofs of the equivalences we will mainly focus on the story of universal $\mathcal{D}$-modules. This is the setting needed for working with universal chiral algebras. Moreover, generally the proofs in the $\mathcal{O}$-module setting are just simpler versions of the $\mathcal{D}$-module proofs. The exception is in the study of the categories of representations; there we will see that it is first necessary to study the groups K and $K^{\text{ét}}$, and then to extend our results to G and $G^{\text{ét}}$.

[1] The reader may wish to pull out the additional copy of the diagram included in Appendix B so that he can refer back to it easily while reading this chapter.

Figure II.1: The main diagram: this chapter in one page.



0.2 The structure of the chapter

We will begin in section 1 by discussing and defining the stacks $\mathcal{M}_n^{(\infty)}$ and $\mathcal{M}_n^{\mathrm{pt},(\infty)}$ of unpointed and pointed étale germs, as well as the necessary variations. In section 2 we introduce the automorphism groups G and K, as well as their quotients $G^{(c)}$ and $K^{(c)}$. We will see that there is a natural map F from our stack $\mathcal{M}_n^{(\infty)}$ to the classifying stack BG, and analogues $F^{(c)}$ for the quotients $G^{(c)}$. Pullback along these maps gives rise to the functors in column (B) of the main diagram. In section 3, we state and prove a generalisation of Artin's approximation theorem [2] to the relative setting, and show as a corollary that the map $F^{(c)}$ is an isomorphism of stacks. It follows that the functors in the front part of column (B) are equivalences.

In section 4, we introduce the group-valued prestack $G^{\text{ét}}$ of étale-type automorphisms of the formal disc; it will be immediate from the relative Artin approximation theorem and the definition of $G^{\text{ét}}$ that the classifying stack $BG^{\text{ét}}$ is equivalent to the stack $\mathcal{M}_n^{(\infty)}$ of étale germs of n-dimensional varieties. We will then study the representation theory of $G^{\text{ét}}$, and identify representations of G as the subcategory of $\mathrm{Rep}(G^{\text{ét}})$ of $K^{\text{ét}}$*-locally-finite* representations. In section 5 we define the categories of universal $\mathcal{D}$- and $\mathcal{O}$-modules, as well as the functor Ψ in the back part of column (C). We then prove that this functor is an equivalence.

In section 6, we define the categories $\mathscr{U}_n^{\mathcal{D},(c)}$ and $\mathscr{U}_n^{\mathcal{D},\mathrm{conv}}$ of cth-order and convergent universal $\mathcal{D}$-modules, and we prove that the functor Ψ restricts to give the equivalences of the front part of column (C). We also characterise convergent universal $\mathcal{D}$-modules as those universal $\mathcal{D}$-modules which are *locally finite* in the sense analogous to that in the study of $\mathrm{Rep}(G) \hookrightarrow \mathrm{Rep}(G^{\text{ét}})$—we shall call these universal $\mathcal{D}$-modules of *ind-finite type* to avoid confusion with the standard use of the word "local" in sheaf theory. In the final section, we discuss the extension of these definitions and results to the setting of ∞-categories.

Combining our results, we obtain the following equivalences of categories:

$$\mathrm{Rep}(K) \xleftarrow{\sim} \mathrm{QCoh}\left(\mathcal{M}_n^{\mathrm{pt}}\right) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{O},\mathrm{conv}} \qquad \mathrm{Rep}(G) \xleftarrow{\sim} \mathrm{QCoh}\left(\mathcal{M}_n\right) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D},\mathrm{conv}}.$$

That is, we have proved a variation of the theorem suggested by Beilinson and Drinfeld [Proposition and Exercise 2.9.9, [4]]:

Theorem 0.2.1. *We have the following equivalences of categories:*

$$\mathscr{U}_n^{\mathcal{O},conv} \simeq \mathrm{Rep}(K)$$
$$\mathscr{U}_n^{\mathcal{D},conv} \simeq \mathrm{Rep}(G).$$

Composing the functors in the main diagram, we obtain a functor

$$\mathrm{Rep}(G) \to \mathscr{U}_n^{\mathcal{D}}.$$

This functor agrees with the functor (2.9.9.1) of [4]. It follows from our results that this functor is a fully faithful embedding, but it is not clear that it is essentially surjective. If it is not, then not all universal $\mathcal{D}$-modules are convergent; we argue that in that case the category of convergent universal $\mathcal{D}$-modules should be the preferred setting.

Note that the statement of Proposition 2.9.9 [4] is also discussed in a more restricted setting by Jordan and Orem (see section 4 of [24]).

0.3 Conventions and notation

We fix $k = \overline{k}$, an algebraically closed field of characteristic zero. By Sch, we will always mean the category of schemes over k. By PreStk, we mean the category of functors

$$\left(\mathrm{Sch}^{\mathrm{Aff}}\right)^{\mathrm{op}} \to \infty\text{-Grpd}.$$

Unlike in the previous chapter, we now work with ordinary categories of sheaves, $\mathcal{D}$-modules, and representations, rather than DG- or ∞-categories. Our categories are cocomplete (i.e. closed under colimits); in particular, colimits of categories are always taken in the ∞-category of cocomplete categories.

Note that each of the stacks appearing in the main diagram can be defined by giving a prestack parametrising only the *trivial* objects; then we take the stackification of the prestack to obtain the stacks in our diagram. (For example, the classifying stack of a group is the stackification of the prestack classifying only the trivial principal bundles; similarly in the third column of the diagram we can consider prestacks classifying the "trivial" pointed n-dimensional variety, $(\mathbb{A}^n, 0)$.) These prestacks will be denoted by adding the subscript $\bullet_{\mathrm{triv}}$ to the symbol for the corresponding stack. In the case of the stacks of germs of varieties, it will also be convenient to consider an intermediate prestack, which has more objects and automorphisms than the trivial version of the prestack, but which still has the same stackification; this prestack will be denoted by decorating the symbol for the corresponding stack with $\widetilde{\bullet}$. Since taking quasi-coherent sheaves is independent of stackification, it does not matter which of these related prestacks we use in the above diagram.

0.4 Acknowledgements

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1 Stacks of étale germs

We fix a natural number n, and are interested in studying smooth pointed varieties of dimension n up to étale morphism. We will define the stack $\mathcal{M}_n^{(\infty)}$ classifying families of such varieties—in fact, we will first introduce the prestack $\left(\mathcal{M}_n^{(\infty)}\right)_{\text{triv}}$ classifying *trivial* n-dimensional pointed families, and then will define $\mathcal{M}_n^{(\infty)}$ to be its stackification. We will also introduce an intermediate prestack $\widetilde{\mathcal{M}_n^{(\infty)}}$, which has $\mathcal{M}_n^{(\infty)}$ as its stackification as well, but which is somewhat more manageable.

We begin in 1.1 with some preliminary definitions on n-dimensional families of varieties and common étale neighbourhoods between them. In 1.2 we discuss equivalence relations which can be imposed on common étale neighbourhoods so that they form the morphisms of a groupoid, and in 1.3 we use these ideas to define stacks of cth-order and étale germs of n-dimensional varieties. In 1.4 we consider the categories of quasi-coherent sheaves on these stacks. All of this material is related to the setting of universal $\mathcal{D}$-modules, but we conclude in 1.5 with some remarks about the *strict* analogues of these definitions, which will be necessary for the setting of universal $\mathcal{O}$-modules.

1.1 Families of pointed varieties and common étale neighbourhoods

Recall the following notion from Proposition I.2.5.6.

Definition 1.1.1. Given two smooth families $\pi_i : X_i \to S_i$ a *fibrewise étale* morphism between them is given by a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_X} & X_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ S_1 & \xrightarrow[f_S]{} & S_2 \end{array}$$

such that for any $s \in S_1$ with $s' := f_S(s) \in S_2$, the induced morphism on fibres $(X_1)_s \to (X_2)_{s'}$ is étale.

Notation 1.1.2. We will often use the subscripts X and S to distinguish between the two maps comprising a fibrewise étale morphism, even when neither of the smooth families involved is X/S.

We are interested in pointed n-dimensional varieties; in the relative setting this is formalised as follows:

Definition 1.1.3. Fix a base scheme $S \in \mathrm{Sch}$. A *pointed n-dimensional family* over S is a scheme X equipped with

- a morphism $\pi : X \to S$, smooth of relative dimension n; and
- a section $\sigma : S \to X$.

Notation 1.1.4. We shall denote such a family by $\pi : X \rightleftarrows S : \sigma$, but will often abbreviate to (π, σ) or $X \rightleftarrows S$ when there is no risk of confusion.

A particular n-dimensional family which will be of special importance to us is the *trivial* n-dimensional family $\mathbb{A}^n_S = S \times \mathbb{A}^n$ over S. We will often work with the *zero section* $z : S \to S \times \mathbb{A}^n$, induced by the inclusion of the origin in $\mathbb{A}^n$. Whenever we write $S \times \mathbb{A}^n \rightleftarrows S$ without specifying the maps, we will always mean the canonical projection and the zero section.

Another important pointed n-dimensional family is the following: let $X \to S$ be any smooth family of relative dimension n, and consider the pointed family

$$\mathrm{pr}_1 : X \times_S X \rightleftarrows X : \Delta. \tag{II.1}$$

We think of this as the universal pointed family over X. Whenever we write $X \times_S X \rightleftarrows X$ without specifying the maps, we will always mean the projection onto the first factor and the diagonal embedding.

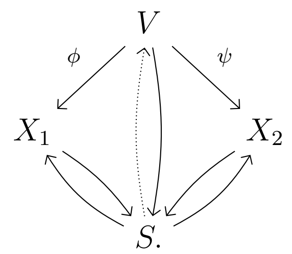
We want to define a groupoid of pointed n-dimensional families over a fixed base scheme S up to étale morphism. When $S = \operatorname{Spec} k$ is a point, we have the notion of a common étale neighbourhood of pointed varieties, which plays the role of isomorphism in the groupoid, and we generalise this notion to the relative setting as follows:

Definition 1.1.5. Let $\pi_i : X_i \rightleftarrows S : \sigma_i$ $(i = 1, 2)$ be smooth n-dimensional families over S. A *common étale neighbourhood* is given by a third pointed n-dimensional family $(\rho : V \rightleftarrows S : \tau)$ together with a pair of étale maps $(\phi : V \to X_1, \psi : V \to X_2)$ such that ϕ and ψ are compatible with the projections, and furthermore are compatible with the sections *on the level of reduced schemes*. That is, we require

1. $\pi_1 \circ \phi = \rho = \pi_2 \circ \psi$

2. $\sigma_1 \circ \iota_S = \phi \circ \tau \circ \iota_S, \qquad \sigma_2 \circ \iota_S = \psi \circ \tau \circ \iota_S,$

where $\iota_S : S_{red} \hookrightarrow S$ denotes the canonical closed embedding. Diagrammatically, we depict this common étale neighbourhood as follows, where the section τ is denoted by a dotted line to remind us that it is only compatible with the sections σ_i on the reduced part of S:



Notation 1.1.6. We will denote a common étale neighbourhood by (V, ϕ, ψ); when no confusion will result, we may use the notation (ϕ, ψ) or simply V.

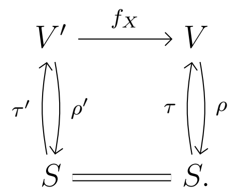
Definition 1.1.7. In the case that the diagram is actually commutative, and not just up to precomposing with ι_S, we will say that the common étale neighbourhood is *strict*, and will use a solid rather than a dotted line for the section $S \to V$.

Remark 1.1.8. We can also introduce another variation on the definition of common étale neighbourhood: rather than requiring the middle family to live over S, we allow smooth families over any scheme T equipped with an étale morphism to S which is compatible with the projections. There are both strict and non-strict versions of such *étale-locally-defined common étale neighbourhoods.*

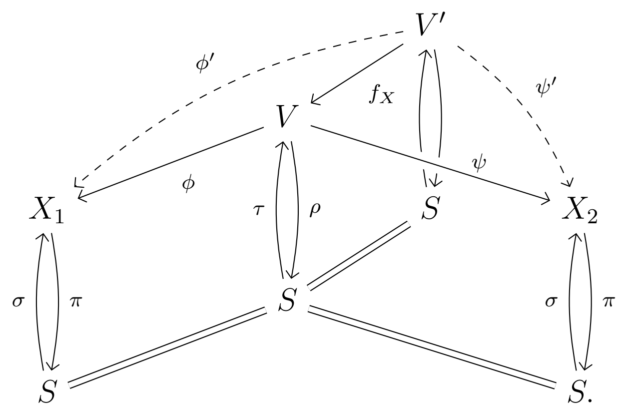
1.2 Groupoids of common étale neighbourhoods

Our goal is to define a groupoid $\widetilde{\mathcal{M}_n^{(\infty)}}(S)$ for each scheme S, whose objects are pointed n-dimensional families over S, and whose morphisms are represented by common étale neighbourhoods. In order to do this, we need to impose an equivalence relation on common étale neighbourhoods, so that the composition of morphisms is well-defined and associative, and the morphisms are invertible.

Moreover, we expect that restricting a common étale neighbourhood by pulling back along another étale morphism should not change the corresponding morphism in our groupoid. More formally, let (V, ϕ, ψ) be a common étale neighbourhood between $X_1 \rightleftarrows S$ and $X_2 \rightleftarrows S$, and suppose that we have a pointed n-dimensional S-family V' étale over V:



Then this yields a second common étale neighbourhood $(V', \phi \circ f_X, \psi \circ f_X)$:



Motivated by this, we introduce the following equivalence relation:

Definition 1.2.1. We will say that two common étale neighbourhoods (V_i, ϕ_i, ψ_i) between $X_1 \rightleftarrows S$ and $X_2 \rightleftarrows S$ are *similar* if there exists a pointed family $W \rightleftarrows S$ and étale maps $f_i : W/S \to V_i/S$ compatible with the sections on the level of S_{red}, such that

$$\phi_1 \circ f_1 = \phi_2 \circ f_2;$$
$$\psi_1 \circ f_1 = \psi_2 \circ f_2.$$

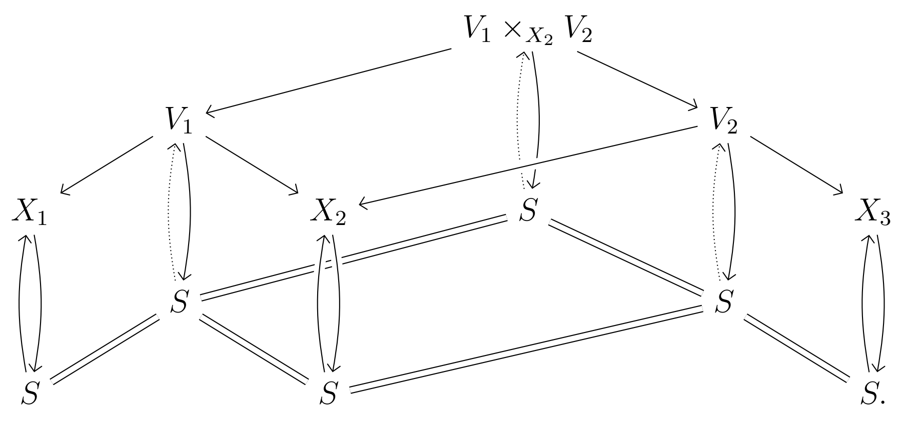
This is an equivalence relation, but it is slightly too restrictive for our purposes. We modify it as follows:

Definition 1.2.2. We will say that two common étale neighbourhoods (V_i, ϕ_i, ψ_i) are (∞)-*equivalent* if for each $s \in S$ there is a Zariski open neighbourhood S' of s such that the restrictions of (V_i, ϕ_i, ψ_i) to S' give similar common étale neighbourhoods between $X_1 \times_S S' \rightleftarrows S'$ and $X_2 \times_S S' \rightleftarrows S'$.

This equivalence relation is exactly what we need to define a groupoid structure. Given two common étale neighbourhoods

$$(V_i/S, \phi_i, \psi_i),\ i = 1, 2,$$

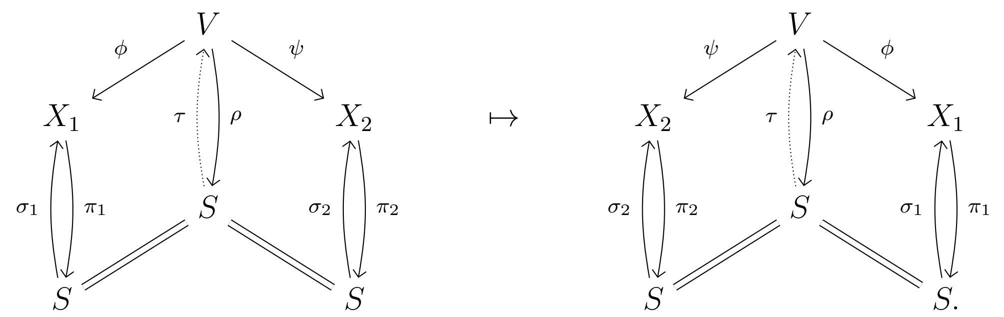
representing morphisms $X_1/S \to X_2/S \to X_3/S$, we would like to define their composition using the fibre product $V_1 \times_{X_2} V_2$, but it requires a little care to show that this is well-defined. It is clear that this is a smooth scheme of relative dimension n over S, but what is not immediate is the existence of a suitable section. However, we can define a map $S_{red} \to V_1 \times_{X_2} V_2$ using the compatibility of the sections $\tau_1 \circ \iota_S$ and $\tau_2 \circ \iota_S$; this map then extends to the desired section using formal smoothness of $V_1 \times_{X_2} V_2 \to S$. Although the choice of extension is not unique, any two choices differ only up to nilpotence, so the resulting common étale neighbourhoods will be equivalent. Therefore, the composition of the morphisms $X_1/S \to X_2/S \to X_3/S$ is indeed represented by the pullback:



It is not hard to check that this is associative.

Next, given $X \rightleftarrows S$, it is straightforward to check that the identity morphism $\mathrm{id}_{X \rightleftarrows S}$ is simply represented by $(X, \mathrm{id}_X, \mathrm{id}_X)$. Moreover, any symmetric common étale neighbourhood (V, ϕ, ϕ) is equivalent to the family $(X, \mathrm{id}_X, \mathrm{id}_X)$ and hence also represents the identity morphism.

Finally, given a common étale neighbourhood, its inverse is represented by the mirror image diagram:



Indeed, to show that the composition of this common étale neighbourhood with its mirror image represents the identity morphism, we need only remark that

$$\Delta : V \to V \times_{X_2} V$$

is an open embedding (because ψ is unramified and locally of finite type), and in particular is étale. Pulling back the composition along Δ gives (V, ϕ, ϕ), which is equivalent to the identity common étale neighbourhood.

We have proved the following:

Proposition 1.2.3. *Under the* (∞)*-equivalence relation and with the composition and inverses described above,* $\widetilde{\mathcal{M}_n}^{(\infty)}(S)$ *is a groupoid.*

Remark 1.2.4. As we will see in Lemma 3.4.3 and Proposition 3.4.4, two common étale neighbourhoods (V_i, ϕ_i, ψ_i) $(i = 1, 2)$ are (∞)-equivalent precisely when they induce the same isomorphism of the formal neighbourhoods of S in the schemes X_1 and X_2:

$$\hat{\psi}_1 \circ \hat{\phi}_1^{-1} = \hat{\psi}_2 \circ \hat{\phi}_2^{-1}.$$

Motivated by this observation, we introduce a family of coarser equivalence relations:

Definition 1.2.5. Let $c \in \mathbb{N}$. Two common étale neighbourhoods (V_i, ϕ_i, ψ_i) are (c)*-equivalent* if they induce the same isomorphisms on the cth infinitesimal neighbourhoods of S in X_1 and X_2:

$$\psi_1^{(c)} \circ \left(\phi_1^{(c)}\right)^{-1} = \psi_2^{(c)} \circ \left(\phi_2^{(c)}\right)^{-1} : X_1^{(c)} \xrightarrow{\sim} X_2^{(c)}.$$

Then we let $\widetilde{\mathcal{M}_n^{(c)}}(S)$ be the groupoid whose objects are pointed n-dimensional families over S and whose morphisms are common étale neighbourhoods up to (c)-equivalence.

Since (c)-equivalence is coarser than $(c+1)$-equivalence for any c, and also than (∞)-equivalence, we obtain morphisms of groupoids

$$\widetilde{\mathcal{M}_n^{(\infty)}}(S) \to \ldots \to \widetilde{\mathcal{M}_n^{(c+1)}}(S) \to \widetilde{\mathcal{M}_n^{(c)}}(S) \to \ldots.$$

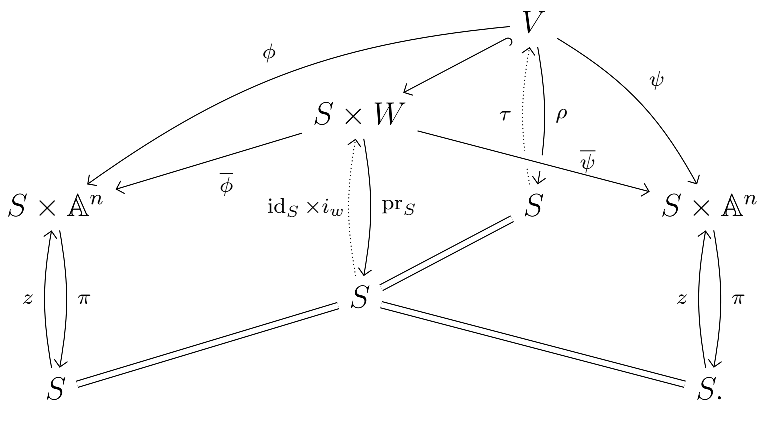
1.3 Stacks of étale and cth-order germs of varieties

With these preliminary notions and definitions established, we can define the prestacks of germs of varieties as follows:

Definition 1.3.1. Given $c \in \mathbb{N} \cup \{\infty\}$, let $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}$ be the prestack that sends a test scheme S to the groupoid whose only object is the trivial pointed n-dimensional variety $\pi : S \times \mathbb{A}^n \rightleftarrows S : z$, and whose automorphisms are given by common étale neighbourhoods of $S \times \mathbb{A}^n$ with itself, modulo (c)-equivalence.

There is a distinguished class of common étale neighbourhoods of $S \times \mathbb{A}^n$, characterised as follows:

Definition 1.3.2. A common étale neighbourhood $V = (V, \phi, \psi)$ between the trivial pointed family and itself will be called *split* if there exists an n-dimensional variety W together with maps $\overline{\phi}, \overline{\psi} : S \times W \to S \times \mathbb{A}^n$ (not necessarily étale), an open embedding $V \hookrightarrow S \times W$, and a point $w \in W$ such that the following diagram commutes:



Remark 1.3.3. Split common étale neighbourhoods can be simpler to work with, and will arise in our discussion of the Artin approximation theorem. Fortunately we will see in Lemma 3.4.2 that all common étale neighbourhoods of the trivial pointed

variety are (∞)-equivalent (and hence (c)-equivalent, for any c) to a common étale neighbourhood which is split.

Definition 1.3.4. Let $\mathcal{M}_n^{(c)}$ be the stackification of $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}$ in the étale topology. When $c = \infty$, we call $\mathcal{M}_n^{(c)}$ the *stack of étale germs of n-dimensional varieties*. For finite c, we call $\mathcal{M}_n^{(c)}$ the *stack of cth-order germs of n-dimensional varieties.*

In fact we will find it convenient to work with the intermediate prestack $\widetilde{\mathcal{M}_n^{(c)}}$, which lies in between $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}$ and its stackification $\mathcal{M}_n^{(c)}$.

Definition 1.3.5. For $c \in \mathbb{N} \cup \{\infty\}$, let $\widetilde{\mathcal{M}_n^{(c)}}$ be the subprestack of $\mathcal{M}_n^{(c)}$ sending a test scheme S to the subgroupoid $\widetilde{\mathcal{M}_n^{(c)}}(S)$ of $\mathcal{M}_n^{(c)}(S)$ defined above. Its objects are pointed n-dimensional varieties over S and its morphisms are represented by common étale neighbourhoods up to (c)-equivalence.

We see that this gives a prestack whose stackification is $\mathcal{M}_n^{(c)}$: indeed, when constructing the stackification of $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}$ explicitly, as in [1, Tag 02ZM], we must add in locally defined objects, which include all of the additional objects of $\widetilde{\mathcal{M}_n^{(c)}}(S)$; we must also add in all of the locally defined morphisms between these new objects, and hence in particular all of the morphisms of $\widetilde{\mathcal{M}_n^{(c)}}(S)$. The next stage in constructing the stackification is to identify all morphisms which agree locally; however, this has already been done in $\widetilde{\mathcal{M}_n^{(c)}}(S)$ by our definition of (c)-equivalence. It follows that we can view $\widetilde{\mathcal{M}_n^{(c)}}(S)$ as a (non-full) sub-groupoid of $\mathcal{M}_n^{(c)}(S)$, and hence by the universal property, we obtain a map from the stackification of $\widetilde{\mathcal{M}_n^{(c)}}$ into $\mathcal{M}_n^{(c)}$. The quasi-inverse to this map is induced by the obvious inclusion of $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}$ into $\widetilde{\mathcal{M}_n^{(c)}}$.

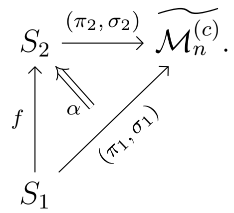
Remark 1.3.6. The crucial difference between the stack $\mathcal{M}_n^{(c)}$ and the prestack $\widetilde{\mathcal{M}_n^{(c)}}$ (and the reason that it is simpler to work with $\widetilde{\mathcal{M}_n^{(c)}}$) is that the groupoid $\mathcal{M}_n^{(c)}(S)$ contains isomorphisms represented by common étale neighbourhoods which are only defined étale-locally over the base as in Remark 1.1.8.

1.4 Quasi-coherent sheaves on $\mathcal{M}_n^{(c)}$

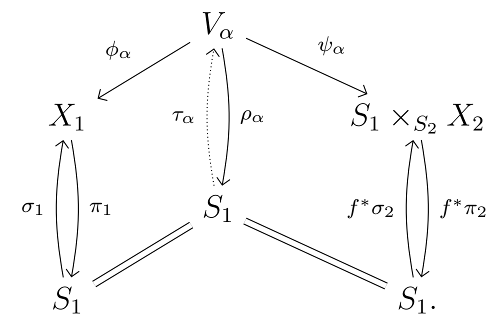
We will be interested in studying the categories of quasi-coherent sheaves on the stacks $\mathcal{M}_n^{(c)}$ for $c \in \mathbb{N} \cup \{\infty\}$. Since the categories of quasi-coherent sheaves on a prestack and its stackification are equivalent, we have the following equivalences:

$$\mathrm{QCoh}\left(\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}\right) \simeq \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right) \simeq \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right).$$

We will find it convenient to work in the realisation of the category given by $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right)$. (See Appendix A.2.1 for an overview of the theory of quasi-coherent sheaves on prestacks. For more details, see Gaitsgory's notes [14].) Concretely, an object M of $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right)$ consists of a collection of quasi-coherent sheaves together with coherences: for each map $S \to \widetilde{\mathcal{M}_n^{(c)}}$ (i.e. for each $X \rightleftarrows S$ smooth of relative dimension n), we have an object $M_{X \rightleftarrows S} \in \mathrm{QCoh}(S)$. Moreover, we require compatibility under pullbacks in the following sense. Suppose that for $i = 1, 2$ we have $S_i \xrightarrow{(\pi_i, \sigma_i)} \widetilde{\mathcal{M}_n^{(c)}}$, two n-dimensional families, together with a map $f : S_1 \to S_2$ and a commutative diagram of prestacks:



Recall that in PreStk, commutativity of a diagram is a structure, not a property, in this case amounting to an automorphism α in $\widetilde{\mathcal{M}_n^{(c)}}(S_1)$ between the objects corresponding to (π_1, σ_1) and $(\pi_2, \sigma_2) \circ f$, represented by a common étale neighbourhood of the form



We require that in such a situation, we have an isomorphism

$$M(f, \alpha) : f^*\left(M_{X_2 \rightleftarrows S_2}\right) \xrightarrow{\sim} M_{X_1 \rightleftarrows S_1}$$

in $\mathrm{QCoh}(S_1)$. This isomorphism must be independent of the choice of representative $(V_\alpha, \phi_\alpha, \psi_\alpha)$ of the isomorphism α in $\widetilde{\mathcal{M}_n^{(c)}}(S_1)$. We also require that these isomorphisms be compatible with compositions $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$.

1.5 Strict analogues, for the $\mathcal{O}$-module setting

Let us introduce the following strict analogues, which will be important in the setting of universal $\mathcal{O}$-modules.

Definition 1.5.1. Fix $c \in \mathbb{N} \cup \{\infty\}$. Let $\left(\mathcal{M}_n^{\mathrm{pt},(c)}\right)_{\mathrm{triv}}$ be the prestack that sends a test scheme S to the groupoid whose only object is the trivial pointed n-dimensional variety $\pi : S \times \mathbb{A}^n \rightleftarrows S : z$, and whose automorphisms are given by strict common étale neighbourhoods of $S \times \mathbb{A}^n$ with itself, up to (c)-equivalence.

Remark 1.5.2. In the case $c = \infty$, one might be tempted to consider a *strict* version of (∞)-equivalence, defined in the obvious way. It is straightforward to check that two strict common étale neighbourhoods are (∞)-equivalent if and only if they are strictly (∞)-equivalent, so in fact it is not necessary to introduce this latter notion.

Definition 1.5.3. Let $\mathcal{M}_n^{\mathrm{pt},(c)}$ be the stackification of $\left(\mathcal{M}_n^{\mathrm{pt},(c)}\right)_{\mathrm{triv}}$ in the étale topology. When $c = \infty$, we call $\mathcal{M}_n^{\mathrm{pt},(c)}$ the *stack of pointed étale germs of n-dimensional varieties*; when c is finite, $\mathcal{M}_n^{\mathrm{pt},(c)}$ is the *stack of pointed cth-order germs of n-dimensional varieties.*

As in the non-strict setting, we will also work with an intermediate prestack $\widetilde{\mathcal{M}_n^{\mathrm{pt},(c)}}$, lying in between the prestack $\left(\mathcal{M}_n^{\mathrm{pt},(c)}\right)_{\mathrm{triv}}$ and its stackification. Namely, for a given test scheme S, an object of the groupoid $\widetilde{\mathcal{M}_n^{\mathrm{pt},(c)}}(S)$ is a pointed n-dimensional family over S, $\pi : X \rightleftarrows S : \sigma$. Given two such pointed families, a morphism between them is represented by a strict common étale neighbourhood (V, ϕ, ψ), modulo (c)-equivalence. Similarly to in the groupoid $\widetilde{\mathcal{M}_n^{(c)}}(S)$, composition is given by pullback and inverses are given by mirror-image diagrams.

The difference between the strict and non-strict definitions lies in whether we require morphisms to preserve the distinguished points of the n-dimensional varieties (in the strict setting), or allow infinitesimal translations (in the non-strict setting). As we will see in section 5, this is what gives quasi-coherent sheaves on $\mathcal{M}_n^{\mathrm{pt},(c)}$ the additional structure of an action of the sheaf of differential operators.

2 Groups of automorphisms and their classifying stacks

In this section, we introduce certain groups G and K of automorphisms of the formal disc. We begin in 2.1 by defining the group formal scheme G and its finite-dimensional quotients $G^{(c)}$; in 2.2 we introduce the reduced part $K = G_{red}$. It is a pro-algebraic group, and contains a pro-unipotent subgroup K_u. In 2.3 we give some general definitions and facts regarding representations of group-valued prestacks and classifying stacks, and in 2.4 we apply these ideas to the groups G and K. We also begin the comparison of the classifying stacks BG and BK with the stacks $\mathcal{M}_n^{(\infty)}$ and $\mathcal{M}_n^{\mathrm{pt},(\infty)}$, which will be the motivation for the next several sections.

2.1 The group G of continuous automorphisms of the formal disc

Definition 2.1.1. Let $\hat{O}_n = k[\![t_1, \dots, t_n]\!]$, and let $G = \underline{\mathrm{Aut}}\hat{O}_n$ be the ind-affine group formal scheme of continuous automorphisms of $\hat{O}_n$. Explicitly, for $S = \mathrm{Spec}(R)$, $G(S)$ is the group of automorphisms of the R-algebra $R[\![t_1, \dots, t_n]\!]$, continuous with respect to the topology corresponding to the ideal $\mathfrak{m}$ generated by $(t_1, \dots, t_n)$.

A continuous homomorphism $\rho : R[\![t_1, \dots, t_n]\!] \to R[\![t_1, \dots, t_n]\!]$ is determined by its values on the topological generators $t_1, \dots, t_n$. Given a multi-index $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$, let us denote by r^k_J the coefficient of $\underline{t}^J = t_1^{j_1} \cdots t_n^{j_n}$ in the series $\rho(t_k) \in R[\![t_1, \dots, t_n]\!]$. For $k' \in \{1, \dots, n\}$, let $e_{k'} = (0, \dots, 0, 1, 0, \dots, 0)$ be the multi-index with 1 only in the k'th place. With this notation,

$$\rho : t_k \mapsto r^k_{\underline{0}} + \sum_{k'=1}^{n} r^k_{e_{k'}} t_{k'} + \text{ higher order terms.} \tag{II.2}$$

The condition that this determines a continuous homomorphism is equivalent to requiring each $r^k_{\underline{0}}$ to be a nilpotent element of R. Then the homomorphism ρ determined by the equations (II.2) is invertible precisely when the matrix $(r^k_{e_{k'}})_{k,k'} \in M_n(R)$ is invertible.

This allows us to describe the indscheme structure of G explicitly:

$$G = \operatorname*{colim}_{N \in \mathbb{N}} \mathrm{Spec}\left(k[a^k_J, (\det (a^k_{e_{k'}})_{k,k'})^{-1}]/((a^k_{\underline{0}})^N)\right).$$

Definition 2.1.2. Given $c \in \mathbb{N}$, we can also consider $G^{(c)}$, the group formal scheme of continuous automorphisms of $\hat{O}_n/\mathfrak{m}^{c+1}$:

$$G^{(c)} = \operatorname*{colim}_{N \in \mathbb{N}} \mathrm{Spec}\left(k[a^k_J, (\det (a^k_{e_{k'}})_{k,k'})^{-1}]_{|J| \leq c}/((a^k_{\underline{0}})^N)\right),$$

where

$$|J| := \sum_{i=1}^{n} j_i.$$

Then $G^{(c)}$ is an indscheme of finite type and a quotient of G, and we can express G as the limit

$$G = \lim_{c \in \mathbb{N}} G^{(c)}.$$

That is, G is a pro-object in the category of ind-affine group formal schemes.

Example 2.1.3. It may be useful to keep in mind the notationally simpler one-dimensional setting. When $n = 1$, an automorphism $\rho : R[\![t]\!] \to R[\![t]\!]$ is determined by its value on the single generator t:

$$\rho : t \mapsto r_0 + r_1 t + r_2 t^2 + \dots,$$

where $r_0 \in \mathrm{Nil}(R)$ and $r_1 \in R^\times$.

The indscheme G is the colimit (of schemes of infinite type)

$$G = \operatorname*{colim}_{N \in \mathbb{N}} \operatorname{Spec} k[a_0, a_1, a_1^{-1}, a_2, a_3, \dots]/(a_0^N).$$

On the other hand, it is also the limit of the indschemes $G^{(c)}$ of finite type, where

$$G^{(c)} = \operatorname*{colim}_{N \in \mathbb{N}} \operatorname{Spec} k[a_0, a_1, a_1^{-1}, a_2, a_3, \dots, a_c]/(a_0^N).$$

The quotient maps $G^{(c)} \to G^{(c-1)}$ correspond to the inclusions

$$k[a_0, a_1, a_1^{-1}, a_2, a_3, \dots, a_{c-1}]/(a_0^N) \hookrightarrow k[a_0, a_1, a_1^{-1}, a_2, a_3, \dots, a_c]/(a_0^N)$$
$$a_i \mapsto a_i.$$

They are clearly smooth of dimension 1.

Note also that it is easy to see from this example that a continuous homomorphism $\rho : R[\![t]\!] \to R[\![t]\!]$ always descends to give a homomorphism $\rho^{(c)} : R[t]/\mathfrak{m}^{c+1} \to R[t]/\mathfrak{m}^{c+1}$. Moreover, ρ is invertible if and only if $\rho^{(c)}$ is invertible for some (or equivalently for all) $c \geq 1$, because this is a condition on the coefficients of the degree 0 and 1 terms only. This is true for $n > 1$ as well, for the same reasons.

2.2 The reduced part $K = G_{red}$

Definition 2.2.1. Let $K = G_{red}$ denote the reduced part of the indscheme G:

$$K = \operatorname{Spec} k[a_J^k, (\det{(a_{e_{k'}}^k)_{k,k'}})^{-1}]_{|J|>0}.$$

It is an affine group scheme of infinite type. Geometrically, ($\operatorname{Spec} R$)-points of K correspond to continuous automorphisms $\rho : R[[t_1, \ldots, t_n]] \to R[[t_1, \ldots, t_n]]$ such that the constant term of each series $\rho(t_k)$ is zero. We think of K as parametrising automorphisms of the formal disc $\operatorname{Spf} k[[t_1, \ldots, t_n]]$ which fix the origin 0, whereas the automorphisms parametrised by the larger group G may involve infinitesimal translations of 0.

We view K as a *pro-algebraic group*: it has finite-dimensional quotients $K^{(c)}$, parametrising automorphisms of $k[[t_1, \ldots, t_n]]/\mathfrak{m}^c$ which preserve the origin.

Definition 2.2.2. Explicitly, $K^{(c)}$ is the algebraic group

$$K^{(c)} = \operatorname{Spec} k[a_J^k, (\det (a_{e_{k'}}^k)_{k,k'})^{-1}]_{0<|J|<c+1},$$

where the group structure comes from composition of the automorphisms ρ.

Note that we have obvious maps

$$K, K^c \to GL_n,$$

where the map on ($\operatorname{Spec} R$)-points sends an automorphism ρ to the matrix

$$(r_{e_{k'}}^k)_{k,k'} \in GL_n(R),$$

in the notation of (II.2). These are homomorphisms of affine group schemes. (Notice that we might try to define a similar map for the groups $G, G^{(c)}$, but that this no longer respects the group structure.)

Definition 2.2.3. Let K_u and $K_u^{(c)}$ denote the kernels of the homomorphisms of group schemes $K \to GL_n$ and $K^{(c)} \to GL_n$ respectively.

Then $K_u^{(c)}$ is a unipotent algebraic group, and $K_u = \lim_{c \in \mathbb{N}} K_u^{(c)}$ is a pro-unipotent group. We can write

$$K = GL_n \ltimes K_u, \qquad K^{(c)} = GL_n \ltimes K_u^{(c)};$$

this will be helpful in section 4 in understanding the representation theory of K.

Example 2.2.4. Let us again consider the case $n = 1$, where the notation is more pleasant. We have

$$K = \operatorname{Spec} k[a_1, a_1^{-1}, a_2, a_3, \ldots],$$
$$K^{(c)} = \operatorname{Spec} k[a_1, a_1^{-1}, a_2, a_3, \ldots, a_c].$$

The maps $K, K^{(c)} \to GL_1 = \mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$ are induced by the algebra homomorphisms given by

$$x \mapsto a_1.$$

The unipotent groups are given by

$$\begin{aligned} K_u &= \operatorname{Spec} k[a_2, a_3, \ldots], \\ K_u^{(c)} &= \operatorname{Spec} k[a_2, a_3, \ldots, a_c]. \end{aligned}$$

Let us consider the coalgebra structure on the algebra of functions $k[a_2, a_3, \ldots]$, induced by the composition of automorphisms $\rho, \sigma \in K_u(k)$. Suppose that

$$\begin{aligned} \rho &: t \mapsto t + r_2 t^2 + r_3 t^3 + \ldots, \\ \sigma &: t \mapsto t + s_2 t^2 + s_3 t^3 + \ldots \end{aligned}$$

Then

$$\begin{aligned} &\rho \circ \sigma : t \mapsto \\ &\qquad t + (r_2 + s_2)t^2 + (r_3 + 2r_2 s_2 + s_2)t^3 + (r_4 + 3r_3 s_2 + r_2 s_2^2 + 2r_2 s_3 + s_4)t^4 + \ldots \end{aligned}$$

From this we see that the comultiplication satisfies

$$\begin{aligned} a_2 &\mapsto a_2 \otimes 1 + 1 \otimes a_2, \\ a_3 &\mapsto a_3 \otimes 1 + 2a_2 \otimes a_2 + 1 \otimes a_3, \\ a_4 &\mapsto a_4 \otimes 1 + 3a_3 \otimes a_2 + a_2 \otimes a_2^2 + 2a_2 \otimes a_3 + 1 \otimes a_4, \end{aligned}$$

and so on.

The action of $\mathbb{G}_m$ on K_u (and similarly on $K_u^{(c)}$ for any c) induces a grading on the algebra of functions as follows: a k-point of $\mathbb{G}_m$ is of the form $z : t \mapsto zt$, for $z \in k^\times$. Conjugating $\rho \in K_u(k)$ by z gives

$$z \circ \rho \circ z^{-1} : t \mapsto t + z r_2 t^2 + z^2 r_3 t^3 + \ldots;$$

that is, the grading on $k[a_2, a_3, \ldots]$ is given by $\deg(a_j) = j - 1$.

Returning to the general setting ($n \geq 1$), note that the diagonal inclusion $\mathbb{G}_m \hookrightarrow GL_n$ results in a grading of the algebra of functions $k[a_J^k]_{|J|>1}$ of K_u (and again, similarly for $K_u^{(c)}$): we have $\deg(a_J^k) = |J| - 1$. It will be important for us that the grading is *non-negative*.

2.3 Representations and classifying stacks

Definition 2.3.1. By a *group-valued prestack*, we mean a functor

$$H : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Grp}.$$

The ordinary prestack underlying H is given by composing with the forgetful functor $\mathrm{Grp} \to \mathrm{Set}$ and the inclusion $\mathrm{Set} \to \infty{-}\mathrm{Grpd}$.

Let H be any group-valued prestack. We wish to consider the category $\mathrm{Rep}(H)$ of representations of H:

Definition 2.3.2. A *representation* of H on a k-vector space V is a morphism of group-valued functors

$$\mathbf{R} : H \to GL_V;$$

that is, for any $S = \operatorname{Spec} R$ we have

$$\mathbf{R}_R : H(S) \to GL(V \otimes_k R),$$

natural in R.

We can reformulate this definition in a more geometric manner as follows. Recall that given a group H we can define the prestack BH_{triv} classifying trivial principal H-bundles: for a test scheme S, $BH_{\mathrm{triv}}(S)$ is a groupoid containing only one object, the trivial bundle $S \times H \to S$. The automorphism group $\mathrm{Aut}_{BH_{\mathrm{triv}}(S)}(S \times H \to S)$ is the group $H(S)$.

Definition 2.3.3. The *classifying stack BH* of H is the stackification of the prestack BH_{triv} in the étale topology.

Remark 2.3.4. If H is an algebraic group, this is the usual classifying stack: that is, S-points of BH are principal H-bundles over S, and automorphisms are morphisms of H-bundles.

Then we have that

$$\mathrm{Rep}(H) \simeq \mathrm{QCoh}\left(BH_{\mathrm{triv}}\right) \simeq \mathrm{QCoh}\left(BH\right),$$

where the second equivalence is due to the fact that $\mathrm{QCoh}\left(\bullet\right)$ is preserved by stackification.

In the case that H is an affine group scheme, say $H = \operatorname{Spec} A$ with A a Hopf algebra, then the data of a representation of H on a vector space V is equivalent to the structure of an A-comodule on V:

$$V \to V \otimes_k A.$$

(See for example [31], Chapter VIII, Prop. 6.1.)

Observation 2.3.5. From this definition we can show that any representation V of an affine group scheme is *locally finite*: that is, every vector $v \in V$ is contained in some finite-dimensional sub-representation. (For example, see [31], Chapter VIII, Prop. 6.6.) This is not true of representations of more general group-valued prestacks, as we will see in section 4.3.

Now suppose that H is a pro-algebraic group, so that

$$H = \lim_i H_i,$$

where H_i runs over all finite-dimensional quotients of H. (The example we have in mind is of course the group K of section 2.2.) If we forget for the moment about the scheme structure on these groups, the pro-structure of H gives it a topology: a base for the open neighbourhoods of 1_H is given by the kernels N_i of the quotient maps $H \twoheadrightarrow H_i$.

We might be interested in restricting our attention to only those representations of H which are continuous with respect to this topology. If we give the vector space V the discrete topology, this amounts to requiring that for each $v \in V$, the action of H on v factors through one of the finite-dimensional quotients H_i, or equivalently, that V is the union of the subrepresentations V_i, where V_i is the largest subspace of V on which the action of H factors through H_i or on which the action of N_i is trivial.

If V is finite-dimensional to begin with, the group-valued prestack GL_V is also a finite-dimensional algebraic group, and so this condition is automatic. Combining this with Observation 2.3.5, we conclude that all representations of H are necessarily continuous with respect to the discrete topology on the underlying vector space. We have proven the following:

Proposition 2.3.6. *For $H = \lim_i H_i$ a pro-algebraic group,*

$$\operatorname{Rep}(H) \simeq \operatorname{colim}_i \operatorname{Rep}(H_i),$$

where the colimit is taken in the ∞-category of cocomplete categories.

Given a pro-algebraic group H we can always write it as the limit of its finite-dimensional quotients as above; however, as with our group $K = \lim_{c\in\mathbb{N}} K^{(c)}$ we can often restrict our attention to a subset of these algebraic quotients. View the collection of all finite-dimensional quotients $H_i = \operatorname{Spec} A_i$ as a category $\mathcal{I}$, whose morphisms are surjections compatible with the quotient maps from H, and suppose that we have a subcategory $\mathcal{J} \hookrightarrow \mathcal{I}$ such that

$$H \simeq \lim_{j\in\mathcal{J}} H_j.$$

Then for any $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ such that H_i is a quotient of H_j: indeed, we know that we have a surjection

$$\lim_{j\in\mathcal{J}} H_j \twoheadrightarrow H_i.$$

This amounts to an inclusion $A_i \hookrightarrow \bigcup_{j\in\mathcal{J}} A_j$. Since A_i is finitely generated and $\mathcal{J}^{\mathrm{op}}$ is filtered, we can find $j \in \mathcal{J}$ such that $A_i \hookrightarrow A_j$, which gives the desired surjection $H_j \twoheadrightarrow H_i$.

It follows that J^{op} is cofinal in I^{op} and in particular

$$\operatorname{Rep}(H) \simeq \operatorname*{colim}_{j\in\mathcal{J}^{\mathrm{op}}} \operatorname{Rep}(H_j).$$

2.4 Application to G and K

The following is immediate from the above discussion:

$$\begin{array}{ccc} \operatorname{Rep}(K) & \xrightarrow{\sim} & \operatorname{QCoh}(BK) \\ \uparrow\wr & & \uparrow\wr \\ \operatorname*{colim}_{c\in\mathbb{N}} \operatorname{Rep}\left(K^{(c)}\right) & \xrightarrow{\sim} & \operatorname*{colim}_{c\in BN} \operatorname{QCoh}\left(BK^{(c)}\right). \end{array}$$

Now we would like to make a similar comparison between the categories $\operatorname{Rep}(G)$ and $\operatorname{colim}_{c\in\mathbb{N}} \operatorname{Rep}(G^{(c)})$; we have the following:

Proposition 2.4.1. *All representations of the group-valued prestack* G *are continuous with respect to the the topology induced by the pro-structure of* G*:*

$$\operatorname*{colim}_{c\in\mathbb{N}} \operatorname{Rep}(G^{(c)}) \xrightarrow{\sim} \operatorname{Rep}(G),$$

where the colimit is taken in the ∞*-category of cocomplete categories.*

Proof. Let V be a vector space and let

$$\mathbf{R} : G \to GL_V$$

be a representation of G on V. Let $v \in V$; we want to show that the action of G on v factors through one of its finite-dimensional quotients $G^{(c)}$, i.e. that there exists some c such that for every $S = \operatorname{Spec} R$ the induced map

$$\begin{aligned} \mathbf{R}_{R,v} : G(S) &\to V \otimes_k R \\ \rho &\mapsto \mathbf{R}_R(\rho)(v \otimes 1_R) \end{aligned}$$

factors through the quotient $G^{(c)}(S)$.

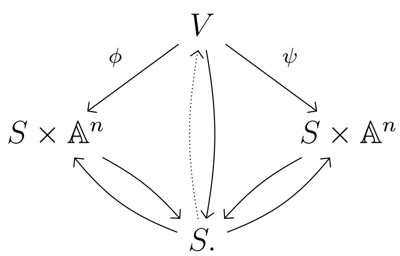
This is equivalent to showing that the restriction of $\mathbf{R}_{R,v}$ to the kernel $N_c(S)$ of the quotient map $G(S) \twoheadrightarrow G^{(c)}(S)$ is the constant map

$$n \mapsto v \otimes 1_R$$

for every $S = \operatorname{Spec} R$.

However, the embedding $K \hookrightarrow G$ allows us to view V as a representation of K; then by Proposition 2.3.6, there exists c such that the restriction of $\mathbf{R}_{R,v}$ to $K(S)$ factors through $K^{(c)}(S)$ for every $S = \operatorname{Spec} R$. This implies that the restriction of $\mathbf{R}_{R,v}$ to $\ker(K(S) \twoheadrightarrow K^{(c)}(S))$ is the constant map—but this kernel is exactly $N_c(S)$. □

Observation 2.4.2. Fix a base scheme $S = \operatorname{Spec}(R)$ and suppose that we have a common étale neighbourhood of the trivial family:



Taking completions along the embeddings of S, we obtain isomorphisms over S

$$\hat{\phi}, \hat{\psi} : V_S^\wedge \xrightarrow{\sim} S \times \hat{\mathbb{A}}^n,$$

and hence, composing, an isomorphism

$$\hat{\phi} \circ \hat{\psi}^{-1} : S \times \hat{\mathbb{A}}^n \to S \times \hat{\mathbb{A}}^n,$$

or equivalently, a continuous automorphism of $\operatorname{Spec}(R[\![t_1, \ldots, t_n]\!])$, i.e. an element ω_V of $G(S)$. Notice that ω_V lies in $K(S)$ precisely if the common étale neighbourhood is strict.

Motivated by this observation we formulate the following:

Proposition 2.4.3. *We have a natural morphism of prestacks:*

$$F^{(\infty)} : \left(\mathcal{M}_n^{(\infty)}\right)_{triv} \longrightarrow BG_{triv}.$$

Proof. On objects, we define $F_S^{(\infty)}(S \times \mathbb{A}^n \rightleftarrows S) := (S \times G \to S)$.

On morphisms, we would like to set $F_S^{(\infty)}([V, \phi, \psi]) := \omega_V$ as in the above discussion. We need to show that this is well-defined and respects composition; for both of these we will use that taking completions of morphisms respects composition.

First, suppose we have (V, ϕ, ψ), and $f_X : V'/S \to V/S$ étale giving rise to a second common étale neighbourhood $(V', \phi \circ f_X, \psi \circ f_X)$ similar to the first. Then we have

$$\begin{aligned} \widehat{\phi \circ f_X} \circ \widehat{\psi \circ f_X}^{-1} &= \hat{\phi} \circ \hat{f_X} \circ \hat{f_X}^{-1} \circ \hat{\psi}^{-1} \\ &= \hat{\phi} \circ \hat{\psi}^{-1}, \end{aligned}$$

and hence $\omega_{V'} = \omega_V$. Since this construction of pulling back along étale morphisms f_X generates the relation of similarity, it follows that any two common étale neighbourhoods which are similar will give rise to the same isomorphism on the completions. Since any two common étale neighbourhoods which are (∞)-equivalent are locally similar, the resulting isomorphisms of completions are locally equal, and hence equal. It follows that $F_S^{(\infty)}$ is well-defined.

Now suppose that we have morphisms $\mathbb{A}^n_S/S \to \mathbb{A}^n_S/S \to \mathbb{A}^n_S/S$ represented by two common étale neighbourhoods $(V_i/S, \phi_i, \psi_i),\ i = 1, 2$. Their composition is represented by the pullback $(V_1 \times_{\mathbb{A}^n} V_2, \phi_1 \circ \mathrm{pr}_{V_1}, \psi_2 \circ \mathrm{pr}_{V_2})$, and we have

$$\begin{aligned} \widehat{\phi_1 \circ \mathrm{pr}_{V_1}} \circ \widehat{\psi_2 \circ \mathrm{pr}_{V_2}}^{-1} &= \hat{\phi}_1 \circ \hat{\mathrm{pr}}_{V_1} \circ \hat{\mathrm{pr}}_{V_2}^{-1} \circ \hat{\psi}_2^{-1} \\ &= \hat{\phi}_1 \circ \hat{\psi}_1^{-1} \circ \hat{\phi}_2 \circ \hat{\psi}_2^{-1}. \end{aligned}$$

Therefore $\omega_{V_1 \times_{\mathbb{A}^n} V_2} = \omega_{V_2} \circ \omega_{V_1}$, and $F_S^{(\infty)}$ respects composition. (Note that the order of composition in $G(S)$ is the opposite of that in $\mathrm{Aut}_S(S \times \hat{\mathbb{A}}^n)$.) □

We also have the following:

Proposition 2.4.4. *Let $c \in \mathbb{N}$. Then we have a natural morphism of prestacks*

$$F^{(c)} : \left(\mathcal{M}_n^{(c)}\right)_{triv} \longrightarrow BG^{(c)}_{triv}.$$

Proof. This morphism is defined analogously to $F^{(\infty)}$; the proof that it is well-defined on morphisms is immediate from the definition of (c)-equivalence, and the proof that it respects composition of morphisms is as above. □

Restricting our attention to strict common étale neighbourhoods, we obtain the following analogous result:

Proposition 2.4.5. *For $c \in \mathbb{N}$, we have morphisms of prestacks*

$$F'^{(c)} : \left(\mathcal{M}_n^{pt,(c)}\right)_{triv} \longrightarrow BK_{triv}^{(c)}.$$

We also have a morphism

$$F' : \left(\mathcal{M}_n^{pt,(\infty)}\right)_{triv} \longrightarrow BK_{triv}.$$

Pulling back along the morphisms of Propositions 2.4.3 and 2.4.4 gives rise to functors

$$F^* : \mathrm{Rep}(G) \to \mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right),$$
$$F^{(c),*} : \mathrm{Rep}(G^{(c)}) \to \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right).$$

In the subsequent sections, we study these functors. We will show that for finite c, $F^{(c)}$ is an equivalence of prestacks and hence $F^{(c),*}$ is an equivalence of categories. On the other hand, we can show only that F^* is a fully faithful embedding, but we give various characterisations of its essential image in remark 3.4.6 and section 6.

Remark 2.4.6 (Remark on Harish-Chandra pairs)**.** Let $\mathfrak{g}$ denote the Lie algebra of G; it is equal to the Lie algebra $\mathrm{Der}\,\hat{O}_n$ of k-linear derivations of $\hat{O}_n$. The pair $(\mathfrak{g}, K = G_{red})$ forms a *Harish-Chandra pair* (see [4], 2.9.7): K is an affine group scheme; $\mathfrak{g}$ is a Lie algebra with a structure of Tate vector space; we have a continuous embedding $\mathrm{Lie}\,K \hookrightarrow \mathfrak{g}$ of Lie algebras with open image; and we have an action of K on $\mathfrak{g}$ which is compatible with the action of $\mathrm{Lie}\,K$ coming from the embedding.

Given a Harish-Chandra pair $(\mathfrak{g}, K)$, we consider the category of $(\mathfrak{g}, K)$-modules: these are algebraic (and hence, by our earlier discussion, discrete) representations V of K equipped with an action of $\mathfrak{g}$ which is compatible with the induced action of $\mathrm{Lie}\,K$. When $K = G_{red}$ as in our setting, this category is equivalent to the category of representations of G.

Thus, for G the group of automorphisms of the formal disc, the data of a representation of G on a vector space V is equivalent to the data of a representation of K on V together with a compatible action of $\mathfrak{g} = \mathrm{Der}\,\hat{O}_n$. This motivates one of the main results of this chapter: we will see that we can associate to a representation of K and a smooth n-dimensional variety X an $\mathcal{O}$-module $\mathscr{F}$ on X. If our representation is in addition a representation of G, this amounts to having a compatible action of $\operatorname{Der} \hat{O}_n$, which in turn gives rise to a $\mathcal{D}$-module structure on $\mathscr{F}$.

3 Relative Artin approximation

In this section, our goal is to show that for finite c the morphisms $F^{(c)}$ and $F'^{(c)}$ from Propositions 2.4.4 and 2.4.5 are in fact isomorphisms of prestacks, and hence that we have

$$BG^{(c)} \simeq \mathcal{M}_n^{(c)}, \qquad BK^{(c)} \simeq \mathcal{M}_n^{\mathrm{pt},(c)}.$$

It suffices to show that the group homomorphisms $F_S^{(c)}$ and $F_S'^{(c)}$ are bijective—that is, given an automorphism of $S \times \hat{\mathbb{A}}^n$ over S, we need to show that we can lift it to a common étale neighbourhood modulo (c)-equivalence; moreover we need to show that if the automorphism preserves the zero section $S \to S \times \hat{\mathbb{A}}^n$ then we can lift it to a *strict* common étale neighbourhood; and finally we need to show that in both cases the lifting is unique up to (c)-equivalence.

Remark 3.0.7. When $c = \infty$, the morphisms of prestacks are not isomorphisms: indeed, we will see that the corresponding group homomorphisms are injective, but not surjective. We will be able to use our understanding of these group homomorphisms to introduce yet another stack, the *stack of formal germs of n-dimensional varieties*, which will be isomorphic to BG. See remark 3.4.6.

In 3.1, we state the main result that we will need, which is a relative version of Artin's approximation theorem. The next two sections are devoted to the proof of this result: in 3.2 we recall some important technical definitions and results, and in 3.3 we apply them to prove our result. Finally, in 3.4 we show how the relative version of Artin's approximation theorem implies that the morphisms $F^{(c)}$ and $F'^{(c)}$ are isomorphisms.

3.1 Statement of the main result

In the case that $S = \operatorname{Spec} k$ is a point, the results that we need follow from a well-known result of Artin:

Theorem 3.1.1 (Corollary 2.6, [2])**.** *Let X_1, X_2 be schemes of finite type over k, and let $x_i \in X_i$ be points. Let $\mathfrak{m}_{x_1}$ denote the maximal ideal in the completed local ring $\widehat{\mathcal{O}}_{X_1,x_1}$, and suppose there is an isomorphism of the formal neighbourhoods*

$$\hat{\alpha} : (X_1)^\wedge_{x_1} \xrightarrow{\sim} (X_2)^\wedge_{x_2}$$

over k. Then X_1 and X_2 are étale locally isomorphic: i.e. there is a common étale neighbourhood (U, u) of $(X_i, x_i), i = 1, 2$, that is, a diagram

$$X_1 \xleftarrow{\ \phi\ } U \xrightarrow{\ \psi\ } X_2$$

with ϕ, ψ étale, such that $\phi(u) = x_1$ and $\psi(u) = x_2$.

Moreover, for any $c \in \mathbb{N}$, we can choose ϕ and ψ such that the resulting maps of completions satisfy

$$\hat{\psi} \circ \hat{\phi}^{-1} \equiv \hat{\alpha} \text{ (modulo } \mathfrak{m}_{x_1}^{c+1}\text{)}.$$

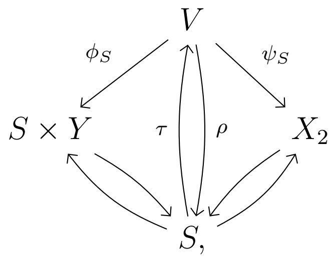
We are interested in the relative setting: $\pi_i : X_i \rightleftarrows S : \sigma_i$ $(i = 1, 2)$ are pointed n-dimensional families, and we ask when an isomorphism of the formal completions $(X_i)^\wedge_S$ can be lifted to an actual morphism of schemes, at least étale locally. We are not able to prove a relative version of Theorem 3.1.1 in full generality; however, we can show that it does hold when X_1 is a product $S \times Y$ for Y any n-dimensional k-variety, and σ_1 is a constant section. This suffices for the applications we have in mind.

Therefore let us fix S an affine scheme over k, and let Y be a smooth n-dimensional variety over k, with $y \in Y$ some fixed point. Then we can form a pointed n-dimensional family $\pi_1 : S \times Y \rightleftarrows S : \sigma_1$, where π_1 is the first projection, and $\sigma_1 = \mathrm{id}_S \times i_y$ is induced by the inclusion of the point y in Y. Let $\hat{Y} := Y^\wedge_y$ denote the completion of Y at the point y, and note that $(S \times Y)^\wedge_S \simeq S \times \hat{Y}$.

Proposition 3.1.2 (Relative Artin Approximation)**.** *Let $(\pi_2 : X_2 \rightleftarrows S : \sigma_2)$ be any pointed n-dimensional family, and suppose that we have an isomorphism $\hat{\alpha} : S \times \hat{Y} \xrightarrow{\sim} (X_2)^\wedge_S$ preserving both the projections to S and the embeddings of S:*

$$\begin{array}{ccc} S \times \hat{Y} & \xrightarrow[\hat{\alpha}]{\sim} & (X_2)^\wedge_S \\ & \searrow\!\nwarrow \quad \swarrow\!\nearrow & \\ & S. & \end{array}$$

Then there exists some affine étale neighbourhood $(U,u) \xrightarrow{\phi} (Y,y)$ *that gives a strict split common étale neighbourhood of the* S*-families of* n*-dimensional varieties as follows:*



(II.3)

where $V \subset S \times U$ *is a Zariski open subset containing* $S \times \{u\}$*,* ϕ_S *is the restriction of* $\mathrm{id}_S \times \phi$ *to* V*, and the section* $\tau : S \hookrightarrow V$ *is induced by the inclusion* i_u *of the point* u *in* U*.*

Furthermore, for any $c \in \mathbb{N}$ *this common étale neighbourhood can be chosen such that when we take completions along the closed embeddings of* S*,*

$$\hat{\psi_S} \circ \hat{\phi_S}^{-1} \equiv \hat{\alpha} \ (\textit{modulo } \mathfrak{m}_S^{c+1}). \tag{II.4}$$

(Here $\mathfrak{m}_S \subset \mathcal{O}_{X_1}$ *is the ideal sheaf corresponding to the closed embedding* $\sigma_1 : S \hookrightarrow X_1$*.)*

The proof is very similar to the original proof of Theorem 3.1.1 in [2]; however we will give the generalisation explicitly below, in particular to demonstrate the equality (II.4), which is only implicit in [2]. Both proofs rely on the notion of a functor locally of finite presentation, which we introduce in the next section.

3.2 Preliminary material

Definition 3.2.1. Let Y be a scheme of finite type over k. A functor

$$F : (\mathrm{Sch}_{/Y})^{\mathrm{op}} \to \mathrm{Set}$$

is said to be *locally of finite presentation* if it maps filtered limits of affine schemes over Y to colimits of sets. That is, if I is a filtered index category and $\{Y_i\}_{i \in I}$ is a diagram of affine schemes over Y, then

$$\operatorname*{colim}_{i \in I} F(Y_i) \simeq F(\lim_{i \in I} Y_i).$$

The following proposition gives a useful class of functors which are locally of finite presentation:

Proposition 3.2.2 (Proposition 2.3, [2]). *Let*

$$\begin{array}{ccc} Y_1 & & Y_2 \\ & \searrow \quad \swarrow & \\ & Z & \\ & \downarrow & \\ & X & \end{array}$$

be a diagram of schemes over X with Z quasi-compact and quasi-separated, and Y_i of finite presentation over Z ($i = 1, 2$). Let $\mathrm{Hom}_Z(Y_1, Y_2)$ *denote the functor:*

$$\begin{aligned} (Sch_{/X})^{op} &\to Set \\ T &\mapsto \mathrm{Hom}_{Z\times_X T}(Y_1 \times_X T, Y_2 \times_X T). \end{aligned}$$

This functor is locally of finite presentation.

Now we give a proposition illustrating the usefulness of functors locally of finite presentation.

Proposition 3.2.3 (Corollary 2.2, [2]). *Fix a base scheme Y over k, choose a point $y \in Y$, and let $\mathfrak{m}_y$ denote the maximal ideal of the completed local ring $\widehat{\mathcal{O}}_{Y,y}$. Let*

$$F : (Sch_{/Y})^{op} \to Set$$

be a contravariant functor locally of finite presentation, and assume we have $\hat{\xi} \in F(\hat{Y})$. Then for any $c \in \mathbb{N}$, there exists an étale neighbourhood (U, u) of y in Y, and an element $\xi' \in F(U)$ such that

$$\xi' \equiv \hat{\xi} \text{ (modulo } \mathfrak{m}_y^{c+1}). \tag{II.5}$$

Here the congruence (II.5) is interpreted as follows: since U is an étale neighbourhood of Y, we have a canonical morphism

$$\epsilon_U : \hat{Y} \to U,$$

inducing a function

$$F(U) \to F(\hat{Y}).$$

The content of (II.5) is that the images of ξ' and $\hat{\xi}$ agree after applying the canonical function

$$F(\hat{Y}) \to F(Y_y^{(c)}),$$

where $Y_y^{(c)}$ denotes the cth infinitesimal neighbourhood of y in Y.

With this result in mind, we can prove Proposition 3.1.2.

3.3 Proof of Proposition 3.1.2

Recalling the notation from the statement of the proposition, we define a functor

$$\begin{aligned} F :& (\mathrm{Sch}_{/Y})^{\mathrm{op}} \to \mathrm{Set} \\ & T \mapsto \mathrm{Hom}_S(S \times T, X_2). \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{Hom}_S(S \times T, X_2) &\simeq \mathrm{Hom}_{S\times T}(S \times T, X_2 \times T) \\ &\simeq \mathrm{Hom}_{(S\times Y)\times_Y T}\left((S \times Y) \times_Y T, (X_2 \times Y) \times_Y T\right). \end{aligned}$$

Therefore, applying Proposition 3.2.2 to the diagram

$$\begin{array}{ccc} S \times Y & & X_2 \times Y \\ & \searrow \quad \swarrow & \\ & S \times Y & \\ & \downarrow & \\ & Y, & \end{array}$$

we conclude that F is locally of finite presentation.

In particular, $F(\hat{Y}) = \mathrm{Hom}_S(S \times \hat{Y}, X_2)$, and we have an element $\hat{\xi} \in F(\hat{Y})$ given by the composition:

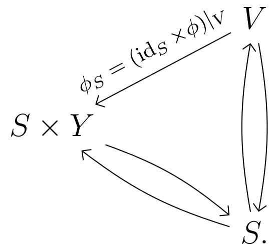
$$S \times \hat{Y} \xrightarrow{\hat{\alpha}} (X_2)^{\wedge}_S \xhookrightarrow{\epsilon_{X_2}} X_2.$$

Now we apply Proposition 3.2.3 and conclude that there exists an étale neighbourhood $\phi : (U, u) \to (Y, y)$ and an element $\xi' \in F(U)$ approximating $\hat{\xi}$ modulo $\mathfrak{m}_y^{c+1}$. The element ξ' corresponds to a diagram of S-schemes:

$$\begin{array}{ccc} S \times U & \xrightarrow{\quad \xi' \quad} & X_2 \\ & \searrow \quad \swarrow & \\ & S. & \end{array}$$

We can find an open neighbourhood V of $S \times \{u\}$ in $S \times U$ such that ξ' is étale on V: indeed, we know that ξ' induces an isomorphism $(S \times U)^{\wedge}_S \simeq S \times U^{\wedge}_u \xrightarrow{\sim} (X_2)^{\wedge}_S$ because it agrees with the isomorphism $\hat{\alpha}$ on the cth infinitesimal neighbourhood. Therefore, for each $s \in S$, ξ' induces an isomorphism $(S \times U)^{\wedge}_{(s,u)} \xrightarrow{\sim} (X_2)^{\wedge}_{(\xi'(s,u))}$, and so ξ' must be étale in some neighbourhood of (s, u), since ξ' is locally of finite presentation.

Having fixed such a neighbourhood V, we have a candidate for the left side of the diagram (II.3) immediately:

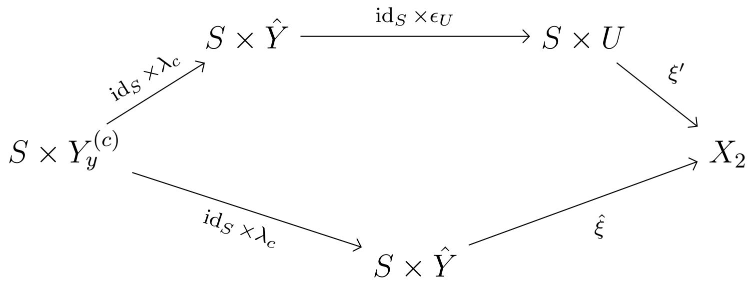


Indeed, it is clear that ϕ_S is étale and respects the sections and the projections.

To complete the right side of the diagram, it remains to show that the restriction ψ_S of ξ' to V commutes with the sections, i.e.

$$\xi' \circ (\mathrm{id}_S \times i_u) = \sigma_2.$$

Observe first that since (U, u) is an étale neighbourhood of (Y, y) we have a canonical morphism $\epsilon_U : \hat{Y} \to U$. Moreover, the fact that $\xi' \equiv \hat{\xi} \pmod{\mathfrak{m}^{c+1}}$ amounts to the commutativity of the following diagram:



Now the result follows easily, once we note that the inclusion $i_u : \mathrm{pt} \hookrightarrow U$ factors through the inclusion of the point y in its cth infinitesimal neighbourhood $Y_y^{(c)}$ and its formal neighbourhood $\hat{Y}$ via the map ϵ_U. Indeed, we have

$$\begin{aligned}
\xi' \circ (\mathrm{id}_S \times i_u) &= \xi' \circ (\mathrm{id}_S \times (\epsilon_U \circ \lambda_c)) \circ \left(\mathrm{id}_S \times i_y^{(c)}\right) \\
&= \hat{\xi} \circ (\mathrm{id}_S \times \lambda_c) \circ \left(\mathrm{id}_S \times i_y^{(c)}\right) \\
&= \hat{\xi} \circ \left(\mathrm{id}_S \times \hat{i_y}\right) \\
&= \epsilon_{X_2} \circ \hat{\alpha} \circ (\mathrm{id}_S \times \hat{i_y}) \\
&= \epsilon_{X_2} \circ \hat{\sigma}_2 \\
&= \sigma_2.
\end{aligned}$$

Finally, we have to check that $\hat{\psi}_S \circ \hat{\phi}_S^{-1} \equiv \hat{\alpha}$. Since V is open in $S \times U$, it suffices to show that $\hat{\xi}' \circ (\mathrm{id}_S \times \hat{\phi}^{-1}) \equiv \hat{\alpha}$. For this, we again use the compatibility of ξ' and

$\hat{\alpha}$, which tells us that the following two compositions are equal:

$$S \times Y_y^{(c)} \to S \times \hat{Y} \xrightarrow{\mathrm{id}_S \times \epsilon_U} S \times U \xrightarrow{\xi'} X_2$$

and

$$S \times Y_y^{(c)} \to S \times \hat{Y} \xrightarrow{\hat{\alpha}} (X_2)_S^\wedge \xrightarrow{\epsilon_{X_2}} X_2.$$

Taking completions along S, we obtain

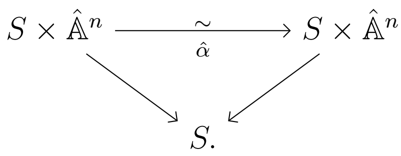
$$\hat{\xi'} \circ (\mathrm{id}_S \times \hat{\epsilon}_U) \equiv \hat{\alpha} \text{ modulo } \mathfrak{m}_S^{c+1}.$$

This completes the proof, because $\hat{\epsilon}_U = \hat{\phi}^{-1}$. □

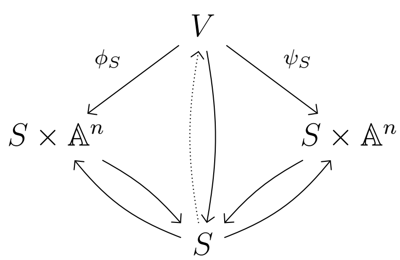
3.4 Applications of the relative Artin approximation theorem

We shall need this theorem in the following two instances:

Corollary 3.4.1. *1. Suppose we have an automorphism of $S \times \hat{\mathbb{A}}^n$ over S which does not necessarily preserve the section $\hat{z}$:*

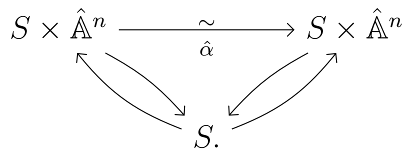


Then for any $c \in \mathbb{N}$ it can be lifted to a common étale neighbourhood



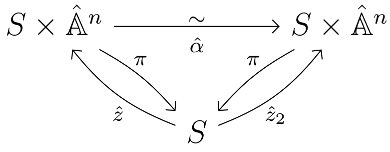
such that $\psi_S^{(c)} \circ (\phi_S^{(c)})^{-1} = \alpha^{(c)}$, as morphisms on the cth infinitesimal neighbourhoods.

2. Suppose we have an automorphism of $S \times \hat{\mathbb{A}}^n$ preserving the section $\hat{z}$:

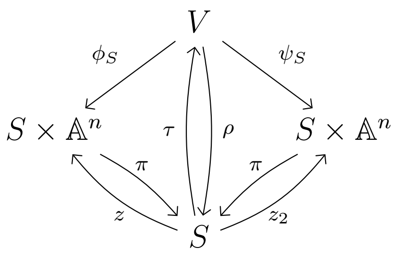


It can be lifted to a common étale neighbourhood as above which is also strict.

Proof. To prove the first part, we apply Proposition 3.1.2 to the following diagram:



where $z_2 = (\mathrm{id}_S \times \epsilon_{\hat{\mathbb{A}}^n}) \circ \hat{\alpha} \circ \hat{z}$. The diagram commutes by construction, and Proposition 3.1.2 yields a strict common étale neighbourhood:



such that $\psi_S^{(c)} \circ (\phi_S^{(c)})^{-1} = \alpha^{(c)}$. Since $\psi_S \circ \tau = z_2$ and $z_2 \circ \iota_S = z \circ \iota_S$, this gives a common étale neighbourhood of the trivial pointed n-dimensional family.

To prove the second part, notice that the additional assumption that $\hat{\alpha}$ preserves the section is equivalent to the statement that $z_2 = z$. It follows that the common étale neighbourhood is strict, as required. □

Applying Propositions 3.1.2, 3.2.2, and 3.2.3, we can also prove the following useful results:

Lemma 3.4.2. *Every common étale neighbourhood of the trivial pointed family over S is (∞)-equivalent (and hence (c)-equivalent for any $c \in \mathbb{N}$) to a split common étale neighbourhood.*

Proof. Let $\rho : V \rightleftarrows S : \tau$ be a pointed n-dimensional family with étale maps $\phi, \psi : V/S \to (S \times \mathbb{A}^n)/S$ giving a common étale neighbourhood. Using classical results on standard smooth and étale n-dimensional morphisms (see for example [30] 3.14), we can show that for every $s \in S$ there is a Zariski open neighbourhood T of s in S, and an open neighbourhood U of $\tau(s) \in \rho^{-1}(T)$ such that we have a commutative diagram as follows, with λ étale:

$$\begin{array}{ccccc} T \times \mathbb{A}^n & \xleftarrow{\lambda} & U & \hookrightarrow & V \\ {\scriptstyle z}\uparrow\downarrow{\scriptstyle \mathrm{pr}_T} & & {\scriptstyle \tau_{|T}}\uparrow\downarrow{\scriptstyle \rho_{|U}} & & {\scriptstyle \tau}\uparrow\downarrow{\scriptstyle \rho} \\ T & = & T & \hookrightarrow & S. \end{array}$$

By Proposition 3.1.2 we can lift $\hat{\lambda}^{-1} : T \times \hat{\mathbb{A}}^n \to U$: that is, we can find (W, w) étale over $(\mathbb{A}^n, 0)$ and $\lambda' : (T \times W)/T \to U/T$. Moreover, λ' is étale on some open neighbourhood V' of $T \times \{w\}$, and $(V', \phi \circ \lambda'_{|V'}, \psi \circ \lambda'_{|V'})$ is a split common étale neighbourhood, similar to the restriction of (V, ϕ, ψ) to T. □

Lemma 3.4.3. *Let (y, Y) be a pointed n-dimensional variety, and $\pi : X \rightleftarrows S : \sigma$ be a pointed n-dimensional family over S. Suppose that we have two morphisms $\phi, \psi : Y \times S \to X$ compatible with the projections, and compatible with the sections on S_{red}, such that $\hat{\phi} = \hat{\psi} : S \times \hat{Y} \xrightarrow{\sim} X^{\wedge}_S$.*

Then there exists some $f_X : U \to Y$ étale such that $\phi \circ (f_X, \mathrm{id}_S) = \psi \circ (f_X, \mathrm{id}_S)$.

In particular, the liftings provided by Corollary 3.4.1 are unique up to equivalence.

Proof. We apply Proposition 3.2.2 to the diagram

$$\begin{array}{ccc} Y \times S & & (Y \times S) \times_X (Y \times S) \\ & {\scriptstyle \Delta}\searrow \quad \swarrow{\scriptstyle (p_1, p_2)} & \\ & (Y \times S) \times (Y \times S) & \\ & \downarrow{\scriptstyle \mathrm{pr}_1^{YSYS}} & \\ & Y & \end{array}$$

and obtain that the functor

$$\begin{aligned} F : \mathrm{Sch}^{\mathrm{op}}_{/Y} &\to \mathrm{Set} \\ (T/Y) &\mapsto \mathrm{Hom}_{T \times_Y (Y \times S \times Y \times S)} (T \times_Y (Y \times S), T \times_Y (Y \times S) \times_X (Y \times S)) \end{aligned}$$

is locally of finite presentation. (Here Δ is the diagonal morphism, p_1 and p_2 are the projections from $(Y \times S) \times_X (Y \times S)$ to $Y \times S$ satisfying $\phi \circ p_1 = \psi \circ p_2$, and pr_1^{YSYS} is the projection onto the first Y factor.)

The fact that $\hat{\phi} = \hat{\psi}$ implies that $\phi \circ (\epsilon_Y, \mathrm{id}_S) = \psi \circ (\epsilon_Y, \mathrm{id}_S)$ as maps from $\hat{Y} \times S$ to X. Hence we obtain a map from $\hat{Y} \times S$ to $(Y \times S) \times_X (Y \times S)$ over Y, which finally gives us a map $\hat{Y} \times S \to \hat{Y} \times_Y (Y \times S) \times_X (Y \times S)$ corresponding to an element $\hat{\xi}$ of $F(\hat{Y})$.

Since F is locally of finite presentation, Proposition 3.2.3 applies, and we obtain an étale neighbourhood $f_X : (U, u) \to (Y, y)$ and an element $\xi' \in F(U)$ which agrees with $\hat{\xi}$ modulo $\mathfrak{m}_y^2$.

Now we remark that for any Y-scheme $f : T \to Y$, $F(T)$ is non-empty if and only if $\phi \circ (f, \mathrm{id}_S) = \psi \circ (f, \mathrm{id}_S)$ (and moreover, in that case $F(T)$ consists of a single point).

Indeed, $F(T)$ is a subset of $\mathrm{Hom}\,(T \times S, T \times_Y (Y \times S) \times_X (Y \times S))$. A map $\alpha : T \times S \to T \times_Y (Y \times S) \times_X (Y \times S)$ is given by three maps

$$\alpha_1 : T \times S \to T$$
$$\alpha_i : T \times S \to Y \times S, \; i = 2, 3,$$

satisfying

$$f \circ \alpha_1 = \mathrm{pr}_Y^{YS} \circ \alpha_2;$$
$$\phi \circ \alpha_2 = \psi \circ \alpha_3. \tag{II.6}$$

This α is an element of $F(T)$ if and only if it is compatible with the maps $T \times S \to T \times_Y (Y \times S \times Y \times S)$ and $T \times_Y (Y \times_S) \times_X (Y \times S)$, or equivalently if and only if

$$\alpha_1 = \mathrm{pr}_T^{TS};$$
$$\alpha_2 = (f, \mathrm{id}_S);$$
$$\alpha_3 = (f, \mathrm{id}_S).$$

Therefore, the only possible candidate for an element of $F(T)$ corresponds to the triple $(\mathrm{pr}_T^{TS}, (f, \mathrm{id}_S), (f, \mathrm{id}_S))$, which only gives a map α in the case that the equations (II.6) are satisfied. This amounts exactly to the condition $\phi \circ (f, \mathrm{id}_S) = \psi \circ (f, id_S)$.

It follows that the existence of $\xi' \in F(U)$ means that $f_X : U \to Y$ gives the desired étale neighbourhood. □

Combining Corollary 3.4.1 and Lemmas 3.4.2 and 3.4.3, we obtain

Proposition 3.4.4. *For any affine base-scheme S and any $c \in \mathbb{N} \cup \{\infty\}$, the group homomorphisms*

$$F_S^{(c)} : Aut_{\left(\left(\mathcal{M}_n^{(c)}\right)_{triv}(S)\right)}(S \times G^{(c)} \to S) \to G^{(c)}(S)$$
$$F_S'^{(c)} : Aut_{\left(\left(\mathcal{M}_n^{pt,(c)}\right)_{triv}(S)\right)}(S \times K^{(c)} \to S) \to K^{(c)}(S)$$

of Propositions 2.4.4 and 2.4.5 are injective.[2] *When c is finite, the homomorphisms are surjective as well.*

Proof. For finite c, injectivity follows immediately from the definition of (c)-equivalence, and Corollary 3.4.1 shows that the homomorphisms are surjective.

Now let $c = \infty$. Lemma 3.4.3 implies that the homomorphisms are injective when restricted to the set of automorphisms represented by split common étale neighbourhoods. By Lemma 3.4.2, this implies that they are injective. □

It follows that for $c \in \mathbb{N}$, $F^{(c)}$ and $F'^{(c)}$ give equivalences of prestacks, and using the uniqueness of stackification, we obtain the following:

Theorem 3.4.5. *Let $c \in \mathbb{N}$. We have isomorphisms of stacks:*

$$\mathcal{M}_n^{(c)} \xrightarrow{\sim} BG^{(c)}$$
$$\mathcal{M}_n^{pt,(c)} \xrightarrow{\sim} BK^{(c)}.$$

Remark 3.4.6. We also have morphisms

$$\mathcal{M}_n^{(\infty)} \to BG$$
$$\mathcal{M}_n^{\mathrm{pt},(\infty)} \to BK,$$

but they are not isomorphisms. There are two approaches to modify the stacks involved to obtain an equivalence: we can either enlarge the automorphism groups of the stacks on the left hand side, or we can restrict the automorphism groups of those on the right hand side.

1. Motivated by the above discussion, we see that we can define yet another stack of germs, this one equivalent to the classifying stack BG. It is the stackification of the prestack $(\mathcal{M}_n)_{\mathrm{triv}}$ which again sends a test scheme S to a groupoid whose only object is the trivial pointed n-dimensional family $S \times \mathbb{A}^n \rightleftarrows S$. However, we would like the morphisms of this groupoid to correspond to elements of $G(S)$, i.e. automorphisms $\hat{\alpha}$ of $S \times \hat{\mathbb{A}}^n$. Proposition 3.1.2 tells us that we can represent such an automorphism by a *sequence* of common étale neighbourhoods $\{(U_c, \phi_c, \psi_c)\}_{c=1}^{\infty}$ such that for each c,

$$\psi_S^{(c)} \circ (\phi_S^{(c)})^{-1} = \alpha^{(c)}.$$

[2] By a slight abuse of notation, we understand $K^{(\infty)}$ and $G^{(\infty)}$ to mean K and G respectively.

It follows that (U_c, ϕ_c, ψ_c) is uniquely determined up to (c)-equivalence, i.e. as a morphism in $\left(\mathcal{M}_n^{(c)}\right)_{\text{triv}}(S)$. Moreover, the sequence $\{(U_c, \phi_c, \psi_c)\}_{c=1}^{\infty}$ determines $\hat{\alpha}$.

That is, we should define $\mathcal{M}_n$ to be

$$\lim_{c\in\mathbb{N}} \mathcal{M}_n^{(c)}.$$

We will call this the *stack of formal germs of n-dimensional varieties.* We can similarly define

$$\mathcal{M}_n^{\text{pt}} := \lim_{c\in\mathbb{N}} \mathcal{M}_n^{\text{pt},(c)}.$$

2. Let $G^{\text{ét}}$ be the group-valued prestack sending a test scheme S to the image of $\text{Aut}_{\left(\mathcal{M}_n^{(\infty)}\right)_{\text{triv}}}(S \times \mathbb{A}^n \rightleftarrows S)$ in $G(S)$ under F_S: i.e. this is the group of all automorphisms of the formal disc which can be lifted precisely to common étale neighbourhoods. Then we have that

$$\mathcal{M}_n^{(\infty)} \xrightarrow{\sim} BG^{\text{ét}}.$$

It follows from Corollary 3.4.1 that $G^{\text{ét}}$ is dense in G, and hence we will be able to show that restriction from G to $G^{\text{ét}}$ gives a fully faithful embedding $\text{Res}_{G,G^{\text{ét}}} : \text{Rep}(G) \hookrightarrow \text{Rep}(G^{\text{ét}})$ (see Corollary 4.4.3).

Similarly, we can define a sub-group $K^{\text{ét}}$ of K such that

$$\mathcal{M}_n^{\text{pt},(\infty)} \xrightarrow{\sim} BK^{\text{ét}}.$$

In section 4 we study the representation theory of these group-valued prestacks $G^{\text{ét}}$ and $K^{\text{ét}}$.

4 Groups of étale automorphisms and their representation theory

This section is about the top rows of the main diagram (Figure 1), when $c = \infty$. We have the following stacks

$$\mathcal{M}_n^{(\infty)} \xrightarrow{\sim} BG^{\text{ét}} \hookrightarrow BG,$$

giving rise to the following categories

$$\text{Rep}(G) \to \text{Rep}(G^{\text{ét}}) \xrightarrow{\sim} \text{QCoh}\left(\mathcal{M}_n^{(\infty)}\right);$$

the composition of the morphisms is the functor F^*. We see that in order to understand the relationship between quasi-coherent sheaves on $\mathcal{M}_n^{(\infty)}$ and representations of G, it suffices to study the restriction functor $\mathrm{Res}_{G,G^{\text{ét}}} : \mathrm{Rep}(G) \to \mathrm{Rep}(G^{\text{ét}})$. In fact we will begin by working with the group K and then applying our results to the group G as well.

In 4.1, we define some subgroups and submonoids of $G, G^{\text{ét}}, K$, and $K^{\text{ét}}$. These will be technically easier to work with than the full groups, as we will see in the subsequent sections. In 4.2 we study the restriction functor $\mathrm{Res}_{K,K^{\text{ét}}}$, and show that it gives an equivalence of the subcategories of finite-dimensional representations. Since K is an affine group scheme, all of its representations are locally finite, from which we conclude that the functor $\mathrm{Res}_{K,K^{\text{ét}}}$ is fully faithful, with essential image the subcategory of locally finite representations of $K^{\text{ét}}$.

We do not know whether there are any representations of $K^{\text{ét}}$ which are *not* locally finite, but in 4.3 we give an example of a pair $H \supset H'$ of a pro-algebraic group H containing a dense group-valued subprestack H', such that H' has representations which are not locally finite, and hence do not extend to representations of H.

Finally, in 4.4 we study the restriction functor $\mathrm{Res}_{G,G^{\text{ét}}} : \mathrm{Rep}(G) \to \mathrm{Rep}(G^{\text{ét}})$. Analogously to 4.2 we show that it is fully faithful, and characterise its essential image as those representations of $G^{\text{ét}}$ satisfying a suitable finiteness condition.

4.1 Unipotent subgroups and polynomial submonoids

Recall from definition 2.2.3 that the pro-unipotent group K_u is the kernel of the natural map

$$K \to GL_n;$$

analogously, we define the sub-group-valued-prestack $K_u^{\text{ét}}$ of $K^{\text{ét}}$ to be the kernel of the the restriction of this map to $K^{\text{ét}}$. We have

$$K^{\text{ét}} = GL_n \ltimes K_u^{\text{ét}},$$

as group-valued prestacks.

Recall also that in the proof of Proposition 2.4.1 we defined for any c a group-valued prestack N_c by setting $N_c(S)$ to be the kernel of the homomorphism

$$G(S) \twoheadrightarrow G^{(c)}(S).$$

We noted then that $N_c(S)$ is also the kernel of the homomorphism

$$K(S) \twoheadrightarrow K^{(c)}(S);$$

now we remark in addition that it is contained in $K_u^{\text{ét}}(S)$, and is in fact also the kernel of the maps

$$K^{\text{ét}}(S) \twoheadrightarrow K^{(c)}(S), \qquad K_u^{\text{ét}} \twoheadrightarrow K_u^{(c)}(S).$$

(That these maps are surjective is a consequence of Corollary 3.4.1.)

The prestack N_c is an affine group scheme of infinite type.

Example 4.1.1. In the case $n = 1$, N_c parametrises automorphisms $\rho : R[\![t]\!] \to R[\![t]\!]$ of the form

$$\rho : t \mapsto t + r_{c+1}t^{c+1} + r_{c+2}t^{c+2} + \dots .$$

Definition 4.1.2. Consider for each $\operatorname{Spec} R$ the set $M(\operatorname{Spec} R) \subset K(\operatorname{Spec} R)$ of *polynomial automorphisms*: these are automorphisms $\rho : R[\![t_1, \dots, t_n]\!] \to R[\![t_1, \dots, t_n]\!]$ such that for each $k = 1, \dots, n$, $\rho(t_k)$ is a polynomial in the variables $\{t_j\}$ with no constant term, rather than a power series. This defines a prestack $M \hookrightarrow K^{\text{ét}} \hookrightarrow K$.

This is a monoid rather than a group-valued prestack, since it is not closed under taking inverses; however, it is in some ways easier to work with than $K^{\text{ét}}$ in that it is an indscheme:

$$M = \operatorname*{colim}_{\alpha \in \mathbb{N}} M_\alpha,$$

where M_α classifies polynomial automorphisms of degree at most α. (Note that M_α is a scheme, but is not even a monoid, since composing two polynomial automorphisms of degree α gives a polynomial automorphism of degree α^2.)

Similarly, we have the unipotent version $M_u = \operatorname{colim}_{\alpha \in \mathbb{N}} M_{u,\alpha}$, where

$$M_{u,\alpha} = \operatorname{Spec} k[a_J^k]_{1<|J|\leq\alpha}$$

classifies unipotent polynomial automorphisms of degree at most α. Although the $M_{u,\alpha}$ are not closed under composition, they are still closed under the action of $\mathbb{G}_m \hookrightarrow GL_n$ by conjugation, and hence each algebra $k[a_J^k]_{1<|J|\leq\alpha}$ is still graded, with $\deg(a_J^k) = |J| - 1$.

We also have a monoid of polynomial automorphisms M^G in G:

$$M^G = \operatorname*{colim}_{\alpha \in \mathbb{N}} M_\alpha^G,$$

where M_α^G parametrises polynomial automorphisms of the formal disc of degree at most α, whose constant terms are nilpotent. It is itself an indscheme:

$$M_\alpha^G = \operatorname*{colim}_N M_{\alpha,N}^G,$$

where $M_{\alpha,N}^G$ parametrises only those polynomial automorphisms whose constant terms all satisfy $a^N = 0$. As a scheme,

$$M_{\alpha,N}^G = \operatorname{Spec} k[a_J^k]_{|J|\leq\alpha}/((a_{\underline{0}}^k)^N).$$

Finally, notice that for any c, the group scheme N_c also contains a monoid $M^{N_c} = \operatorname{colim}_{\alpha\geq c+1} M_\alpha^{N_c}$ of polynomial automorphisms.

Having established this notation, we can prove the following:

Lemma 4.1.3. *Let $H \in \{K, K_u, G, N_c\}$. Then the inclusion*

$$H^{\acute{e}t} \hookrightarrow H$$

induces a map of sets

$$\operatorname{Hom}_{PreStk}(H, \mathbb{A}^1) \to \operatorname{Hom}_{PreStk}(H^{\acute{e}t}, \mathbb{A}^1).$$

This map is injective.

Moreover, the same is true of the restriction map

$$\operatorname{Hom}_{PreStk}(H \times H, \mathbb{A}^1) \to \operatorname{Hom}_{PreStk}(H^{\acute{e}t} \times H^{\acute{e}t}, \mathbb{A}^1).$$

Proof. The intuition behind this statement is that Artin's approximation theorem tells us that $H^{\text{ét}}$ is dense in H. In order to give a rigorous proof it is useful to restrict further to the monoid introduced above: this is still dense in H and we can exploit its indscheme structure to study its functions. It is clearly sufficient to show that restriction from H to the monoid, which we'll denote by M^H, (respectively from $H \times H$ to $M^H \times M^H$) is injective: if two maps agree on $H^{\text{ét}}$, they certainly agree on M^H. We will carry out the proof for the case $H = G$; the remaining cases are very similar (and where different, simpler).

By the universal property of colimits, a map $\phi : M^G \to \mathbb{A}^1$ is given by a compatible family of polynomials

$$\left(f_{\alpha,N} \in k[a_J^k]_{|J|\leq\alpha}/((a_{\underline{0}}^k)^N)\right)_{\alpha,N}.$$

For fixed N, the compatibility between $f_{\alpha,N}$ is the following: if $\alpha_1 > \alpha_2$, we require that the polynomial obtained from $f_{\alpha_1,N}$ by setting $a_J^K = 0$ for all J with $|J| > \alpha_2$ be equal to $f_{\alpha_2,N}$.

Similarly, a map $\psi : G \to \mathbb{A}^1$ is defined by a compatible family of polynomials

$$\left(g_N \in k[a_J^k]_{J\in\mathbb{Z}^n_{\geq 0}}/((a_{\underline{0}}^k)^N)\right)_N.$$

Let $\phi^1 = (f^1_{\alpha,N})$ and $\phi^2 = (f^2_{\alpha,N})$ be the restriction of two maps $\psi^1 = (g^1_N)$ and $\psi^2 = (g^2_N)$ to M. For any α, the polynomial $f^i_{\alpha,N}$ is obtained from g^i_N by setting $a_J^k = 0$ for all J with $|J| > \alpha$. It is clear that if $f^1_{\alpha,N} = f^2_{\alpha,N}$ for every α, then $g^1_N = g^2_N$. That is, if $\phi^1 = \phi^2$, then $\psi^1 = \psi^2$, and so the restriction map is injective, as required.

The argument for $G^{\text{ét}} \times G^{\text{ét}} \hookrightarrow G \times G$ is similar. □

4.2 Representations of $K^{\text{ét}}$

Theorem 4.2.1. *Let V be a finite-dimensional vector space, and let $\mathbf{R} : K^{\acute{e}t} \to GL_V$ be a representation of $K^{\acute{e}t}$ on V. Then the natural transformation $\mathbf{R}$ extends uniquely to a representation*

$$\overline{\mathbf{R}} : K \to GL_V.$$

Equivalently, there exists some c such that $\mathbf{R}$ factors through the finite-dimensional quotient $K^{(c)}$; then the extension $\overline{\mathbf{R}}$ is defined via the quotient $K \twoheadrightarrow K^{(c)}$.

Proof. Choose a basis $\{v_1, \dots, v_m\}$ of V such that the action of $\mathbb{G}_m \hookrightarrow GL_n \hookrightarrow K^{\text{ét}}$ is diagonal:

$$z \mapsto \begin{pmatrix} z^{d_1} & & & \\ & z^{d_2} & & \\ & & \ddots & \\ & & & z^{d_m} \end{pmatrix}$$

with $d_1 \leq d_2 \leq \ldots \leq d_m$ an increasing sequence of integers.

Now consider the restriction of $\mathbf{R}$ to the monoid $M_u = \operatorname{colim}_\alpha M_{u,\alpha} \hookrightarrow K_u^{\text{ét}}$, and let $\mathbf{R}_{ij}$ denote the (i, j)th matrix coefficient with respect to the basis $\{v_1, \dots, v_m\}$:

$$\mathbf{R}_{ij} : M_u \to \mathbb{A}^1.$$

As in the proof of Proposition 4.1.3, $\mathbf{R}_{ij}$ is given by an infinite family of polynomials

$$\left\{f_{ij,\alpha} \in k[a_J^k]_{1<|J|\leq\alpha}\right\}_\alpha,$$

satisfying the compatibility condition

$$f_{ij,\alpha+1}|_{a_J^k \equiv 0, |J|=\alpha+1} = f_{ij,\alpha}.$$

Step 1 Let $\alpha_0 = d_m - d_1 + 1$. We will show that in fact the polynomials $f_{ij,\alpha}$ depend only on the variables $\{a_1, \ldots, a_{\alpha_0}\}$. In particular,

$$f_{ij,\alpha_0+1} = f_{ij,\alpha_0+2} = \cdots ,$$

and the function $\mathbf{R}_{ij}$ is the restriction of a function $\overline{\mathbf{R}_{ij}} : K_u \to \mathbb{A}^1$, corresponding to the polynomial

$$f_{ij,\alpha_0+1} \in k[a_J^k]_{1<|J|}.$$

To prove this claim, let $z \in \mathbb{G}_m(k)$ be an arbitrary k-point, and recall that conjugation by z gives maps

$$\gamma_z : M_u \to M_u, \qquad \gamma_{z,\alpha} : M_{u,\alpha} \to M_{u,\alpha},$$

or equivalently maps $\gamma_z^\sharp$ and $\gamma_{z,\alpha}^\sharp$ on the corresponding algebras of functions.

We know that for any k-algebra R and for any $m \in M_u(\operatorname{Spec} R)$, we have

$$\mathbf{R}(zmz^{-1}) = \mathbf{R}(z)\mathbf{R}(m)\mathbf{R}(z)^{-1}.$$

In terms of matrix coefficients, this implies that

$$\mathbf{R}_{ij} \circ \gamma_z = z^{d_i - d_j}\mathbf{R}_{ij},$$

or equivalently that for every α,

$$\gamma_{z,\alpha}^\sharp(f_{ij,\alpha}) = z^{d_i-d_j} f_{ij,\alpha}.$$

It follows that $f_{ij,\alpha}$ is homogeneous of degree $d_i - d_j$, and hence cannot depend on any variable a_J^k with $|J| > d_i - d_j + 1$.

Allowing (i, j) to vary, we obtain the global bound $\alpha_0 = d_m - d_1 + 1 = \max\{d_i - d_j + 1\}$ on the degree of the variables appearing in the matrix coefficients of $\mathbf{R}$.

Step 2 We have shown that each matrix coefficient extends to a function on K_u, but it is not a priori clear that these assemble to give a representation $\overline{\mathbf{R}}$ of K_u, which in turn extends to all of K. We show this now.

By Step 1 and Lemma 4.1.3, if we take $c > \alpha_0$, the restriction of $\mathbf{R}$ to N_c is constant. That is, N_c is in the kernel of the representation $\mathbf{R}$, and hence we have a factorisation

$$\begin{array}{ccc} K^{\text{ét}} & \xrightarrow{\mathbf{R}} & GL_V. \\ & \searrow & \nearrow_{\mathbf{R}^{(c)}} \\ & K^{(c)} & \end{array}$$

So we can define an extension of $\mathbf{R} : K^{\text{ét}} \to GL_V$ to all of K by defining $\overline{\mathbf{R}}$ to be the composition

$$K \twoheadrightarrow K^{(c)} \to GL_V.$$

By applying Lemma 4.1.3 to each of the matrix coefficients of $\overline{\mathbf{R}}$, we see that this extension is unique.

□

Corollary 4.2.2. *The restriction functor*

$$\text{Res}_{K,K^{\acute{e}t}} : \text{Rep}(K) \to \text{Rep}(K^{\acute{e}t})$$

is fully faithful. Its essential image is the full subcategory $\text{Rep}^{l.f.}(K^{\acute{e}t})$ *of locally finite representations of* $K^{\acute{e}t}$.

Proof. Let $(V, \overline{\mathbf{R}})$ and $(W, \overline{\mathbf{S}})$ be two representations of K, and let $(V, \mathbf{R}), (W, \mathbf{S})$ denote their image in $\text{Rep}(K^{\text{ét}})$. We consider the map

$$\text{Hom}_K(V, W) \to \text{Hom}_{K^{\text{ét}}}(V, W).$$

It is clear that this is injective. To see that it is surjective, notice that a map $V \to W$ compatible with $\mathbf{R}$ and $\mathbf{S}$ is necessarily compatible with the extensions $\overline{\mathbf{R}}$ and $\overline{\mathbf{S}}$ of the representations to K: this amounts to the fact that for sufficiently large c, we have a commutative diagram

$$\begin{array}{ccc} K^{(c)} & \xrightarrow{\mathbf{R}^{(c)}} & GL_V \\ {\scriptstyle \mathbf{S}^{(c)}}\downarrow & & \downarrow \\ GL_W & \longrightarrow & \text{End}(V, W). \end{array}$$

So $\text{Res}_{K,K^{\text{ét}}}$ is fully faithful as claimed.

To identify the essential image, first recall that because K is an affine group scheme, all of its representations are locally finite: any representation V can be written as a union of finite-dimensional representations. This clearly still gives a decomposition of V when we restrict to the action of $K^{\text{ét}}$, and so the essential image is a subcategory of $\text{Rep}^{\text{l.f.}}(K^{\text{ét}})$.

On the other hand, suppose that $(V, \mathbf{R}) \in \text{Rep}^{\text{l.f.}}(K^{\text{ét}})$. Then we can write $V = \bigcup_i V_i$, where $V_i \subset V$ is a finite-dimensional subrepresentation of $K^{\text{ét}}$. Let $\mathbf{R}_i : K^{\text{ét}} \to GL_{V_i}$ denote the restriction of $\mathbf{R}$ to the subspace V_i. Since V_i is finite-dimensional, Theorem 4.2.1 provides us with a unique extension of $\mathbf{R}_i$:

$$\overline{\mathbf{R}_i} : K \to GL_{V_i}.$$

The uniqueness of these extensions means that they agree on intersections $V_i \cap V_j$, and hence give a representation

$$\overline{\mathbf{R}} : K \to GL_V.$$

By construction, $\text{Res}_{K,K^{\text{ét}}}(V, \overline{\mathbf{R}}) = (V, \mathbf{R})$ and hence $(V, \mathbf{R})$ is indeed in the essential image of $\text{Res}_{K,K^{\text{ét}}}$. □

Remark 4.2.3. At the time of writing, we do not know whether there exist any representations of $K^{\text{ét}}$ which are *not* locally finite. If there are no such representations, then the functor

$$\text{Res}_{K,K^{\text{ét}}} : \text{Rep}(K) \to \text{Rep}(K^{\text{ét}})$$

is an equivalence.

4.3 Non-locally-finite representations

In this section, we work in a similar set-up to show that it is possible to have a group-valued prestack which is dense in a pro-algebraic group and which still has representations which are not locally finite. This means that if the functor $\text{Res}_{K,K^{\text{ét}}}$ is to be an equivalence, and all representations of $K^{\text{ét}}$ are locally finite, it is a property very particular to these group-valued prestacks.

Consider $\mathbb{A}^\infty = \text{colim}_{i\in\mathbb{N}} \mathbb{A}^i \hookrightarrow \overline{\mathbb{A}^\infty} = \lim_{i\in\mathbb{N}} \mathbb{A}^i$, viewed as additive groups in the obvious way: for $S = \text{Spec}\, R$, $\mathbb{A}^\infty(S)$ is the set of finite sequences in R, while $\overline{\mathbb{A}^\infty}(S)$ is the set of infinite sequences, both equipped with term-wise addition.

Just as in the situation of $K^{\text{ét}} \hookrightarrow K$ the maps from the subgroup to the finite-dimensional quotients of the pro-group are all surjective: we have

$$\mathbb{A}^\infty \twoheadrightarrow \mathbb{A}^c,$$

corresponding to truncating the sequences at the cth term. Restriction of functions from $\overline{\mathbb{A}^\infty}$ to $\mathbb{A}^\infty$ is still injective, by the same argument as in the proof of Lemma 4.1.3.

Now consider the regular representation of $\mathbb{A}^\infty$: this is the infinite-dimensional vector space

$$\begin{aligned} V &= \mathrm{Hom}(\mathbb{A}^\infty, \mathbb{A}^1) \\ &= \lim_{n\in\mathbb{N}}(k[t_1, \ldots, t_n]), \end{aligned}$$

with the action of $\mathbb{A}^\infty$ given by precomposition with the addition map. Consider the element $v \in V$ given by the infinite family of polynomials

$$f_n = \sum_{j=1}^{n} \prod_{i=1}^{j} t_i \in k[t_1, \ldots, t_n].$$

Then the subspace generated by $v \in V$ under the action of $\mathbb{A}^\infty(k)$ is infinite-dimensional. For example, if $\mathbf{e}_{i_0} \in \mathbb{A}^\infty(k)$ is the sequence with 1 in the i_0th position and 0 everywhere else, then $\mathbf{e}_{i_0}.v$ is given by the infinite sequence of polynomials

$$\sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq i_0}}^{j} t_i,$$

and so the $\mathbf{e}_{i_0}.v$ are linearly independent as i_0 varies. Hence the representation V is not locally finite.

We conclude that

$$\mathrm{Rep}^{\mathrm{l.f.}}(\mathbb{A}^\infty) \subsetneq \mathrm{Rep}(\mathbb{A}^\infty).$$

On the other hand, there even exist finite-dimensional representations of $\mathbb{A}^\infty$ which do not extend to $\overline{\mathbb{A}^\infty}$, which cannot happen for the case of $K^{\text{ét}} \hookrightarrow K$. Indeed, consider the two-dimensional unipotent representation corresponding to the assignment

$$(x_i)_i \mapsto \begin{pmatrix} 1 & \sum_i x_i \\ 0 & 1 \end{pmatrix}.$$

It cannot be extended to $\overline{\mathbb{A}^\infty}$.

4.4 Representations of $G^{\text{ét}}$

In this subsection, we compare the representations of G to those of $G^{\text{ét}}$. We distinguish the category $\mathrm{Rep}^{K^{\text{ét}}\text{-l.f.}}(G^{\text{ét}})$ of representations of $G^{\text{ét}}$ which are locally finite when viewed as representations of $K^{\text{ét}}$ via the inclusion

$$K^{\text{ét}} \hookrightarrow G^{\text{ét}}.$$

Definition 4.4.1. We shall refer to such representations as $K^{\acute{e}t}$*-locally-finite* representations of $G^{\text{ét}}$.

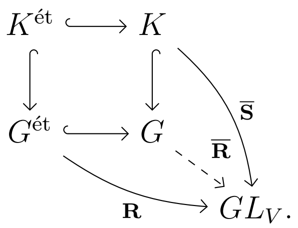
Proposition 4.4.2. *Let* $(V, \mathbf{R}) \in \mathrm{Rep}^{K^{\acute{e}t}\text{-}l.f.}(G^{\acute{e}t})$ *where* V *is a vector space and* $\mathbf{R} : G^{\acute{e}t} \to GL_V$*. Then* $\mathbf{R}$ *extends uniquely to a morphism*

$$\overline{\mathbf{R}} : G \to GL_V$$

of group-valued prestacks.

Proof. Let $\mathbf{S} : K^{\text{ét}} \to V$ be the composition of $\mathbf{R}$ with the inclusion $K^{\text{ét}} \hookrightarrow G^{\text{ét}}$. Then $(V, \mathbf{S})$ is a locally finite representation of $K^{\text{ét}}$, and hence extends uniquely to a representation $\overline{\mathbf{S}} : K \to GL_V$ by Corollary 4.2.2.

Consider the following diagram of group-valued prestacks:



We wish to define the group homomorphism $\overline{\mathbf{R}}$ completing this diagram, and to show that it is unique.

If there is to be a function $\overline{\mathbf{R}}$ which respects the group structures as in the above diagram, it must satisfy

$$\overline{\mathbf{R}}(xk) = \mathbf{R}(x)\overline{\mathbf{S}}(k) \tag{II.7}$$

for every k-algebra R and for all R-points $x \in G^{\text{ét}}(\mathrm{Spec}\, R), k \in K(\mathrm{Spec}\, R)$. Notice that the map

$$G^{\text{ét}} \times K \to G$$

given by composition of automorphisms (i.e. multiplication inside G) is surjective on $(\operatorname{Spec} R)$-points: every automorphism can be written as the composition of one whose constant terms are zero and an étale (and in fact polynomial) automorphism. It follows that equation (II.7) determines $\overline{\mathbf{R}}$ uniquely, if it exists. We need to prove that the assignment

$$xk \mapsto \mathbf{R}(x)\overline{\mathbf{S}}(k)$$

gives a well-defined function $G(\operatorname{Spec} R) \to GL(V \otimes_k R)$, and moreover that this is functorial in the k-algebra R.

For the first point, suppose that

$$xk = x'k',$$

for some $x, x' \in G^{\text{ét}}(\operatorname{Spec} R)$, $k, k' \in K(\operatorname{Spec} R)$. Then $(x')^{-1}x = k'k^{-1}$, and this is an element of $K^{\text{ét}}(\operatorname{Spec} R)$, so that

$$\mathbf{R}(x')^{-1}\mathbf{R}(x) = \mathbf{S}(x')^{-1}\mathbf{S}(x) = \mathbf{S}(k')\mathbf{S}(k)^{-1} = \overline{\mathbf{S}}(k')\overline{\mathbf{S}}(k)^{-1},$$

and hence

$$\mathbf{R}(x)\overline{\mathbf{S}}(k) = \mathbf{R}(x')\overline{\mathbf{S}}(k'),$$

as required.

To show functoriality, let $f : R \to R'$ be a homomorphism of k-algebras, and let $g \in G(\operatorname{Spec} R)$ be an R-point, with g' its image in $G(\operatorname{Spec} R')$. We choose $x \in G^{\text{ét}}(\operatorname{Spec} R)$ and $k \in K(\operatorname{Spec} R)$ such that $g = xk$; then $\overline{R}(g) := \mathbf{R}(x)\overline{\mathbf{S}}(k)$. Letting $x' \in G^{\text{ét}}(\operatorname{Spec} R')$ and $k' \in K(\operatorname{Spec} R')$ be the images of x and k respectively, we see that $g' = x'k'$, and hence that $\overline{\mathbf{R}}(g') = \mathbf{R}(x')\overline{\mathbf{S}}(k')$. It is clear that this is equal to the image of $\overline{\mathbf{R}}(g)$ in $GL(V \otimes_k R')$.

We conclude that there is a unique morphism of prestacks $\overline{\mathbf{R}} : G \to GL_V$ making the above diagram commute. It remains to show that this morphism respects the group structures. That is, we need to show that the following diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ \downarrow{\scriptstyle \overline{\mathbf{R}} \times \overline{\mathbf{R}}} & & \downarrow{\scriptstyle \overline{\mathbf{R}}} \\ GL_V \times GL_V & \xrightarrow[m_{GL_V}]{} & GL_V. \end{array}$$

Fix a (possibly infinite) basis $\{v_i\}_{i\in I}$ for V, and for any pair $(i,j) \in I \times I$ define the projection

$$\pi_{ij} : GL_V \to \mathbb{A}^1$$

in the obvious way. To show that $m_{GL_V} \circ \overline{\mathbf{R}} = (\overline{\mathbf{R}} \times \overline{\mathbf{R}}) \circ m_G$, it suffices to show that

$$\pi_{ij} \circ m_{GL_V} \circ \overline{\mathbf{R}} = \pi_{ij} \circ (\overline{\mathbf{R}} \times \overline{\mathbf{R}}) \circ m_G : G \times G \to \mathbb{A}^1$$

for every pair (i,j). By Lemma 4.1.3, it suffices to show that these functions agree when restricted to $G^{\text{ét}} \times G^{\text{ét}}$; but this is clear because the restriction of $\overline{\mathbf{R}}$ to $G^{\text{ét}}$ is the homomorphism $\mathbf{R}$. □

We have the following analogue to Corollary 4.2.2:

Corollary 4.4.3. *The restriction functor*

$$\operatorname{Res}_{G,G^{\acute{e}t}} : \operatorname{Rep}(G) \to \operatorname{Rep}(G^{\acute{e}t})$$

is fully faithful, with essential image $\operatorname{Rep}^{K^{\acute{e}t}\text{-}l.f.}(G^{\acute{e}t})$.

Proof. The only part that is not immediate is the surjectivity of the maps of hom-spaces. For $t = 1, 2$, let $(V_t, \overline{\mathbf{R}}_t)$ be representations of G, with $(V_t, \mathbf{R}_t)$ the restrictions to $G^{\text{ét}}$. Suppose that we have a linear map $f : V_1 \to V_2$ compatible with the maps $\mathbf{R}_t$; then we need to show that it is also compatible with $\overline{\mathbf{R}}_t$. That is, we need to show that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\overline{\mathbf{R}}_1} & GL_{V_1} \\ {\scriptstyle \overline{\mathbf{R}}_2}\downarrow & & \downarrow \\ GL_{V_2} & \longrightarrow & \operatorname{End}(V_1, V_2). \end{array}$$

This is similar to the argument in the last part of the proof of Proposition 4.4.2. Fix bases $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ for V_1 and V_2; then it suffices to show that the diagram commutes after composing with the (i,j)th projection $\operatorname{End}(V_1, V_2) \to \mathbb{A}^1$ for every $(i,j) \in I \times J$. But then by Lemma 4.1.3 once more, it is enough to show that each of the resulting diagrams commutes after pre-composing with the inclusion $G^{\text{ét}} \hookrightarrow G$, and this is true by our assumption on f. □

5 Universal modules

In this section, we finally begin our analysis of universal $\mathcal{D}$-modules. We will see that they are equivalent to quasi-coherent sheaves on the stack $\mathcal{M}_n^{(\infty)}$ of étale germs. Similarly, we will see that universal $\mathcal{O}$-modules are just quasi-coherent sheaves on the stack $\mathcal{M}_n^{\mathrm{pt},(\infty)}$. We begin by recalling the definitions of universal modules, given by Beilinson and Drinfeld (see [4] 2.9.9); in 5.1, 5.2, and 5.3 we prove that the category $\mathscr{U}_n^{\mathcal{D}}$ of universal $\mathcal{D}$-modules of dimension n is equivalent to $\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$. In 5.4 we discuss the corresponding equivalence in the case of universal $\mathcal{O}$-modules.

Definition 5.0.4. A *universal $\mathcal{O}$-module* $\mathscr{F}$ of dimension n consists of the following data:

1. For each smooth family $X \xrightarrow{\pi} S$ of relative dimension n, we have an $\mathcal{O}_X$-module $\mathscr{F}_{X/S} \in \mathrm{QCoh}\,(X)$.
2. For each fibrewise étale morphism $f = (f_X, f_S) : (X/S) \to (X'/S')$ of smooth n-dimensional families, we have an isomorphism

$$\mathscr{F}(f) : \mathscr{F}_{X/S} \xrightarrow{\sim} (f_X)^* \mathscr{F}_{X'/S'}$$

   of $\mathcal{O}_X$-modules.

These isomorphisms are required to be compatible with composition in the following sense. Suppose we are given three smooth n-dimensional families with fibrewise étale morphisms between them:

$$\begin{array}{ccccc} X & \xrightarrow{f_X} & X' & \xrightarrow{g_X} & X'' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi''} \\ S & \xrightarrow[f_S]{} & S' & \xrightarrow[g_S]{} & S''. \end{array}$$

We require the following diagram of isomorphisms to commute:

$$\begin{array}{ccc} \mathscr{F}_{X/S} & \xrightarrow{\mathscr{F}(f)} & f_X^* \mathscr{F}_{X'/S'} \\ \downarrow{\scriptstyle \mathscr{F}(g \circ f)} & & \downarrow{\scriptstyle f_X^* \mathscr{F}(g)} \\ (g_X \circ f_X)^* \mathscr{F}_{X''/S''} & \xrightarrow{\sim} & f_X^* g_X^* \mathscr{F}_{X''/S''}. \end{array}$$

A morphism $\phi : \mathscr{F} \to \mathscr{G}$ of universal $\mathcal{O}$-modules is a collection of morphisms

$$\phi_{X/S} : \mathscr{F}_{X/S} \to \mathscr{G}_{X/S},$$

indexed by n-dimensional families, compatible with the structure isomorphisms. That is, for any fibrewise étale morphism $f = (f_X, f_S) : (X'/S') \to (X/S)$, the following diagram should commute:

$$\begin{array}{ccc} \mathscr{F}_{X/S} & \xrightarrow{\sim} & f_X^* \mathscr{F}_{X'/S'} \\ {\scriptstyle \phi_{X/S}} \downarrow & & \downarrow {\scriptstyle f_X^* \phi_{X'/S'}} \\ \mathscr{G}_{X/S} & \xrightarrow{\sim} & f_X^* \mathscr{G}_{X'/S'}. \end{array}$$

In this way we obtain a category $\mathscr{U}_n^{\mathcal{O}}$ of universal $\mathcal{O}$-modules of dimension n.

Similarly, we can define the category $\mathscr{U}_n^{\mathcal{D}}$ of universal $\mathcal{D}$-modules of dimension n:

Definition 5.0.5. A *universal $\mathcal{D}$-module* of dimension n is a rule $\mathscr{F}$ which assigns:

1. to each smooth $X \to S$ of relative dimension n a left $\mathcal{D}_{X/S}$-module $\mathscr{F}(X/S)$;
2. to each fibrewise étale morphism $f = (f_X, f_S) : (X/S) \to (X'/S')$ of smooth n-dimensional families, an isomorphism

   $$\mathscr{F}(f) : \mathscr{F}(X/S) \xrightarrow{\sim} f^* \mathscr{F}(X'/S'),$$

   in a way compatible with composition.

Let us be a little more precise. The category $\mathcal{D}(X/S)$ is (by definition) the category of quasi-coherent sheaves on $(X/S)_{\mathrm{dR}}$, where $(X/S)_{\mathrm{dR}}$ is the following fibre product:

$$\begin{array}{ccc} (X/S)_{\mathrm{dR}} & \longrightarrow & X_{\mathrm{dR}} \\ \downarrow & & \downarrow {\scriptstyle \pi_{\mathrm{dR}}} \\ S & \xrightarrow[p_{\mathrm{dR},S}]{} & S_{\mathrm{dR}} \end{array}$$

(Recall that we associate to any prestack $\mathcal{Y}$ its *de Rham prestack* $\mathcal{Y}_{\mathrm{dR}}$:

$$\mathcal{Y}_{\mathrm{dR}} : T \mapsto \mathcal{Y}_{\mathrm{dR}}(T) := \mathcal{Y}(T_{red}).$$

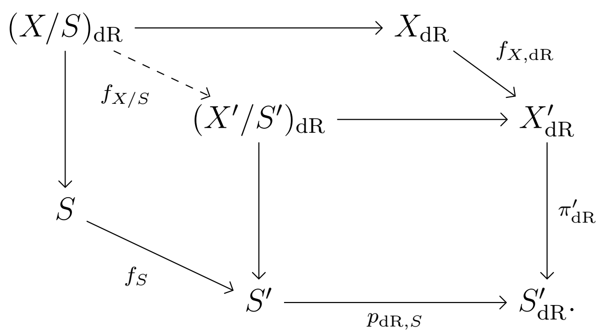
Then we define the category of *left $\mathcal{D}$-modules* on $\mathcal{Y}$ to be the category $\mathrm{QCoh}\,(\mathcal{Y}_{\mathrm{dR}})$. The forgetful functor from $\mathcal{D}$-modules to $\mathcal{O}$-modules is given by pullback along the natural map $p_{\mathrm{dR},\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}}$. See Appendix A.2.3 for an overview, or see [18] for more details, as well as the full definition in the $(\infty, 1)$-categorical and derived setting.)

This means that the object $\mathscr{F}(X/S)$ is given by a collection of objects

$$\mathscr{F}(X/S)_{T\to(X/S)_{\mathrm{dR}}} \in \mathrm{QCoh}\,(T)$$

indexed by affine schemes T and morphisms $T \to (X/S)_{\mathrm{dR}}$, along with isomorphisms describing compatibility with pullbacks.

Recall from the proof of Proposition I.2.5.6 that if we have a fibrewise étale morphism of n-dimensional families $f = (f_X, f_S) : (X/S) \to (X'/S')$, we obtain a morphism $f_{X/S} : (X/S)_{\mathrm{dR}} \to (X'/S')_{\mathrm{dR}}$ as follows:



Then the compatibility isomorphism $\mathscr{F}(f)$ associated to $\mathscr{F}$ is an isomorphism between $\mathscr{F}(X/S)$ and $f^*_{X/S}\mathscr{F}(X'/S')$ in $\mathrm{QCoh}\,((X/S)_{\mathrm{dR}})$.

Notation 5.0.6. We will always use the subscript $\bullet_{X/S}$ to denote the morphism of relative de Rham prestacks induced by a fibrewise étale morphism between two smooth families, even when neither of the smooth families involved is actually denoted by X/S.

Remark 5.0.7. At this stage, the reader may wonder why we have chosen to use *left* relative $\mathcal{D}$-modules, rather than *right*, which is the more usual category in which to work. (See for example the discussion of the category $\mathrm{IndCoh}\,((X/S)_{\mathrm{dR}})$ of *relative crystals* in [20], part III, chapter 4, section 3.3.) We have several reasons for this choice.

1. We wish, at least for the moment, to remain consistent with the definition of universal $\mathcal{D}$-module given by Beilinson and Drinfeld.

2. Also for the moment, we wish to work with abelian categories rather than DG-categories; the !-pullback functors needed in the definition of the category of ind-coherent sheaves on a prestack are inherently derived.

3. Suppose for the sake of argument that we are working with DG-categories rather than abelian categories. As will be seen in the following sections, we are going to compare universal $\mathcal{D}$-modules to sheaves on the stack $\mathcal{M}_n^{(\infty)}$. We will see that, given a universal $\mathcal{D}$-module $\mathscr{F}$ and a map $S \to \mathcal{M}_n^{(\infty)}$ corresponding to a pointed family $\pi : X \rightleftarrows S : \sigma$, we pull back the $\mathcal{D}$-module $\mathscr{F}(X/S)$ by the section σ, thus obtaining a $\mathcal{D}$-module on S. These should be compatible with pullback along maps $S \to S'$.

   If we try to work with right $\mathcal{D}$-modules, we should obtain a family of ind-coherent sheaves on S, compatible under !-pullback, and we might be tempted to call this an ind-coherent sheaf on $\mathcal{M}_n^{(\infty)}$. However, at this stage we run into technical difficulties. The category $\operatorname{IndCoh}(S)$ is only well-behaved for schemes S of finite type—for example, the pull-back $f^!$ is defined only for morphisms of finite type.

   As a consequence, the category $\operatorname{IndCoh}(\mathcal{Y})$ is defined only for prestacks which are *locally of finite type.* (See [15] 1.3.9 for the definition of prestacks locally of finite type (or l.f.t.), and [17] 10.1 for the definition of $\operatorname{IndCoh}(\mathcal{Y})$—or, for an overview, see Appendix A.1 and A.2.2.) The reason that the definition works for an l.f.t. prestack $\mathcal{Y}$ is that such a prestack is determined by its S-points for S of finite type. Our problems arise because the stack $\mathcal{M}_n^{(\infty)}$ is not l.f.t.

   On the other hand, $\operatorname{QCoh}(\mathcal{Y})$ is defined for any prestack, so we do not have the same difficulties when working with left $\mathcal{D}$-modules.

4. One might have the intuition that the structure of a universal $\mathcal{D}$-module is determined by its behaviour associated to the trivial pointed neighbourhood $\mathbb{A}^n \rightleftarrows \mathrm{pt}$, or perhaps at worst $\mathbb{A}^n \times \mathbb{A}^m \rightleftarrows \mathbb{A}^m$ and to common étale neighbourhoods of this neighbourhood, and consequently that it should be enough to consider base schemes of finite type. However, in order to make this intuition more rigorous, we will need to formulate the notion of a *convergent* universal $\mathcal{D}$-module, which we will do in section 6. We will see then that we can construct sheaves corresponding to convergent universal $\mathcal{D}$-modules from quasi-coherent sheaves on the stacks $\mathcal{M}_n^{(c)}$ or equivalently $BG^{(c)}$. Since these stacks *are* l.f.t.,

it is possible to formulate an ind-coherent version as well as a quasi-coherent version, in the DG setting. We will discuss this more in 7.3.

Theorem 5.0.8. *The category $\mathscr{U}_n^{\mathcal{D}}$ of universal $\mathcal{D}$-modules of dimension n is equivalent to the category of quasi-coherent sheaves on $\mathcal{M}_n^{(\infty)}$.*

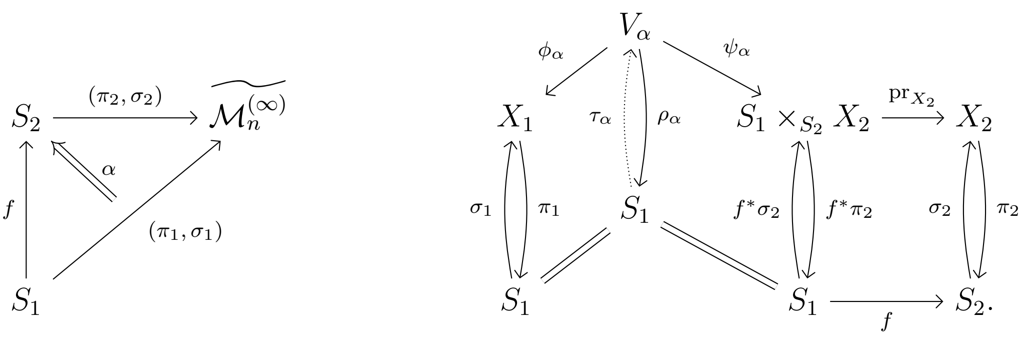
Proof. Recall that it suffices to show that $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right) \simeq \mathscr{U}_n^{\mathcal{D}}$. The idea behind the proof is quite simple, but there are many technical details to be checked. We proceed by defining functors in both directions and checking that they are quasi-inverse to each other.

5.1 From universal $\mathcal{D}$-modules to quasi-coherent sheaves

First we define the functor $\theta : \mathscr{U}_n^{\mathcal{D}} \to \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$. Given a universal $\mathcal{D}$-module $\mathscr{F}$, we wish to define a quasi-coherent sheaf $\theta(\mathscr{F})$ on $\widetilde{\mathcal{M}_n^{(\infty)}}$. That is, given any S and any morphism $(\pi, \sigma) : S \to \widetilde{\mathcal{M}_n^{(\infty)}}$ representing a pointed n-dimensional family $(\pi : X \rightleftarrows S : \sigma)$, we need to define a quasi-coherent sheaf $\theta(\mathscr{F})_{X \rightleftarrows S}$ on S. We have $\mathscr{F}(X/S) \in \mathrm{QCoh}((X/S)_{\mathrm{dR}})$, and $\sigma : S \to X$ induces a section $\overline{\sigma} : S \to (X/S)_{\mathrm{dR}}$, so we simply set

$$\theta(\mathscr{F})_{X \rightleftarrows S} := \overline{\sigma}^* \mathscr{F}(X/S).$$

(We adopt the convention of denoting the map into the relative de Rham stack induced by a section by $\overline{\bullet}$.) Next we need to define the compatibility isomorphisms. Suppose we have a commutative diagram in PreStk of schemes mapping to $\widetilde{\mathcal{M}_n^{(\infty)}}$:



We need to specify an isomorphism

$$\theta(\mathscr{F})(f, \alpha) : f^* \left(\theta(\mathscr{F})_{X_2 \rightleftarrows S_2}\right) \xrightarrow{\sim} \theta(\mathscr{F})_{X_1 \rightleftarrows S_1};$$

it arises naturally from the universality of $\mathscr{F}$. Indeed, from the definition of $\mathscr{F}$ we have isomorphisms

$$\begin{aligned}\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1}) : \mathscr{F}(V_\alpha/S_1) &\xrightarrow{\sim} (\phi_\alpha, \mathrm{id}_{S_1})^*_{X/S}\mathscr{F}(X_1/S_1); \\ \mathscr{F}(\mathrm{pr}_{X_2}\circ\psi_\alpha, f) : \mathscr{F}(V_\alpha/S_1) &\xrightarrow{\sim} \left(\mathrm{pr}_{X_2}\circ\psi_\alpha, f\right)^*_{X/S}\mathscr{F}(X_2/S_2).\end{aligned}$$

Note that $(\phi_\alpha, \mathrm{id}_{S_1})_{X/S}\circ\overline{\tau_\alpha} = \overline{\sigma_1}$ and $(\mathrm{pr}_{x_2}\circ\psi_\alpha, f)_{X/S}\circ\overline{\tau_\alpha} = \overline{\sigma_2}\circ f$, so that setting

$$\theta(\mathscr{F})(f,\alpha) := \overline{\tau_\alpha}^*\left(\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(\mathrm{pr}_{X_2}\circ\psi_\alpha, f)^{-1}\right),$$

we obtain an isomorphism

$$f^*\overline{\sigma_2}^*\mathscr{F}(X_2/S_2) \xrightarrow{\sim} \overline{\sigma_1}^*\mathscr{F}(X_1/S_1),$$

as required.

Let us now check that $\theta(\mathscr{F})(f,\alpha)$ is independent of the choice of common étale neighbourhood $(V_\alpha, \phi_\alpha, \psi_\alpha)$ taken to represent the isomorphism α between $(X_1 \rightleftarrows S_1)$ and $(S_1\times_{S_2} X_2 \rightleftarrows S_1)$ in $\widetilde{\mathcal{M}_n^{(\infty)}}(S_1)$. It suffices to show that if $(V_\alpha, \phi_\alpha, \psi_\alpha)$ and $(V'_\alpha, \phi'_\alpha, \psi'_\alpha)$ are (∞)-equivalent common étale neighbourhoods, then

$$\overline{\tau}^*\left(\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})^{-1}\right) = \overline{\tau}^*\left(\mathscr{F}(\phi'_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(\psi'_\alpha, \mathrm{id}_{S_1})^{-1}\right)$$

as morphisms of quasi-coherent sheaves on S_1. It is enough to show that they agree on an open cover of S_1, and hence we can assume that $(V_\alpha, \phi_\alpha, \psi_\alpha)$ and $(V'_\alpha, \phi'_\alpha, \psi'_\alpha)$ are in fact similar. Finally, we can assume that $\phi'_\alpha = \phi_\alpha\circ g$ and $\psi'_\alpha = \psi_\alpha\circ g$ for some $g : V'/S_1 \to V/S_1$ étale and compatible with the sections on $S_{1,red}$. This is because this relation generates the equivalence relation of similarity.

In this case, using the compatibility of $\mathscr{F}(\bullet)$ with respect to composition, we have

$$\begin{aligned}&\overline{\tau'}^*\left(\mathscr{F}(\phi_\alpha\circ g, \mathrm{id}_{S_1})\circ\mathscr{F}(\psi_\alpha\circ g, \mathrm{id}_{S_1})^{-1}\right) \\ &\quad= \overline{\tau'}^*\left(\left(g^*_{X/S}\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(g, \mathrm{id}_{S_1})\right)\circ\left(g^*_{X/S}\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(g, \mathrm{id}_{S_1})\right)^{-1}\right) \\ &\qquad= \overline{\tau'}^*\left(g^*_{X/S}\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1})\circ g^*_{X/S}\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})^{-1}\right) \\ &\qquad\quad= \overline{\tau}^*\left(\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1})\circ\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})^{-1}\right),\end{aligned}$$

and so the assignment $(f, [(V_\alpha, \phi_\alpha, \psi_\alpha)]) \mapsto \theta(\mathscr{F})(f,\alpha)$ is well-defined with respect to the (∞)-equivalence relation on common étale neighbourhoods.

Remark 5.1.1. Note that in general this assignment is *not* well-defined with respect to (c)-equivalence for any finite c. We will return to this point in section 6.

One can also check that the $\theta(\mathscr{F})(f,\alpha)$ are compatible under composition, and hence $\theta(\mathscr{F})$ is indeed an object of $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$. See Appendix C for the details of the proof.

The definition of θ on morphisms is straightforward: given a morphism $F : \mathscr{F} \to \mathscr{G}$ of universal $\mathcal{D}$-modules, the morphism $\theta(F) : \theta(\mathscr{F}) \to \theta(\mathscr{G})$ of quasi-coherent sheaves is given by $\theta(F)_{X \rightleftarrows S} := \overline{\sigma}^*(F(X/S))$:

$$\theta(\mathscr{F})_{X \rightleftarrows S} = \overline{\sigma}^*(\mathscr{F}(X/S)) \longrightarrow \theta(\mathscr{G})_{X \rightleftarrows S} = \overline{\sigma}^*(\mathscr{G}(X/S)).$$

This completes the construction of the functor

$$\theta : \mathscr{U}_n^{\mathcal{D}} \to \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right).$$

5.2 From quasi-coherent sheaves to universal $\mathcal{D}$-modules

Now we will construct the quasi-inverse functor

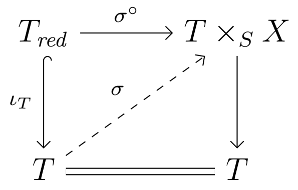
$$\Psi : \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right) \to \mathscr{U}_n^{\mathcal{D}}.$$

Let $M \in \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$, and let $X \to S$ be smooth of relative dimension n. We need to define an object $\Psi(M)(X/S) \in \mathrm{QCoh}\left((X/S)_{\mathrm{dR}}\right)$. More precisely, for any $T \to (X/S)_{\mathrm{dR}}$, we need to define $\Psi(M)(X/S)_{T \to (X/S)_{\mathrm{dR}}}$, together with isomorphisms describing the compatibility with pullbacks.

By definition of $(X/S)_{\mathrm{dR}}$, a morphism $T \to (X/S)_{\mathrm{dR}}$ is given by a pair of morphisms (g,h) as in the following commutative diagram:

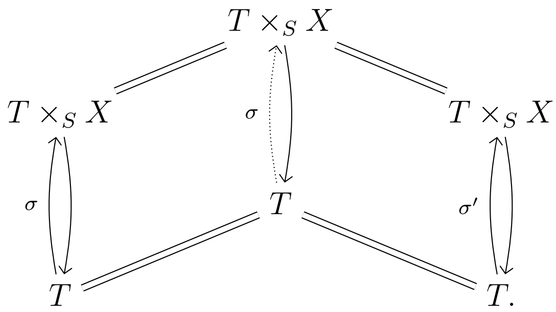
$$\begin{array}{ccc} T_{red} & \xrightarrow{\;g\;} & X \\ {\scriptstyle \iota_T}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ T & \xrightarrow[\;h\;]{} & S. \end{array}$$

To define an object of $\mathrm{QCoh}\,(T)$ using M, we need an object of $\widetilde{\mathcal{M}_n^{(\infty)}}(T)$, i.e. a pointed n-dimensional family over T. An obvious candidate is $T \times_S X$, which is smooth of dimension n over T. To define a section, note that we can define $\sigma^\circ := (\iota_T, g) : T_{red} \to T \times_S X$. Then we have the following commutative diagram:



where formal smoothness of $T \times_S X \to T$ allows us to lift σ° to a section σ. This gives us an object of $\widetilde{\mathcal{M}_n^{(\infty)}}(T)$.

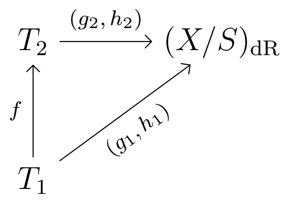
Of course, σ is not unique, but any other choice σ' of lifting will yield an object of $\widetilde{\mathcal{M}_n^{(\infty)}}(T)$ canonically isomorphic to the original one. Indeed, we have the following common étale neighbourhood:



Hence up to a canonical isomorphism, we obtain $M_{T\times_S X \rightleftarrows T} \in \mathrm{QCoh}\,(T)$, and we define

$$\Psi(M)(X/S)_{T \to (X/S)_{\mathrm{dR}}} := M_{T\times_S X \rightleftarrows T}.$$

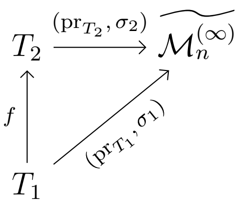
To complete the construction of $\Psi(M)(X/S) \in \mathrm{QCoh}\,((X/S)_{\mathrm{dR}})$, we need to specify the compatibilities under pullback. Assume we have a commutative diagram in PreStk:



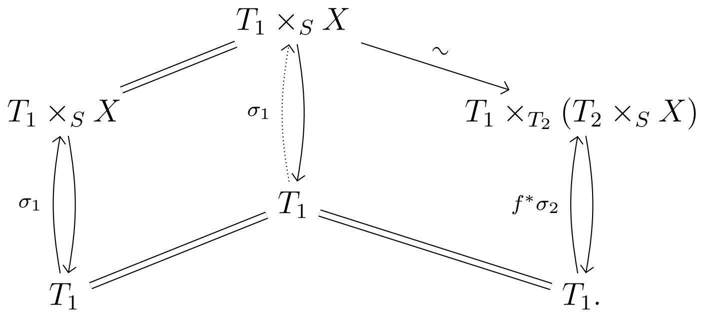
i.e. $(g_1, h_1) = f^*(g_2, h_2) = (g_2 \circ f_{red}, h_2 \circ f)$. We need to exhibit an isomorphism

$$f^*\left(\Psi(M)(X/S)_{T_2 \to (X/S)_{\mathrm{dR}}}\right) \xrightarrow{\sim} \Psi(M)(X/S)_{T_1 \to (X/S)_{\mathrm{dR}}}$$

or equivalently $f^* M_{T_2\times_S X \rightleftarrows T_2} \xrightarrow{\sim} M_{T_1\times_S X \rightleftarrows T_1}$. To do this, it suffices to show that the following diagram commutes canonically in PreStk:



i.e. to exhibit a canonical (up to (∞)-equivalence) common étale neighbourhood between $T_1 \times_S X \rightleftarrows T_1$ and $T_1 \times_{T_2} (T_2 \times_S X) \rightleftarrows T_1$. The obvious candidate is



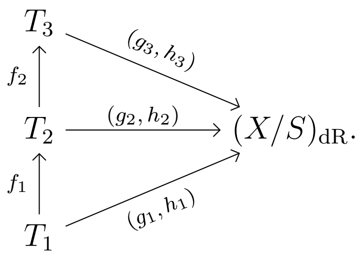
(II.8)

The (reduced) commmutativity of this diagram follows from noting that $\sigma_1 \circ \iota_{T_1} = (\iota_{T_1}, g_1)$ and $f^*\sigma_2 \circ \iota_{T_1} = (\iota_{T_1}, g_2 \circ f_{red})$ as maps $(T_1)_{red} \to T_1 \times_S X$.

This yields the desired isomorphism[3]

$$M(f, T_1 \times_S X) : f^* M_{T_2 \times_S X \rightleftarrows T_2} \xrightarrow{\sim} M_{T_1 \times_S X \rightleftarrows T_1},$$

and the compatibility of these isomorphisms comes from the structure of M. Indeed, suppose we have the following commutative diagram



Then we need to show that the isomorphism $f_1^* f_2^* M_{T_3 \times_S X \rightleftarrows T_3} \xrightarrow{\sim} M_{T_1 \times_S X \rightleftarrows T_1}$ is equal to the composition

$$f_1^* f_2^* M_{T_3 \times_S X \rightleftarrows T_3} \xrightarrow{\sim} f_1^* M_{T_2 \times_S X \rightleftarrows T_2} \xrightarrow{\sim} M_{T_1 \times_S X \rightleftarrows T_1}.$$

[3]Here and in the following we suppress the étale morphisms and write simply $(T_1 \times_S X)$ for the common étale neighbourhood (II.8).

That is, we need to prove that

$$M(f, T_1 \times_S X \rightleftarrows T_1) \circ f_1^* M(f_2, T_2 \times_S X \rightleftarrows T_2) = M(f_2 \circ f_1, T_1 \times_S X \rightleftarrows T_1).$$

This follows from the compatibility of M with respect to pullbacks, and the fact that the composition of the morphisms represented by the common étale neighbourhoods given by $(T_1 \times_S X)$ and $(T_2 \times_S X)$ is represented by the common étale neighbourhood $(T_1 \times_S X)$ between $(T_1 \times_S X \rightleftarrows T_1)$ and $(f_2 \circ f_1)^* (T_3 \times_S X \rightleftarrows T_3)$.

Therefore $\Psi(M)(X/S) \in \mathrm{QCoh}\left((X/S)_{\mathrm{dR}}\right)$, as claimed.

Finally, to show that $\Psi(M) \in \mathscr{U}_n^{\mathcal{D}}$ we need to define the isomorphisms $\Psi(M)(f)$ associated to fibrewise étale morphisms $f = (f_X, f_S) : (X/S) \to (X'/S')$. We need to define an isomorphism $\Psi(M)(X/S) \xrightarrow{\sim} f_{X/S}^* \Psi(M)(X'/S')$ of sheaves on $(X/S)_{\mathrm{dR}}$, i.e. a compatible family of isomorphisms

$$\Psi(M)(X/S)_{T \to (X/S)_{\mathrm{dR}}} \xrightarrow{\sim} \left(f_{X/S}^* \Psi(M)(X'/S')\right)_{T \to (X/S)_{\mathrm{dR}}} \tag{II.9}$$

for each $T \to (X/S)_{\mathrm{dR}}$.

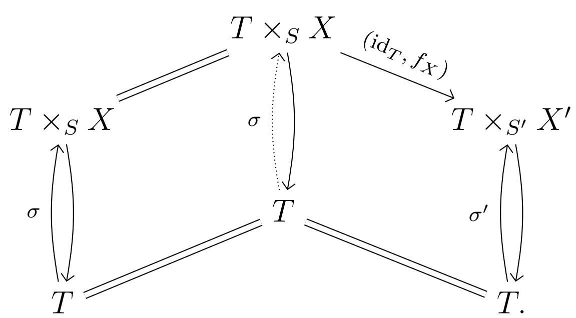
Let $T \to (X/S)_{\mathrm{dR}}$ be the morphism corresponding to the pair $(g : T_{red} \to X, h : T \to S)$. Unwinding the definitions, we see that

$$\left(f_{X/S}^* \Psi(M)(X'/S')\right)_{T \to (X/S)_{\mathrm{dR}}} = \Psi(M)(X'/S')_{T \to (X'/S')_{\mathrm{dR}}}$$

where the morphism $T \to (X'/S')_{\mathrm{dR}}$ corresponds to the pair $(f_X \circ g, f_S \circ h)$. So we can rewrite (II.9) as

$$M_{T \times_S X \rightleftarrows T} \xrightarrow{\sim} M_{T \times_{S'} X' \rightleftarrows T}.$$

To define such an isomorphism, it suffices to exhibit an isomorphism of the corresponding objects in $\widetilde{\mathcal{M}_n^{(\infty)}}(T)$, i.e. a common étale neighbourhood between $(T \times_S X \rightleftarrows T)$ and $(T \times_{S'} X' \rightleftarrows T)$. We can take the following representative:



The reduced commutativity of the right side of the diagram follows from noting that $\sigma \circ \iota_T = (\iota_T, g)$ while $\sigma' \circ \iota_T = (\iota_T, f_X \circ g)$.

Because of the structure of M, these isomorphisms are compatible with pullback along maps $T' \to T$, and hence give the desired isomorphism

$$\Psi(M)(f) : \Psi(M)(X/S) \xrightarrow{\sim} f^*_{X/S}\Psi(M)(X'/S').$$

Indeed, this compatibility with pullbacks amounts to the commutativity of the following diagram:

$$\begin{array}{ccc}
f^*\left(M_{T_2 \times_S X \rightleftarrows T_2}\right) & \xrightarrow{f^*\left(M(\mathrm{id}_{T_2}, T_1 \times_S X)\right)} & f^*\left(M_{T_2 \times_{S'} X' \rightleftarrows T_2}\right) \\
{\scriptstyle M(f, T_1 \times_S X)}\big\uparrow & & \big\uparrow{\scriptstyle M(f, T_1 \times_{S'} X')} \\
M_{T_1 \times_S X \rightleftarrows T_1} & \xrightarrow[M(\mathrm{id}_{T_1}, T_1 \times_{S'} X')]{} & M_{T_1 \times_{S'} X' \rightleftarrows T_1}.
\end{array}$$

In turn, the maps $\Psi(M)(f)$ are themselves compatible with composition: this amounts to the fact that given any two fibrewise étale morphisms $X/S \to X'/S' \to X''/S''$ and any $T \to (X/S)_{\mathrm{dR}}$, we have

$$M\left(\mathrm{id}_T, T \times_{S'} X'\right) \circ M\left(\mathrm{id}_T, T \times_S X\right) = M\left(\mathrm{id}_T, T \times_S X\right),$$

where $T \times_S X$ represents a morphism from $T \times_S X/T$ to $T \times_{S'} X'/T$ in the first instance and from $T \times_S X/T$ to $T \times_{S''} X''/T$ in the second, and $T \times_{S'} X'$ represents a morphism from $T \times_{S'} X'/T$ to $T \times_{S''} X''/T$. We conclude that $\Psi(M)$ is indeed a universal $\mathcal{D}$-module.

The definition of Ψ on morphisms of $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$ is clear: a morphism $F : M \to N$ of quasi-coherent sheaves on $\widetilde{\mathcal{M}_n^{(\infty)}}$ amounts to a compatible family of morphisms $F_{X \rightleftarrows S} : M_{X \rightleftarrows S} \to N_{X \rightleftarrows S} \in \mathrm{QCoh}\,(S)$ indexed by morphisms $S \to \widetilde{\mathcal{M}_n^{(\infty)}}$. Then we define $\Psi(F) : \Psi(M) \to \Psi(N)$ by setting

$$\Psi(F)(X/S)_{T \to (X/S)_{\mathrm{dR}}} : M_{T \times_S X \rightleftarrows T} \to N_{T \times_S X \rightleftarrows T}$$

to be equal to $F_{T \times_S X \rightleftarrows T}$. It is not hard to see that this definition is compatible with pullback by morphisms $T' \to T$ as well as with the structure morphisms $\Psi(M)(f)$ and $\Psi(N)(f)$ corresponding to fibrewise étale morphisms $f = (f_X, f_S) : X'/S' \to X/S$. It is also immediate that $\Psi(F \circ G) = \Psi(F) \circ \Psi(G)$, and so Ψ gives a functor $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right) \to \mathscr{U}_n^{\mathcal{D}}$.

5.3 Compatibility of θ and Ψ

It remains to check that θ and Ψ are indeed quasi-inverse. First suppose that we have $M \in \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$ and consider $\theta \circ \Psi(M) \in \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$. For $(\pi : X \rightleftarrows S : \sigma) \in \widetilde{\mathcal{M}_n^{(\infty)}}$, we have

$$(\theta \circ \Psi(M))_{X \rightleftarrows S} = \overline{\sigma}^* \Psi(M)(X/S).$$

Here $\overline{\sigma} : S \to (X/S)_{\mathrm{dR}}$ corresponds by definition to the pair $(\sigma \circ \iota_S, \mathrm{id}_S)$, so it follows that

$$\overline{\sigma}^* \Psi(M)(X/S) = M_{S \times_S X \rightleftarrows S},$$

and $(S \times_S X \rightleftarrows S) \simeq (X \rightleftarrows S)$. Therefore

$$(\theta \circ \Psi(M))_{X \rightleftarrows S} \simeq M_{X \rightleftarrows S},$$

which gives the natural isomorphism between $\theta \circ \Psi$ and $\mathrm{Id}_{\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)}$.

Conversely, let $\mathscr{F} \in \mathscr{U}_n^{\mathcal{D}}$ and consider $\Psi \circ \theta(\mathscr{F})$. Take $\pi : X \to S$ smooth of dimension n and $T \to (X/S)_{\mathrm{dR}}$ corresponding to a compatible pair of morphisms $(g : T_{red} \to X, h : T \to S)$. Then

$$\begin{aligned} (\Psi \circ \theta(\mathscr{F}))\,(X/S)_{T \to (X/S)_{\mathrm{dR}}} &= \theta(\mathscr{F})_{\mathrm{pr}_T : T \times_S X \rightleftarrows T : \sigma} \\ &= \overline{\sigma}^* \left(\mathscr{F}(T \times_S X/T)\right), \end{aligned}$$

where σ is a section $T \to T \times_S X$ such that $\sigma \circ \iota_T = (\mathrm{id}_T, g)$. Notice that $f := (\mathrm{pr}_X, h)$ gives a fibrewise étale map $(T \times_S X)/T \to (X/S)$, so that we have

$$\mathscr{F}(f) : \mathscr{F}(T \times_S X/T) \xrightarrow{\sim} f_{X/S}^* \mathscr{F}(X/S).$$

Finally, unwinding the definitions of $f_{X/S}$ and $\overline{\sigma}$ shows that $f_{X/S} \circ \overline{\sigma} : T \to (X/S)_{\mathrm{dR}}$ agrees with (g, h); hence we have

$$\begin{aligned} \overline{\sigma}^* \left(\mathscr{F}(T \times_S X/T)\right) &\simeq \overline{\sigma}^* f_{X/S}^* \mathscr{F}(X/S) \\ &\simeq \mathscr{F}(X/S)_{T \to (X/S)_{\mathrm{dR}}} \end{aligned}$$

as required. These isomorphisms gives the desired natural isomorphisms between $\Psi \circ \theta$ and $\mathrm{Id}_{\mathscr{U}_n^{\mathcal{D}}}$. The proof is complete. □

5.4 The $\mathcal{O}$-module setting

We have an analogous result in the case of universal $\mathcal{O}$ modules:

Theorem 5.4.1. *The category $\mathscr{U}_n^{\mathcal{O}}$ of universal $\mathcal{O}$-modules of dimension n is equivalent to the category QCoh* $\left(\widetilde{\mathcal{M}_n^{pt,(\infty)}}\right)$ *and hence to the category QCoh* $\left(\mathcal{M}_n^{pt,(\infty)}\right)$.

Proof. The idea behind the proof is similar to the case of universal $\mathcal{D}$-modules: we proceed by defining functors in both directions and checking that they are quasi-inverse to each other. In brief, we have

$$\theta : \mathscr{U}_n^{\mathcal{O}} \to \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{\mathrm{pt},(\infty)}}\right) \qquad \text{(II.10)}$$
$$\mathscr{F} \mapsto \left((\pi : X \rightleftarrows S : \sigma) \mapsto \sigma^*(\mathscr{F}_{X/S})\right);$$

$$\Psi : \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{\mathrm{pt},(\infty)}}\right) \to \mathscr{U}_n^{\mathcal{O}} \qquad \text{(II.11)}$$
$$M \mapsto \left((X \to S) \mapsto M_{\mathrm{pr}_1 : X\times_S X \rightleftarrows X : \Delta}\right).$$

We refrain from spelling out the details—the definitions and arguments are along the same lines as those used in the proof of Theorem 5.0.8, but simpler. □

6 Convergent and ind-finite universal modules

So far, we have identified the category of universal $\mathcal{D}$-modules with the category of representations of the group-valued prestack $G^{\text{ét}}$. Furthermore, we have identified $\mathrm{Rep}(G)$ as a full subcategory of $\mathrm{Rep}(G^{\text{ét}})$. In this section, we study the corresponding full subcategory of $\mathscr{U}_n^{\mathcal{D}}$.

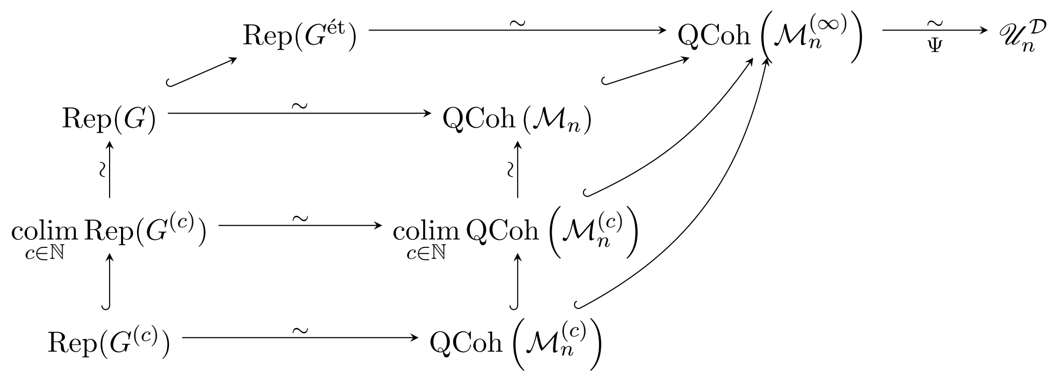
We have two approaches: in 6.1 we take the first approach, via the description of $\mathrm{Rep}(G)$ as $\mathrm{colim}_{c\in\mathbb{N}}\,\mathrm{Rep}(G^{(c)})$, as in Proposition 2.4.1. The second method uses the characterisation of representations of G as those representations of $G^{\text{ét}}$ which are locally finite when viewed as representations of $K^{\text{ét}}$, as in Corollary 4.4.3. We discuss this in 6.2 and 6.3. Comparing the results obtained from each of these approaches allows us to provide two characterisations of those universal $\mathcal{D}$-modules which lie in the essential image of $\mathrm{Rep}(G)$ under the equivalence $\mathrm{Rep}(G^{\text{ét}}) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D}}$. We will call these the *convergent* universal $\mathcal{D}$-modules.

6.1 Convergent universal modules

Recall the stack of formal germs introduced in Remark 3.4.6:

$$\mathcal{M}_n := \lim_{c\in\mathbb{N}} \mathcal{M}_n^{(c)}.$$

Combining the results of Proposition 2.4.1, Corollary 4.4.3, and Theorem 5.0.8, we obtain the following diagram:



Since our goal is to identify Rep(G) with a category of universal $\mathcal{D}$-modules, we must now study the essential image of QCoh$\left(\mathcal{M}_n^{(c)}\right)$ under Ψ for each $c \in \mathbb{N}$.

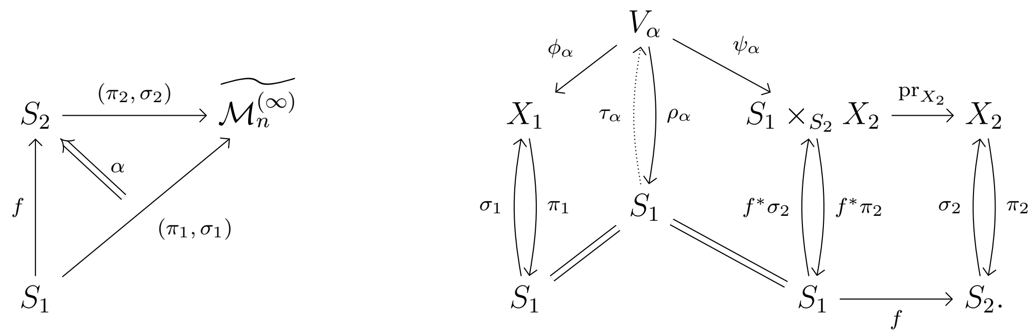
Suppose that $M \in \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right) \hookrightarrow \mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$. Then we know that M consists of the data of a sheaf

$$M_{X\rightleftarrows S} \in \mathrm{QCoh}\,(S)$$

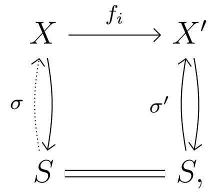
for each object $(X \rightleftarrows S) \in \widetilde{\mathcal{M}_n^{(\infty)}}$, together with isomorphisms

$$M(f,\alpha) : f^* M_{X'\rightleftarrows S'} \xrightarrow{\sim} M_{X\rightleftarrows S} \in \mathrm{QCoh}\,(S)\,,$$

for any commutative diagram in PreStk of the form



The fact that M is an object of $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right)$ amounts to the condition that $M(f,\alpha) = M(f,\alpha')$ whenever any representatives of α and α' are (c)-equivalent (c.f. Remark 5.1.1). Let us consider the implications of this condition for the corresponding universal $\mathcal{D}$-module $\Psi(M)$. Suppose that we have two étale maps f_1, f_2 of n-dimensional families over some base scheme S



inducing the same isomorphism of the cth infinitesimal neighbourhoods of S in X and X':

$$f_1^{(c)} = f_2^{(c)} : X_S^{(c)} \xrightarrow{\sim} X_S'^{(c)}.$$

Then we obtain isomorphisms

$$\Psi(M)(f_i) : \Psi(M)(X/S) \xrightarrow{\sim} f_{i,X/S}^{*}\Psi(M)(X'/S) \in \mathrm{QCoh}\left((X/S)_{\mathrm{dR}}\right), \quad i = 1,2,$$

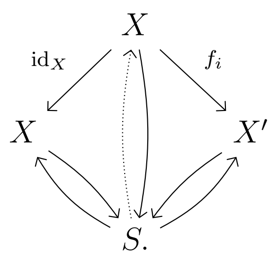
and pulling back along $\overline{\sigma} : S \to (X/S)_{\mathrm{dR}}$ yields maps

$$\overline{\sigma}^{*}\Psi(M)(f_i) : \overline{\sigma}^{*}\Psi(M)(X/S) \xrightarrow{\sim} \overline{\sigma}^{*} f_{i,X/S}^{*}\Psi(M)(X'/S) \in \mathrm{QCoh}\left(S\right), \quad i = 1,2.$$

Note that $f_{1,X/S} \circ \overline{\sigma} = \overline{\sigma'} = f_{2,X/S} \circ \overline{\sigma}$, and so the maps $\overline{\sigma}^{*}\Psi(M)(f_i)$ are maps between the same sheaves on S. From the definition of $\Psi(M)$, we identify

$$\overline{\sigma}^{*}\Psi(M)(X/S) = M_{X \rightleftarrows S}; \qquad \overline{\sigma'}^{*}\Psi(M)(X'/S\prime) = M_{X' \rightleftarrows S},$$

and we see that for $i = 1,2$ the map $\overline{\sigma}^{*}\Psi(M)(f_i)$ is given by the structure isomorphism $M(\mathrm{id}_S, \alpha_i)$ of M, where α_i is the isomorphism in $\widetilde{\mathcal{M}_n^{(\infty)}}$ represented by the common étale neighbourhood



Since these two common étale neighbourhoods are (c)-equivalent, it follows that $M(\mathrm{id}_S, \alpha_1) = M(\mathrm{id}_S, \alpha_2)$.

Motivated by this observation, we formulate the following definition:

Definition 6.1.1. A universal $\mathcal{D}$-module $\mathscr{F}$ is *of cth order* if whenever we have two étale morphisms f_1, f_2 of n-dimensional families over S

$$\begin{array}{ccc} X & \xrightarrow{f_i} & X' \\ \sigma \updownarrow & & \updownarrow \sigma' \\ S & = & S \end{array}$$

such that

$$f_1^{(c)} = f_2^{(c)} : X_S^{(c)} \xrightarrow{\sim} X_S'^{(c)},$$

then we have that

$$\overline{\sigma}^* \mathscr{F}(f_1) = \overline{\sigma}^* \mathscr{F}(f_2) : \overline{\sigma}^* \mathscr{F}_{X/S} \xrightarrow{\sim} \overline{\sigma'}^* \mathscr{F}_{X'/S}.$$

We let $\mathscr{U}_n^{\mathcal{D},(c)}$ denote the full subcategory of $\mathscr{U}_n^{\mathcal{D}}$ whose objects are the universal $\mathcal{D}$-modules of cth order.

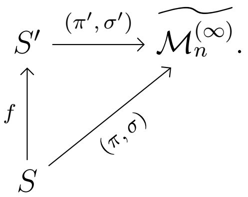
Proposition 6.1.2. *The functor* $\Psi : QCoh\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D}}$ *restricts to an equivalence*

$$\Psi^{(c)} : QCoh\left(\widetilde{\mathcal{M}_n^{(c)}}\right) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D},(c)}.$$

Proof. By the above discussion, the restriction $\Psi^{(c)}$ of Ψ to $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right)$ is a fully faithful embedding into $\mathscr{U}_n^{\mathcal{D},(c)}$. To complete the proof, it suffices to show that the functor $\theta : \mathscr{U}_n^{\mathcal{D}} \xrightarrow{\sim} \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$ restricts to a functor

$$\theta^{(c)} : \mathscr{U}_n^{\mathcal{D},(c)} \to \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right).$$

So let us assume that $\mathscr{F} \in \mathscr{U}_n^{\mathcal{D},(c)}$, and consider the quasi-coherent sheaf $\theta(\mathscr{F}) \in \mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(\infty)}}\right)$. Suppose that we have a diagram in PreStk of the form:

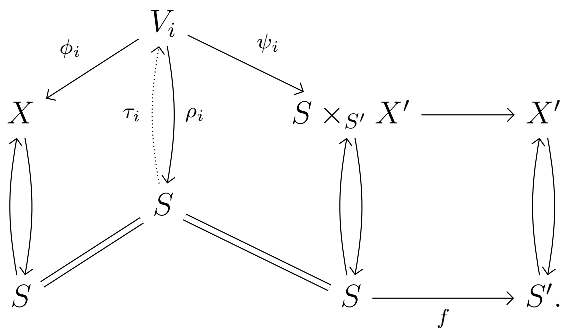


Assume in addition that we have two isomorphisms

$$\alpha_i \in \operatorname{Hom}_{\widetilde{\mathcal{M}_n^{(\infty)}}(S)} \left((\pi, \sigma), (\pi', \sigma') \circ f\right)$$

which make this diagram commute, and which are (c)-equivalent, although not necessarily (∞)-equivalent. In order to show that $\theta(\mathscr{F})$ is a quasi-coherent sheaf on $\widetilde{\mathcal{M}_n^{(c)}}$, we need to show that the structure isomorphisms $\theta(\mathscr{F})(f, \alpha_i)$ agree with each other.

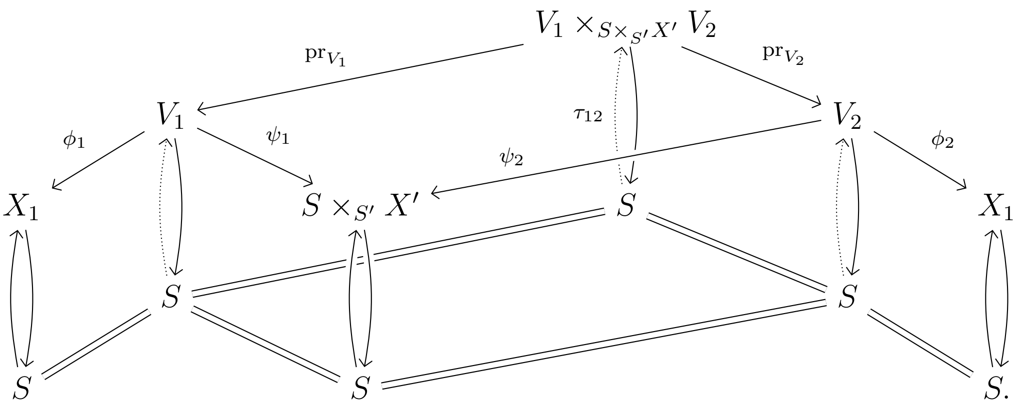
Let us choose representatives of the isomorphisms α_i as follows:



Then it suffices to show that

$$\overline{\tau_1}^* \left(\mathscr{F}(\phi_1, \mathrm{id}_S) \circ \mathscr{F}(\psi_1, \mathrm{id}_S)^{-1}\right) = \overline{\tau_2}^* \left(\mathscr{F}(\phi_2, \mathrm{id}_S) \circ \mathscr{F}(\psi_2, \mathrm{id}_S)^{-1}\right).$$

Equivalently, we can compose the representatives of α_1 and α_2^{-1} to obtain a common étale neighbourhood which we'll call α_{12}:



Now we need to show that

$$\overline{\tau_{12}}^{*}\left(\mathscr{F}(\phi_1 \circ \mathrm{pr}_{V_1}, \mathrm{id}_S) \circ \mathscr{F}(\phi_2 \circ \mathrm{pr}_{V_2}, \mathrm{id}_S)^{-1}\right) = \mathrm{id}_{\overline{\tau}^{*}\mathscr{F}(X/S)}\,.$$

We can use the fact that $\mathscr{F}$ is of cth order: it suffices to show that α_{12} is (c)-equivalent to the identity. But of course

$$\begin{aligned}\left(\phi_1 \circ \mathrm{pr}_{V_1}\right)^{(c)} \circ \left(\left(\phi_2 \circ \mathrm{pr}_{V_2}\right)^{(c)}\right)^{-1} &= \phi_1^{(c)} \circ \mathrm{pr}_{V_1}^{(c)} \circ \left(\mathrm{pr}_{V_2}^{(c)}\right)^{-1} \circ \left(\phi_2^{(c)}\right)^{-1} \\ &= \phi_1^{(c)} \circ \left(\psi_1^{(c)}\right)^{-1} \circ \psi_2^{(c)} \circ \left(\phi_2^{(c)}\right)^{-1} \\ &= \mathrm{id}_{X_S^{(c)}}\,.\end{aligned}$$

So $\theta(\mathscr{F})$ is indeed an object of the subcategory $\mathrm{QCoh}\left(\widetilde{\mathcal{M}_n^{(c)}}\right)$, and the proof is complete. ☐

The following is immediate:

Corollary 6.1.3. *We have an equivalence of categories*

$$\mathscr{U}_n^{\mathcal{D},(c)} \simeq \mathrm{Rep}(G^{(c)}).$$

We have the following nested sequence of subcategories of $\mathscr{U}_n^{\mathcal{D}}$:

$$\ldots \hookrightarrow \mathscr{U}_n^{\mathcal{D},(c)} \hookrightarrow \mathscr{U}_n^{\mathcal{D},(c+1)} \hookrightarrow \ldots \hookrightarrow \mathscr{U}_n^{\mathcal{D}}.$$

Definition 6.1.4. Let

$$\mathscr{U}_n^{\mathcal{D},\mathrm{conv}} := \operatorname*{colim}_{c\in\mathbb{N}} \mathscr{U}_n^{\mathcal{D},(c)}.$$

It is a full subcatgory of $\mathscr{U}_n^{\mathcal{D}}$. An object of $\mathscr{U}_n^{\mathcal{D},\mathrm{conv}}$ will be called a *convergent* universal $\mathcal{D}$-module of dimension n.

Corollary 6.1.5. *The essential image of* $\mathrm{colim}_{c\in\mathbb{N}}$ *QCoh* $\left(\mathcal{M}_n^{(c)}\right)$ *in* $\mathscr{U}_n^{\mathcal{D}}$ *is* $\mathscr{U}_n^{\mathcal{D},conv}$. *We have an equivalence of categories*

$$\mathrm{Rep}(G) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D},conv}.$$

We can similarly define the category $\mathscr{U}_n^{\mathcal{O},(c)}$ of cth-order universal $\mathcal{O}$-modules and can show that

$$\mathscr{U}_n^{\mathcal{O},(c)} \simeq \mathrm{QCoh}\left(\mathcal{M}_n^{\mathrm{pt},(c)}\right)$$

for any $c \in \mathbb{N}$. Letting $\mathscr{U}_n^{\mathcal{O},\mathrm{conv}} := \mathrm{colim}_{c\in\mathbb{N}} \mathscr{U}_n^{\mathcal{O},(c)}$, we obtain the following:

Proposition 6.1.6. *We have an equivalence of categories*

$$\mathrm{Rep}(K) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{O},conv}.$$

6.2 Ind-finite universal $\mathcal{O}$-modules

In this subsection, we take a different approach, beginning with the identification

$$\operatorname{Rep}(G) \xrightarrow{\sim} \operatorname{Rep}_{K^{\text{ét}}\text{-l.f.}}(G^{\text{ét}}).$$

As has generally been the case when working with the groups $G^{\text{ét}}$ and $K^{\text{ét}}$, we will begin by studying the picture for $K^{\text{ét}}$, and then extend our results to $G^{\text{ét}}$.

Our first step is to identify the subcategory of $\operatorname{QCoh}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ corresponding to the locally finite representations of K. First notice that a representation V of $K^{\text{ét}}$ is finite-dimensional if and only if the corresponding sheaf M on $\operatorname{QCoh}\left(\mathrm{BK}^{\text{ét}}\right)$ and hence on $\operatorname{QCoh}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ is of *finite type*: that is, for every $S = \operatorname{Spec} R \to \mathrm{BK}^{\text{ét}} \simeq \mathcal{M}_n^{\text{pt},(\infty)}$, the corresponding sheaf $M_S \in \operatorname{QCoh}(S)$ is of finite type.

Remark 6.2.1. This is equivalent to requiring the sheaf M_S to be of finite presentation, and in fact to be locally free. This follows from three facts: these properties are all equivalent for $\operatorname{QCoh}(\mathrm{pt}) \simeq \mathrm{Vect}$; they are preserved by pullback; and every morphism $S \to \left(\mathcal{M}_n^{\text{pt},(\infty)}\right)_{\text{triv}}$ factors through $S \to \mathrm{pt}$.

However, it is not equivalent to requiring the sheaf M_S to be coherent, because coherence is not preserved under arbitrary pullbacks. For example, if $V = k$ is the trivial representation, with M the corresponding sheaf on S, then $M_S = R$ for every $S = \operatorname{Spec} R$. If R is a k-algebra which is not coherent as an R-module, then M_S is finitely generated, but it is not coherent.

It follows that the essential image of $\operatorname{Rep}^{\text{l.f.}}(K^{\text{ét}})$ in $\operatorname{QCoh}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ is the full subcategory generated by those sheaves $M \in \operatorname{QCoh}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ which can be written as a union $M = \bigcup_i M_i$ of sheaves M_i of finite type.

Definition 6.2.2. Let $\mathcal{Y}$ be a prestack, and $M \in \operatorname{QCoh}(\mathcal{Y})$ be a sheaf that can be written as the colimit of its subsheaves of finite type. Then we say that M is of *ind-finite type*. We denote the full subcategory of ind-finite sheaves by $\operatorname{QCoh}^{\text{i.f.}}(\mathcal{Y})$.

Of course, for any sheaf $M \in \operatorname{QCoh}(\mathcal{Y})$ and for any $S = \operatorname{Spec} R$, we can always write M_S as a union of finitely generated subsheaves; however, this cannot always be done in a way compatibly with all pullbacks and automorphisms of S-points of $\operatorname{QCoh}(\mathcal{Y})$.

Example 6.2.3. Recall the notation of section 4.3. Let $\mathcal{Y} = B\mathbb{A}^{\infty}$, and let $M \in \operatorname{QCoh}(\mathcal{Y})$ be the sheaf corresponding to the regular representation V. Then M is not an object of $\operatorname{QCoh}^{\text{i.f.}}(\mathcal{Y})$.

We do not know if $\mathrm{QCoh}^{\text{i.f.}}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ is equal to $\mathrm{QCoh}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$, or if it is a proper subcategory. By construction, this question is equivalent to the question of whether $\mathrm{Rep}^{\text{l.f.}}(K^{\text{ét}})$ is a proper subcategory of $\mathrm{Rep}(K^{\text{ét}})$.

Now we can study the essential image of $\mathrm{QCoh}^{\text{i.f.}}\left(\mathcal{M}_n^{\text{pt},(\infty)}\right)$ in $\mathscr{U}_n^{\mathcal{O}}$. It is the full subcategory whose objects are those universal $\mathcal{O}$-modules which can be written as a union of their submodules of finite type:

$$\mathscr{F} = \bigcup_i \mathscr{F}_i,$$

where $\mathscr{F}_i \in \mathscr{U}_n^{\mathcal{O}}$ is such that for any X/S smooth of dimension n, $\mathscr{F}_{i,X/S} \in \mathrm{QCoh}\,(X)$ is of finite type. This is equivalent to requiring $\mathscr{F}_{i,X/S}$ to be locally free of finite rank: this is because the translation-invariance of $\mathscr{F}$ ensures that $\mathscr{F}_{i,X/S}$ is of constant rank.

Definition 6.2.4. If $\mathscr{F}$ is a universal $\mathcal{O}$-module satisfying this condition, then we shall say that $\mathscr{F}$ is a universal $\mathcal{O}$-module of *ind-finite type*. We denote the subcategory of universal $\mathcal{O}$-modules of ind-finite type by $\mathscr{U}_n^{\mathcal{O},\text{i.f.}}$.

The following result is clear by definition:

Proposition 6.2.5. *The equivalence* $\mathrm{Rep}(K^{\acute{e}t}) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{O}}$ *restricts to give an equivalence of categories*

$$\mathrm{Rep}(K) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{O},i.f.}.$$

6.3 Ind-finite universal $\mathcal{D}$-modules

Similarly, it is clear that the essential image in $\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$ of $\mathrm{Rep}\,G$ is the subcategory of sheaves $M \in \mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$ such that the pullback of M along the map

$$\mathcal{M}_n^{\text{pt},(\infty)} \to \mathcal{M}_n^{(\infty)}$$

is of ind-finite type. We denote this category by

$$\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)_{\mathcal{M}_n^{\text{pt},(\infty)}\text{-i.f.}} \hookrightarrow \mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right).$$

Again, we do not know whether this is in fact a proper subcategory.

We can again characterise the image of this subcategory in $\mathscr{U}_n^{\mathcal{D}}$:

Definition 6.3.1. A universal $\mathcal{D}$-module $\mathscr{F}$ is of *ind-finite type* if it is of ind-finite type when regarded as a universal $\mathcal{O}$-module. We denote the full subcategory of universal $\mathcal{D}$-modules of ind-finite type by $\mathscr{U}_n^{\mathcal{D},\text{i.f.}}$.

Remark 6.3.2. Let us emphasise that the decomposition of $\mathscr{F}$ into subsheaves of finite type only needs to respect the $\mathcal{O}$-module structures; we do not expect the subsheaves $\mathscr{F}_i \subset \mathscr{F}$ to be sub-$\mathcal{D}$-modules of $\mathscr{F}$.

Proposition 6.3.3. *The equivalence* $\mathrm{Rep}(G^{\acute{e}t}) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D}}$ *restricts to an equivalence of subcategories*

$$\mathrm{Rep}(G) \xrightarrow{\sim} \mathscr{U}_n^{\mathcal{D},i.f.}.$$

Combining this with our previous results (essentially, travelling to the left and then back again to the right along the middle two rows of the main diagram in Figure 1), we deduce the following:

Proposition 6.3.4. *A universal $\mathcal{O}$- or $\mathcal{D}$-module is of ind-finite type if and only if it is convergent.*

Our proposal is that these categories of convergent universal modules, rather than the full categories of universal modules as defined in [4], are the more natural categories with which to work. Of course, we do not know if in fact the categories are equivalent.

7 Remarks on ∞-categories

In this section, we extend our results to the ∞-categories of representations and universal modules. In fact, none of these categories are previously well-established in the literature, so we have some freedom to choose our definitions to allow our results to extend. We will provide some justification of our choices as we proceed, but the very fact that our results extend so naturally is in itself a good defence for these definitions.

The motivation for working with ∞-categories rather than ordinary categories is the following. Recall that we are interested in the study of universal chiral algebras of dimension n, under the hypothesis that these give the correct notion of an n-dimensional vertex algebra. In particular, a universal chiral algebra of dimension n is a universal $\mathcal{D}$-module. Although the (ordinary) categories of universal $\mathcal{D}$-modules and representations of G may be interesting in their own right, if we wish to work with universal chiral algebras of dimension two or higher, we immediately see that it is necessary to work in the derived setting. For example, if we insist on remaining in the abelian categories, the definitions of universal chiral algebras and Lie $\star$ algebras (as in [9]) become equivalent.

7.1 Conventions

Henceforth all categories of sheaves, modules, vector spaces, and $\mathcal{D}$-modules will be assumed to be the $(\infty, 1)$-categories, unless otherwise specified. We shall appropriate notation established earlier in the chapter for abelian categories without further decoration by symbols such as "d.g." or "∞"; when we wish to refer to the abelian hearts of these categories, we shall indicate it with a superscript $\heartsuit$. When we say "category", we mean cocomplete $(\infty, 1)$-category; it is in this sense that we take colimits, for example.

7.2 ∞-categories of universal modules and representations

We can begin by naïvely extending the definition of a universal $\mathcal{D}$-module (and similarly a universal $\mathcal{O}$-module), repeating the definition 5.0.5 in the setting of ∞-categories. (We will carry this out explicitly for cth-order $\mathcal{D}$-modules in 7.3.) The functors θ and Ψ of Theorem 5.0.8 admit ∞-categorical extensions, and provide an equivalence between the categories $\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$ and $\mathscr{U}_n^{\mathcal{D}}$. The equivalence between $\mathcal{M}_n^{(\infty)}$ and $BG^{\text{ét}}$ is purely geometric, valid before considering categories or ∞-categories of sheaves, and hence we still have the equivalence

$$\mathrm{QCoh}\left(BG^{\text{ét}}\right) \simeq \mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right).$$

In fact, apart from the first column, the entire content of the main diagram lifts immediately from Cat to DGCat with no serious modifications.

However, it is not immediately clear what we should take for the ∞-category of representations of our groups. For an algebraic group H, we take

$$\mathrm{Rep}(H) \coloneqq \mathrm{QCoh}\,(BH)\,,$$

as in e.g. 6.4.3, [7]. Since BH is a smooth Artin stack, we can show that

$$\Upsilon_{BH} : \mathrm{QCoh}\,(BH) \to \mathrm{IndCoh}\,(BH)$$

is an equivalence of categories, so we could also have defined

$$\mathrm{Rep}(H) = \mathrm{IndCoh}\,(BH)\,.$$

Recall that the functors Υ intertwine the $*$-pullback on $\mathrm{QCoh}\,(\bullet)$ with the !-pullback on $\mathrm{IndCoh}\,(\bullet)$.

On the other hand, given a pro-algebraic group H, we can consider the category $\mathrm{QCoh}\,(BH)$, but the category $\mathrm{IndCoh}\,(BH)$ is not defined, because the stack BH is not locally of finite type. It can, however, be written as the limit of stacks which are locally of finite type, and this leads us to a second potential definition for the category of representations.

More precisely, recall that if we write $H = \lim_i H_i$, with H_i finite-dimensional quotients, and all maps $H_i \to H_j$ smooth surjections of algebraic groups, then by Proposition 2.3.6

$$\operatorname{Rep}^{\heartsuit}(H) \simeq \operatorname*{colim}_i \operatorname{Rep}^{\heartsuit}(H_i).$$

Motivated by this fact, it is natural to consider the category

$$\operatorname*{colim}_i \operatorname{Rep}(H_i) = \operatorname*{colim}_i \operatorname{QCoh}^*(BH_i) \simeq \operatorname*{colim}_i \operatorname{IndCoh}^!(BH_i),$$

and unlike in the abelian categories, this category is not all of $\operatorname{QCoh}(BH)$. Instead, we think of $\operatorname{QCoh}(BH)$ as the DG-category of representations of H, and $\operatorname{colim}_i \operatorname{Rep}(H_i)$ as the subcategory of representations of H which are locally finite. In the ∞-categorical setting, this condition is not automatically satisfied, but it is one which we are happy to impose. In other words, we set

$$\operatorname{Rep}(H) := \operatorname*{colim}_i \operatorname{Rep}(H_i).$$

Similarly, for a group formal scheme that can be written as $L = \lim_i L_i$ (such as the group G of automorphisms of the formal disc) with $H = L_{red} = \lim_i L_{i,red}$ a pro-algebraic group, we set

$$\operatorname{Rep}(L) := \operatorname*{colim}_i \operatorname{Rep}(L_i).$$

Although the L_i are themselves indschemes, they are of finite type, and hence BL_i is locally of finite type and $\operatorname{Rep}(L_i)$ is given by $\operatorname{QCoh}(BL_i) \simeq \operatorname{IndCoh}(BL_i)$.

We do not know how to define a corresponding category for an arbitrary group-valued prestack. In the case of $G^{\text{ét}}$ and $K^{\text{ét}}$, for example, the stacks $BG^{\text{ét}}$ and $BK^{\text{ét}}$ seem to be quite intractable. Fortunately the relative Artin approximation theorem and its corollaries from section 3.4 allow us to approximate these stacks using the stacks $BG^{(c)}$ and $BH^{(c)}$, which are easier to work with. By restricting our attention to representations which are sufficiently finite-dimensional in flavour, we can avoid working with the stacks $BG^{\text{ét}}$ and $BK^{\text{ét}}$ entirely. The cost, however, is that the corresponding categories of representations do not correspond to the categories of arbitrary universal modules, but only those of convergent universal modules. In fact, though, we view this as an advantage rather than a cost.

As a consequence of this discussion, we shall henceforth ignore the back two rows of the main diagram, and shall only work with the front part of the diagram, which can be described entirely using stacks which are locally of finite type. Thus far, we have the following:

$$\begin{array}{ccccc} \mathrm{Rep}(G) & & & & \\ \| & & & & \\ \operatorname*{colim}_{c\in\mathbb{N}} \mathrm{Rep}\left(G^{(c)}\right) & = & \operatorname*{colim}_{c\in\mathbb{N}} \mathrm{QCoh}\left(BG^{(c)}\right) & \xrightarrow{\sim} & \operatorname*{colim}_{c\in\mathbb{N}} \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Rep}\left(G^{(c)}\right) & = & \mathrm{QCoh}\left(BG^{(c)}\right) & \xrightarrow{\sim} & \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right). \end{array}$$

Recall the discussion in remark 5.0.7 on the use of the categories QCoh (•) as compared to IndCoh (•). At that stage, we defended the use of $\mathrm{QCoh}((X/S)_{\mathrm{dR}})$, corresponding to relative left $\mathcal{D}$-modules, rather than its more well-studied and better-behaved counterpart $\mathrm{IndCoh}((X/S)_{\mathrm{dR}})$, corresponding to relative right $\mathcal{D}$-modules. Our reasons were threefold: we wished to remain consistent with the definitions of Beilinson and Drinfeld and to work with abelian categories rather than ∞-categories, and moreover we needed to work with prestacks which were not locally of finite type, and so we could not rely on the theory of ind-coherent sheaves.

Indeed, all of the stacks appearing in the back rows of the main diagram from Figure 1 are of infinite type and hence not well-suited to being studied using the theory of ind-coherent sheaves—but by restricting our attention to the category of convergent universal $\mathcal{D}$-modules, as we have just decided to do, we can avoid using these stacks, instead using only the stacks in the front rows of the diagram, which are locally of finite type. In particular, we can work with ind-coherent sheaves on these prestacks.

Furthermore, we argued that the correct notion of universal $\mathcal{D}$-module should include the convergence condition, regardless of whether it agrees with the definition given by Beilinson and Drinfeld even in the abelian setting. In other words, our three motivations for working with quasi-coherent rather than ind-coherent sheaves have disappeared, and consequently we now feel free to use the better-behaved theory of ind-coherent sheaves in the ∞-categorical setting and to define an ∞-category which will correspond to universal right $\mathcal{D}$-modules.

By taking ind-coherent sheaves rather than quasi-coherent sheaves at each stage, we obtain the "right $\mathcal{D}$-module" version of the above diagram. However, note that each of the stacks appearing in the diagram is (equivalent to) a smooth Artin stack, so that the categories of quasi-coherent sheaves and ind-coherent sheaves are in fact equivalent via the funtors Υ. In other words, the diagrams are actually equivalent, termwise, and the functors Υ between the terms intertwine the morphisms of the diagram as well.

7.3 ∞-categories of convergent universal modules

We have (equivalent) categories

$$\operatorname*{colim}_{c\in\mathbb{N}} \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right),$$
$$\operatorname*{colim}_{c\in\mathbb{N}} \mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right)$$

which should, formally, correspond to categories of universal left and right $\mathcal{D}$-modules, but which in flavour belong to the third column of the main diagram from Figure 1. In order to give an equivalent description of these categories in the language of the fourth column, of “universal modules”, we must apply a construction analogous to that of the functor Ψ. We first study the ∞-categorical analogue of universal right $\mathcal{D}$-modules of cth order; that is, we apply a version of the functor Ψ to the category $\mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right)$. (The $\mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right)$ setting is completely analogous.) We do this here only informally, as the full technical definition is no more enlightening.

Given an object $M \in \mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right)$, we begin to argue as in the proof of Theorem 5.0.8 in subsection 5.2 and obtain the following data:

1. For any $X \to S$ smooth of dimension n with S a scheme of finite type, we have
$$\mathscr{F}(X/S) \in \mathrm{IndCoh}\left((X/S)_{\mathrm{dR}}\right),$$
given by the compatible family
$$\left\{\mathscr{F}(X/S)_{T\to(X/S)_{\mathrm{dR}}} := M_{T\times_S X\rightleftarrows T} \in \mathrm{IndCoh}\,(T)\right\}_{T\in\mathrm{Sch}_{/(X/S)_{\mathrm{dR}}}}.$$

2. For any any pair $(X/S), (X', S')$ of smooth n-dimensional families, and for any fibrewise étale morphism $f : (X/S) \to (X'/S')$, an isomorphism
$$\mathscr{F}(f) : \mathscr{F}_{X/S} \xrightarrow{\sim} f^!_{X/S}\mathscr{F}_{X'/S'}$$
in $\mathrm{IndCoh}\left((X/S)_{\mathrm{dR}}\right)$, defined for each T-point $T \to (X/S)_{\mathrm{dR}}$ to be equal to the compatibility isomorphism
$$M_{T\times_S X\rightleftarrows T} \xrightarrow{\sim} M_{T\times_{S'} X'\rightleftarrows T} \in \mathrm{IndCoh}\,(T).$$

The fact that M depends only on morphisms in $\mathcal{M}_n^{(c)}(T)$ (as compared to in $\mathcal{M}_n^{(\infty)}(T)$) tells us that for any sections $\sigma : S \to X$, $\sigma : S' \to X'$ compatible with f on the level of S'_{red}, the map

$$\overline{\sigma}^! \mathscr{F}(f) : \overline{\sigma}^! \mathscr{F}(X/S) \xrightarrow{\sim} \overline{\sigma'}^! \mathscr{F}(X'/S')$$

depends only on the restriction of f to the cth infinitesimal neighbourhood of $S \hookrightarrow X$.

In subsection 5.2 we then showed that the isomorphisms $\mathscr{F}(f)$ were compatible with composition. Since we are now working in ∞-categories, this compatibility is now a structure rather than a condition, and so we obtain additional data. The first few stages look like this:

3. Given three smooth families with fibrewise maps between them

$$(X/S) \xrightarrow{f} (X'/S') \xrightarrow{g} (X''/S''),$$

we have a natural isomorphism

$$a_{f,g} : f^!_{X/S} \mathscr{F}(g) \circ \mathscr{F}(f) \Rightarrow \mathscr{F}(g \circ f)$$

of isomorphisms $\mathscr{F}(X/S) \to (g \circ f)^! \mathscr{F}(X''/S'')$. (Note that we have omitted from our notation the canonical isomorphism $(g \circ f)^! \Rightarrow f^! \circ g^!$.)

This natural isomorphism is defined for each T-point $T \to (X/S)_{\mathrm{dR}}$ to be the natural transformation between the two maps

$$M_{T \times_S X \rightleftarrows T} \to M_{T \times_{S'} X' \rightleftarrows T} \to M_{T \times_{S''} X'' \rightleftarrows T}$$

and

$$M_{T \times_S X \rightleftarrows T} \to M_{T \times_{S''} X'' \rightleftarrows T},$$

which comes from the structure of M as an object of $\mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right)$.

4. Given four smooth families with fibrewise maps between them

$$(X/S) \xrightarrow{f} (X'/S') \xrightarrow{g} (X''/S'') \xrightarrow{h} (X'''/S'''),$$

the data of (3) gives us two natural isomorphisms between the maps

$$f^! g^! \mathscr{F}(h) \circ f^! \mathscr{F}(g) \circ \mathscr{F}(f) \text{ and } \mathscr{F}(h \circ g \circ f)$$

as follows:

$$\begin{array}{ccc}
f^! g^! \mathscr{F}(h) \circ f^! \mathscr{F}(g) \circ \mathscr{F}(f) & \overset{a_{f,g}}{\Longrightarrow} & f^! g^! \mathscr{F}(h) \circ \mathscr{F}(g \circ f) \\
{\scriptstyle a_{g,h}} \Downarrow & & \Downarrow {\scriptstyle a_{g \circ f,h}} \\
f^! \mathscr{F}(h \circ g) \circ \mathscr{F}(f) & \underset{a_{f,h \circ g}}{\Longrightarrow} & \mathscr{F}(h \circ g \circ f)
\end{array}$$

There is a 3-morphism $b_{f,g,h}$ making this diagram commute.

5. ... and so on ...

Definition 7.3.1. A collection $\mathscr{F} = (\{\mathscr{F}(X/S)\}, \{\mathscr{F}(f)\}, \{a_{f,g}\}, \{b_{f,g,h}\}, \ldots)$ as above will be called a *universal right $\mathcal{D}$-module of cth order.* Such objects form an ∞-category ${}^r\mathscr{U}_n^{\mathcal{D},(c)}$, equivalent by construction to the category $\mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right)$.

Definition 7.3.2. The ∞-category of *convergent universal right $\mathcal{D}$-modules* is by definition the colimit

$${}^r\mathscr{U}_n^{\mathcal{D},\mathrm{conv}} := \operatorname*{colim}_{c \in \mathbb{N}} {}^r\mathscr{U}_n^{\mathcal{D},(c)}.$$

We think of an object of ${}^r\mathscr{U}_n^{\mathcal{D},\mathrm{conv}}$ as a family

$$\mathscr{F} = (\{\mathscr{F}(X/S)\}, \{\mathscr{F}(f)\}, \{a_{f,g}\}, \{b_{f,g,h}\}, \ldots)$$

as in the above description, except with the condition in (2) pertaining to (c)-equivalence for morphisms omitted; instead, we assume that $\mathscr{F}$ has an exhaustive filtration by subobjects $\mathscr{F}^{(c)}$, where for each c, $\mathscr{F}^{(c)}$ does satisfy the condition in (2).

Remark 7.3.3. Of course, it is impossible to specify such an object completely in this manner. However, this description in terms of families of sheaves has a significantly more geometric feel than that of the category of representations of G. This is a particular aspect of the difference between the study of vertex algebras (living on the same side of the story as $\mathrm{Rep}(G)$) and the study of chiral algebras (living on the geometric side).

We have, immediately, the following diagram (with equivalences along the rows):

$$\begin{array}{ccccccc}
\mathrm{Rep}(G) & \dashrightarrow & & & & & {}^r\mathscr{U}_n^{\mathcal{D},\mathrm{conv}} \\
\| & & & & & & \| \\
\operatorname*{colim}_{c\in\mathbb{N}} \mathrm{Rep}\left(G^{(c)}\right) & = & \operatorname*{colim}_{c\in\mathbb{N}} \mathrm{IndCoh}\left(BG^{(c)}\right) & \longrightarrow & \operatorname*{colim}_{c\in\mathbb{N}} \mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right) & \longrightarrow & \operatorname*{colim}_{c\in\mathbb{N}} {}^r\mathscr{U}_n^{\mathcal{D},(c)} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Rep}\left(G^{(c)}\right) & = & \mathrm{IndCoh}\left(BG^{(c)}\right) & \longrightarrow & \mathrm{IndCoh}\left(\mathcal{M}_n^{(c)}\right) & \longrightarrow & {}^r\mathscr{U}_n^{\mathcal{D},(c)}.
\end{array}$$

We have a completely analogous diagram for convergent universal left $\mathcal{D}$-modules, using quasi-coherent sheaves. Because all of the categories in the first three columns are equivalent whether we use IndCoh$(\bullet)$ or QCoh$(\bullet)$, we deduce that the categories of universal right and left $\mathcal{D}$-modules are equivalent as well.

This is somewhat surprising: we do not have, in general, that the categories $\mathrm{QCoh}((X/S)_{\mathrm{dR}})$ and $\mathrm{IndCoh}((X/S)_{\mathrm{dR}})$ are equivalent. We can only say that

$$\Upsilon_{(X/S)_{\mathrm{dR}}} : \mathrm{QCoh}((X/S)_{\mathrm{dR}}) \to \mathrm{IndCoh}((X/S)_{\mathrm{dR}})$$

is a fully faithful embedding. However, as a consequence of the convergence condition, we can see that any universal family $(X/S) \mapsto \mathscr{F}(X/S) \in \mathrm{IndCoh}((X/S)_{\mathrm{dR}})$ will actually take values in the essential image of these functors Υ. Consequently, the categories of convergent universal right and left $\mathcal{D}$-modules are in fact canonically equivalent.

A final remark on the convergence condition is the following: even in the right $\mathcal{D}$-modules setting, where we do work with ind-coherent sheaves, we still do not obtain a category whose objects are all compatible families of ind-coherent sheaves (or relative right $\mathcal{D}$-modules) indexed by smooth n-dimensional families X/S. Instead, what we obtain is, informally, closer to the ind-completion of a category of universal families of coherent sheaves. This is close in spirit to the description of convergent universal $\mathcal{D}$-modules as ind-finite families of modules from section 6.3.

7.4 An example: $\mathcal{A}_{X/S}$

Recall from Proposition I.2.5.6 that the assignment

$$\mathcal{A} : X/S \mapsto \mathcal{A}_{X/S} := (g_{X/S})_* \omega_{\mathscr{H}ilb_{X/S}} \in \mathcal{D}(X/S)$$

is compatible with pullback by étale morphisms. We claim now that it gives a convergent universal (right) $\mathcal{D}$-module of any fixed dimension n.

Fix $c \in \mathbb{N}$, and consider the assignment

$$X/S \mapsto \mathcal{A}^{(c)}_{X/S} := \left(g^{\leq c}_{X/S}\right)_* \omega_{\mathscr{H}ilb^{\leq c}_{X/S}},$$

where $\mathscr{H}ilb^{\leq c}_{X/S}$ is the union

$$\bigsqcup_{k=0}^{c} \mathscr{H}ilb^{k}_{X/S},$$

and $g^{\leq c}_{X/S}$ is the restriction of $g_{X/S}$ to $\mathscr{H}ilb^{\leq c}_{X/S}$.

It is clear that the isomorphisms $\mathcal{A}(\varphi)$ of Proposition I.2.5.6 restrict to give isomorphisms of the submodules $\mathcal{A}_\bullet^{(c)} \hookrightarrow \mathcal{A}_\bullet$. It is also clear that $\mathcal{A}_\bullet = \operatorname{colim}_{c\in\mathbb{N}} \mathcal{A}_\bullet^{(c)}$, so in order to show that $\mathcal{A}$ is a convergent universal $\mathcal{D}$-module, it suffices to show that each $\mathcal{A}^{(c)}$ is a universal $\mathcal{D}$-module of cth order. This follows from observation I.2.5.8: we noted that the isomorphisms

$$\mathscr{H}ilb^{\leq c}_{X/S} \xrightarrow{\sim} X \times_{X'} \mathscr{H}ilb^{\leq c}_{X'/S}$$

near a point $x \in X$ depend only on the restriction of φ_X to the cth infinitesimal neighbourhood of x. It follows that the same is true of the corresponding morphism $\mathcal{A}^{(c)}(\varphi)$ of $\mathcal{D}$-modules.

Now we begin the identification of the universal $\mathcal{D}$-module $\mathcal{A}^{(c)}$ as a representation of $G^{(c)}$. Let us first determine the underlying complex of vector spaces $V^{(c)} \in \mathrm{Vect}$. The universal $\mathcal{D}$-module $\mathcal{A}^{(c)}$ determines an ind-coherent sheaf $M^{(c)}$ on the stack $\mathcal{M}_n^{(c)}$ given by

$$(\pi : X \rightleftarrows S : \sigma) \mapsto M^{(c)}_{X\rightleftarrows S} := \overline{\sigma}^! \mathcal{A}^{(c)}_{X/S}.$$

In turn, $M^{(c)}$ gives rise to a sheaf on the stack $BG^{(c)}$, or equivalently on the prestack $BG^{(c)}_{\mathrm{triv}}$, and hence corresponds to a representation of $G^{(c)}$ with underlying vector space $V^{(c)} := M_{\mathrm{pt}\to BG^{(c)}_{\mathrm{triv}}}$.

Here the map $\mathrm{pt} \to BG^{(c)}_{\mathrm{triv}}$ corresponds to the trivial principal $G^{(c)}$-bundle given by $G^{(c)} \to \mathrm{pt}$. Under the equivalence of $BG^{(c)}$ and $\mathcal{M}_n^{(c)}$, it corresponds to the map $\mathrm{pt} \to \mathcal{M}_n^{(c)}$ given by the trivial n-dimensional family

$$\pi : \mathbb{A}^n \rightleftarrows \mathrm{pt} : z.$$

In other words, $V^{(c)} = \overline{z}^! \mathcal{A}^{(c)}_{\mathbb{A}^n/\mathrm{pt}} = \overline{z}^! (f_{\mathbb{A}^n})_* \omega_{\mathscr{H}ilb^{\leq c}_{\mathbb{A}^n}}$. We have the following Cartesian diagram:

$$\begin{array}{ccc} \left(\mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0}\right)_{\mathrm{dR}} & \xrightarrow{\overline{z}'} & \left(\mathscr{H}ilb^{\leq c}_{\mathbb{A}^n}\right)_{\mathrm{dR}} \\ {\scriptstyle p_{\left(\mathscr{H}ilb^{\leq c}_{\mathbb{A}^n}\right)_{\mathrm{dR}}}} \downarrow & & \downarrow {\scriptstyle (f^{\leq c}_{\mathbb{A}^n})_{\mathrm{dR}} = g^{\leq c}_{\mathbb{A}^n/\mathrm{pt}}} \\ \mathrm{pt} & \xrightarrow[\overline{z} = z_{\mathrm{dR}}]{} & \mathbb{A}^n_{\mathrm{dR}}. \end{array}$$

By base-change, we have

$$\begin{aligned} V^{(c)} &\simeq (p_{\left(\mathscr{H}ilb^{\leq c}_{\mathbb{A}^n}\right)_{\mathrm{dR}}})_* (\overline{z}')^! \omega_{\mathscr{H}ilb^{\leq c}_{\mathbb{A}^n}} \\ &\simeq \mathrm{H}_\bullet(\mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0}). \end{aligned}$$

Here $\mathrm{Hilb}_{\mathbb{A}^n,0}$ is the *punctual Hilbert scheme*, parametrising closed subschemes supported at the origin. For example, when $n = 2$, each component $\mathrm{Hilb}_{\mathbb{A}^n,0}$ is an irreducible variety, of dimension $c - 1$ (for $c \geq 1$.) In that case we have

$$H_*(\mathrm{Hilb}_{\mathbb{A}^2,0}) = \mathrm{Sym}(k[t]),$$

where by H_* we mean the homology of the complex $\mathrm{H}_\bullet$.

Then it is easy to see from the proof of Proposition I.2.5.6 that the action of $G^{(c)}$ on $V^{(c)} = \mathrm{H}_\bullet(\mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0})$ is induced from the action of $G^{(c)}$ on the variety $\mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0}$. Recall that $V^{(c)}$ is the complex of vector spaces used to compute the homology, but that we haven't actually taken homology yet. If we do, it is straightforward to show (for example by modifying the proof of Proposition 6.4 of [6] to the setting of ind-affine group formal schemes) that the action of $G^{(c)}$ will be trivial: $G^{(c)}$ is connected, so the action of $G^{(c)}$ on the variety $\mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0}$ induces the trivial action on homology. Since we are interested in the action of $G^{(c)}$ on the complex $V^{(c)}$ before taking homology, we must be slightly more careful. We sketch an argument below.

Let us begin by giving some general background on the DG-category $\mathrm{Rep}(H)$ of representations of H, an algebraic group or ind-affine group formal scheme of finite type. (In particular, we could take H to be $K^{(c)}$ or $G^{(c)}$.)

Let the identity of the group be denoted by

$$e_H : \{1\} \to H,$$

and let $H_e^\wedge$ denote the completion of H along this map. We have a exact sequence of group-valued functors

$$1 \to H_e^\wedge \to H \to H_{\mathrm{dR}} \to 1,$$

inducing maps of the classifying stacks which give a Cartesian diagram

$$\begin{array}{ccc} BH_e^\wedge & \longrightarrow & \mathrm{pt} \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ BH & \xrightarrow[\beta]{} & BH_{\mathrm{dR}}. \end{array}$$

This allows us to identify the DG-category $\mathrm{Rep}(H_{\mathrm{dR}}) := \mathrm{IndCoh}\,(H_{\mathrm{dR}})$ with the category of pairs $(\mathscr{F}, t)$, where $\mathscr{F} \in \mathrm{IndCoh}\,(H)$ is a representation of H, and t is a trivialisation of $\mathscr{F}$ when viewed as a representation of $\mathfrak{h}$. That is, $\mathrm{Rep}(H_{\mathrm{dR}})$ is the category of *infinitesimally trivial representations* of H.

Let us emphasise that since we are working the DG-setting, the data of a trivialisation is indeed data, not simply a condition on the sheaf $\mathscr{F}$. More precisely, let L be an arbitrary group-valued prestack satisfying the above conditions. Then a *trivialisation* of $\mathscr{G} \in \mathrm{Rep}(L) = \mathrm{IndCoh}\,(BL)$ is an isomorphism

$$t : \mathscr{F} \to (p_{BL})^!(Be_L)^!\mathscr{G} \in \mathrm{IndCoh}\,(Be_L)\,,$$

where $Be_L : \mathrm{pt} = B\{1\} \to BL$ and $p_{BL} : BL \to \mathrm{pt}$ are the obvious maps. In this language, we have

$$\mathrm{Rep}(H_{\mathrm{dR}}) \simeq \left\{ (\mathscr{F}, t) \;\middle|\; \begin{array}{l} \mathscr{F} \in \mathrm{IndCoh}\,(BH)\,, \\ t : \alpha^!\mathscr{F} \xrightarrow{\sim} (p_{BH_e^\wedge})^!(Be_{H_e^\wedge})^!\alpha^!\mathscr{F} \in \mathrm{IndCoh}\,(BH_e^\wedge) \end{array} \right\}.$$

Claim 7.4.1. *Suppose H is a connected algebraic group.*

1. *If $\mathscr{F} \in$ IndCoh(BH), then all trivialisations of $\mathscr{F}$ are canonically isomorphic.*
2. *Suppose H is the semi-direct product of a unipotent group by a semi-simple group. Then $\alpha^! :$ IndCoh$(BH) \to$ IndCoh$(BH_e^\wedge)$ is fully faithful.*

Remarks on the proof.

1. See 20.8, [11] for a discussion related to the first part of the claim. Frenkel and Gaitsgory use the language of weakly and strongly H-equivariant objects of Vect, rather than working with sheaves on BH and BH_{dR}.
2. We do not know in what generality the second part of the claim holds, although we suspect that some results are known already to experts. The statement is that the natural map

$$\mathrm{Rep}(H) \to \mathrm{Rep}(\mathfrak{h})$$

is fully faithful, or equivalently that given two representations V, W of H the natural map

$$\mathrm{Ext}^\bullet_H(V, W) \to \mathrm{Ext}^\bullet_{\mathfrak{h}}(V, W)$$

is an equivalence. For the abelian categories, this is known, but since the DG-categories $\mathrm{Rep}(H)$ and $\mathrm{Rep}(\mathfrak{h})$ are not simply the DG versions of their hearts, we cannot extend the result immediately.

On the other hand, if we assume that the connected group H is a semi-simple or unipotent algebraic group, we have $\mathrm{Rep}^\heartsuit_{\mathrm{f.d.}}(H) \xrightarrow{\sim} \mathrm{Rep}^\heartsuit_{\mathrm{f.d}}(\mathfrak{h})$. We can use this together with the fact that $\mathrm{Rep}(H)$ and $\mathrm{Rep}(\mathfrak{h})$ are subcategories of $D(\mathrm{Rep}^\heartsuit(H))$ and $D(\mathrm{Rep}^\heartsuit(\mathfrak{h}))$ to obtain our result.

Then we can extend to the case where H is a semi-direct product of a semi-simple and a unipotent algebraic group. The case that we are interested in is the group $K^{(c)} = GL_n \ltimes K_u^{(c)}$, which is not quite of this form; however, at least we will be able to use these results to study the action of its subgroup $SL_n \ltimes K_u^{(c)}$.

Notice that $\alpha \circ Be_{H_e^\wedge} \circ p_{BH_e^\wedge} = Be_H \circ p_{BH} \circ \alpha$. Then it follows from Claim 7.4.1 that for H satisfying the conditions of the claim

$$\begin{aligned} \operatorname{Rep}(H_{\mathrm{dR}}) &\simeq \left\{ (\mathscr{F}, s) \,\middle|\, \begin{array}{l} \mathscr{F} \in \operatorname{IndCoh}(BH), \\ s : \mathscr{F} \xrightarrow{\sim} (p_{BH})^!(Be_H)^!\mathscr{F} \in \operatorname{IndCoh}(BH) \end{array} \right\} \\ &\simeq \{\mathscr{F} \in \operatorname{IndCoh}(BH) \mid \text{ there exists a trivialisation of } \mathscr{F}\}. \end{aligned}$$

That is, the infinitesimally trivial representations of H are just the trivial representations of H.

To see how this applies to our situation, note the following: suppose H acts on a proper scheme Y. The induced action on $\mathrm{H}_\bullet(Y) = (p_Y)_!\omega_Y = (p_Y)_*\omega_Y$ is encoded in the sheaf $\gamma_*\omega_{Y/H} \in \mathcal{D}(BH) = \operatorname{IndCoh}(BH_{\mathrm{dR}})$, where γ is the map in the following Cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{p_Y} & \mathrm{pt} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle Be_H} \\ Y/H & \xrightarrow[\gamma]{} & BH. \end{array}$$

Indeed, by base-change, the underlying vector space is

$$(Be_H)^!(\gamma)_*\omega_{(Y/H)} = (p_Y)_*\omega_Y.$$

To view $\gamma_*\omega_{Y/H}$ as a representation of H rather than H_{dR}, we pull back by the map $\beta : BH \to BH_{\mathrm{dR}}$. In the language of $\mathcal{D}$-module theory, this amounts to forgetting the $\mathcal{D}$-module structure; in the language of infinitesimally trivial representations, it amounts to forgetting the trivialisation of the $\mathfrak{h}$ action. Either way, we see that if H is a connected algebraic group satifying the conditions of Claim 7.4.1 (2), the induced representation on $\mathrm{H}_\bullet(Y)$ is equipped with a canonical trivialisation.

We conclude that because the action of $K^{(c)} = GL_n \ltimes K_u^{(c)}$ on the complex $V^{(c)}$ is induced by the action on the variety $Y = \mathrm{Hilb}^{\leq c}_{\mathbb{A}^n,0}$, the restriction of this action to

$SL_n \ltimes K_u^{(c)}$ is trivial. To further deduce that the action of $SL_n \ltimes G_u^{(c)}$ on $V^{(c)}$ is trivial, we note that the above arguments show that it is infinitesimally trivial. Hence $V^{(c)}$ is trivial as a $(\mathrm{Lie}(SL_n \ltimes G_u^{(c)}), SL_n \ltimes K_u^{(c)})$-module, and thus as a representation of $SL_n \ltimes G_u^{(c)}$.

Finally, we deduce that $V = \bigcup_{c \in \mathbb{N}} V^{(c)}$ is (canonically) trivial as a representation of $SL_n \ltimes G_u^{(c)}$. Therefore, in order to identify V as a representation of G, it remains to determine the action of $\mathbb{G}_m$ on the complexes $V^{(c)}$. At the time of writing, the author can say nothing about this action for general n and general c before taking homology.

Appendix A

Preliminaries: the geometry of prestacks

In this section, we collect together some necessary definitions and results on the geometry of prestacks.

We work in classical, rather than derived, algebraic geometry.[1] In particular, we work with the category Sch of schemes over k, and its full subcategory $\mathrm{Sch}^{\mathrm{Aff}} \simeq k\text{-alg}^{\mathrm{op}}$ of k-algebras. We will be interested in the ∞-category of *prestacks*, which are simply functors:

$$\mathrm{PreStk} \coloneqq \mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}})^{\mathrm{op}}, \infty\text{-Grpd}\right).$$

We view prestacks as geometric objects, rather than just as categorical gadgets: they are generalisations of more familiar objects from algebraic geometry (including schemes, formal schemes, and stacks), and we can study them using the tools of classical algebraic geometry as well, by defining categories of sheaves and $\mathcal{D}$-modules on them. There are for us two main advantages of working in the generality of the category of prestacks: first is that it is cocomplete, that is, we can take arbitrary small colimits. The second is that working with ∞-groupoids rather than sets allows us to encompass the study of stacks. Let us now discuss these ideas briefly.

[1] That is, we work with ordinary schemes and algebras, rather than their DG generalisations, and we work with the category of *classical* prestacks, which Gaitsgory et al. often denote by ${}^{\leq 0}\mathrm{PreStk}$ or ${}^{\mathrm{cl}}\mathrm{PreStk}$. Although all of the definitions and results in this section follow their work, we have simplified the exposition to leave out the technicalities necessary for the DG setting.

1 Special classes of prestacks

1.1 Schemes

First, we view schemes as particular examples of prestacks via the Yoneda embedding: a scheme X gives rise to a functor, which we will also denote by X, in the following way:

$$\begin{aligned}(\mathrm{Sch}^{\mathrm{Aff}})^{\mathrm{op}} &\to \mathrm{Set} \subset \infty\text{-Grpd}\\ S &\to X(S) := \mathrm{Hom}_{\mathrm{Sch}}(S, X).\end{aligned}$$

If a prestack $\mathcal{Y}$ is equivalent to the functor defined by a scheme X, we say that $\mathcal{Y}$ is *representable* by the scheme X. Even if $\mathcal{Y}$ is not representable, for any affine scheme S we have a canonical identification between the groupoid $\mathcal{Y}(S)$, and the groupoid of maps of prestacks $S \to \mathcal{Y}$. We call these the *S-points* of $\mathcal{Y}$.

Definition 1.1.1. Let $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. We say that F is *schematic* if for any base scheme S and any S-point $f \in \mathcal{Y}_2(S)$, the pullback $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ is representable. We say that F is a *closed embedding* (resp. separated, proper, etc.) if it is schematic and in addition the projection maps

$$S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to S$$

are closed embeddings (resp. separated, proper, etc. maps) of schemes.

1.2 Indschemes

A particular class of prestacks that we will find it easy to work with is the class of indschemes. For example, we will see that it is much easier to work with $\mathcal{D}$-modules over indschemes than over arbitrary prestacks.

Definition 1.2.1. Let $\mathcal{Y}$ be a prestack such that

$$\mathcal{Y} \simeq \operatorname*{colim}_{I \in \mathcal{S}} Z(I),$$

where $\mathcal{S}$ is a filtered category, and $Z : \mathcal{S} \to \mathrm{Sch}_{\mathrm{f.t.}}$ is a functor such that for all $\alpha : J \to I \in \mathcal{S}$, the morphism

$$Z(\alpha) : Z(J) \to Z(I)$$

is a closed embedding of schemes of finite type. Then $\mathcal{Y}$ is an *indscheme*; we may also say that $\mathcal{Y}$ is *ind-representable*. We denote by IndSch the full subcategory of PreStk whose objects are indschemes.

Example 1.2.2. Familiar examples of indschemes are given by schemes of infinite type and formal completions of schemes.

Definition 1.2.3. Let $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. We say that F is *indschematic* if for any scheme S mapping to $\mathcal{Y}_2$, the pullback $S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ is an indscheme.

Example 1.2.4. Any morphism between indschemes is indschematic. This uses the fact that indschemes are given by colimits over *filtered* index categories.

So suppose that $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a morphism of indschemes, and consider $S \in \mathrm{Sch}_{/\mathcal{Y}_2}$. Then we can write

$$S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \simeq \operatorname*{colim}_{I \in \mathcal{S}} Z(I) \tag{A.1}$$

for some filtered $\mathcal{S}$ and functor Z, and we have tautological maps $Z(I) \to S \times_{\mathcal{Y}_2} \mathcal{Y}_1$ for each $I \in \mathcal{S}$. Composition with the projection yields maps

$$Z(I) \to S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to S.$$

Definition 1.2.5. We say that F as above is *ind-proper* (resp. *ind-closed*) if for every $S \in \mathrm{Sch}_{/\mathcal{Y}_2}$ and for every presentation as in (A.1), all of the maps $Z(I) \to S$ are proper (resp. closed).

Remark 1.2.6. There is a more general definition of an ind-proper or ind-closed morphism of general prestacks (see for example section 1.1 of [16]), but we will not need it.

1.3 Pseudo-indschemes

Definition 1.3.1. Let $\mathcal{Y}$ be a prestack such that

$$\mathcal{Y} \simeq \operatorname*{colim}_{I \in \mathcal{S}} Z(I), \tag{A.2}$$

where now $\mathcal{S}$ is an arbitrary index category (not necessarily filtered) and the functor Z has image in IndSch. Whereas in the definition of an indscheme we required morphisms α in $\mathcal{S}$ to be mapped to closed embeddings of schemes, we now require that $Z(\alpha)$ be an ind-proper morphism of indschemes.

Then we say that $\mathcal{Y}$ is a *pseudo-indscheme*.

Example 1.3.2. Our first example of a pseudo-indscheme will be the *Ran space* of a separated scheme X. See I.1.1.

Remark 1.3.3. Every pseudo-indscheme $\mathcal{Y}$ can in fact be expressed as a (not necessarily filtered) colimit of schemes, simply by expanding a colimit expression of each indscheme in the presentation (A.2) of $\mathcal{Y}$, so that we could have given a definition of pseudo-indschemes without referring to indschemes. However, many of the many of the pseudo-indschemes that we use in this thesis will be defined as colimits of indschemes, so it is convenient for us to work with the definition given above.

There are several good properties that are satisfied by indschemes which are not satisfied by pseudo-indschemes, arising from the fact that filtered colimits are much better behaved than arbitrary colimits. For example:

1. A functor which is ind-representable always takes values in $\mathrm{Set} \subset \infty\text{-Grpd}$; this is not true of a functor representable by a pseudo-indscheme.

2. As mentioned in example 1.2.4, a morphism between two indschemes is always indschematic. By contrast, there is in general no nice expression for the pullback of a morphism between two pseudo-indschemes, because arbitrary colimits do not commute with finite limits.

However, it is still possible to study pseudo-indschemes because they have reasonable categories of $\mathcal{D}$-modules. This is roughly because the ind-proper morphisms $Z(\alpha) : Z(J) \to Z(I)$ in the colimit expression (A.2) induce well-behaved pushforward and pullback functors between the categories of $\mathcal{D}$-modules on $Z(I)$ and $Z(J)$.

1.4 Stacks

We will mostly be interested in *étale* stacks, i.e. prestacks satisfying descent in the étale topology. Let us be a little more precise. Suppose that $\mathcal{Y}$ is a prestack, T is an affine scheme, and

$$f : S \to T \in \mathrm{Sch}^{\mathrm{Aff}}$$

is an étale cover.

Then we form the Čech nerve $S^\bullet/T$:

$$\cdots S \times_T S \times_T S \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} S \times_T S \rightrightarrows S,$$

and consider the corresponding cosimplicial object $\mathcal{Y}(S^\bullet/T)$. We have a canonical map

$$\mathcal{Y}(T) \to \mathrm{Tot}(\mathcal{Y}(S^\bullet/T)),$$

and we say that $\mathcal{Y}$ *satisfies étale descent* if this map is an equivalence of ∞-groupoids for every T and étale cover $f : S \to T$ as above.

Example 1.4.1. Every indscheme satisfies étale descent; this is because the finite limits in the Čech nerve commute with the filtered colimits in the presentation of the indscheme. On the other hand, not all pseudo-indschemes are stacks.

Given any prestack $\mathcal{Y}$, there exists a unique (up to equivalence) stack $\mathcal{Y}^+$ and a morphism $F : \mathcal{Y} \to \mathcal{Y}^+$ of prestacks such that any morphism from $\mathcal{Y}$ to another stack $\mathcal{Z}$ factors uniquely (again up to equivalence) through $\mathcal{Y} \to \mathcal{Y}^+$.

Definition 1.4.2. We call $\mathcal{Y}^+$ the *(étale) stackification* of $\mathcal{Y}$.

We will use this notion many times, in particular in constructing stacks of étale germs of varieties. An explicit construction of the stackification is given in Lemma 8.8.1 of [1] for prestacks with values in Grpd (as compared to ∞-Grpd).

1.5 Prestacks locally of finite type

In order to define certain categories of ind-coherent sheaves and $\mathcal{D}$-modules on our prestacks (following Gaitsgory and Rozenblyum), we will find it necessary to impose the following finiteness condition:

Definition 1.5.1 (1.3.2, [15])**.** A prestack $\mathcal{Y}$ is *locally of finite type* if it is the left Kan extension of its own restriction along the embedding

$$\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}} \hookrightarrow \mathrm{Sch}^{\mathrm{Aff}}.$$

That is, we have a functor

$$\mathrm{Res} : \mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}})^{\mathrm{op}}, \infty\text{-Grpd}\right) \to \mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}})^{\mathrm{op}}, \infty\text{-Grpd}\right),$$

given by restriction. It has a left adjoint

$$\mathrm{LKE} : \mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}})^{\mathrm{op}}, \infty\text{-Grpd}\right) \to \mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}})^{\mathrm{op}}, \infty\text{-Grpd}\right),$$

and $\mathcal{Y}$ is locally of finite type if the natural map

$$\mathrm{LKE}(\mathrm{Res}(\mathcal{Y})) \to \mathcal{Y}$$

is an equivalence.

The functor LKE is a fully faithful embedding, so that we can equivalently define the ∞-category $\mathrm{PreStk}_{\mathrm{l.f.t.}}$ of locally finite-type prestacks to be the ∞-category of functors

$$\mathrm{Fun}\left((\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}})^{\mathrm{op}}, \infty\text{-Grpd}\right).$$

2 Sheaves on prestacks

We will be interested in studying categories (and $(\infty,1)$-categories, or rather DG-categories) of sheaves on prestacks: in particular, we wish to define categories of quasi-coherent and ind-coherent sheaves, and of $\mathcal{D}$-modules. The theory has been developed by Gaitsgory and Rozenblyum in a series of papers and notes—here we attempt only to give the most important definitions and ideas used in this thesis. In the following, we will give definitions and results for ∞-categories, and will mention explicitly (for example, by decorating the category with a $\heartsuit$) when results apply specifically to the abelian hearts.

2.1 Quasi-coherent sheaves

Let $\mathcal{Y}$ be an arbitrary prestack, and consider the category $\mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}}$ of affine schemes equipped with a map to $\mathcal{Y}$. We view $\mathrm{QCoh}(\bullet)$ as a functor on this category:

$$\begin{aligned}(\mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}})^{\mathrm{op}} &\to \mathrm{DGCat}^{\mathrm{sym.mon.}}\\ (S\to\mathcal{Y}) &\mapsto \mathrm{QCoh}(S)\\ (f: S\to T) &\mapsto (f^*:\mathrm{QCoh}(T)\to\mathrm{QCoh}(S)).\end{aligned}$$

Definition 2.1.1. The symmetric monoidal DG-category of quasi-coherent sheaves on $\mathcal{Y}$ is given by the limit

$$\mathrm{QCoh}(\mathcal{Y}) := \lim_{S\in(\mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)\,.$$

That is, we think of a quasi-coherent sheaf M on $\mathcal{Y}$ as the following collection of data:

1. For each $f: S\to\mathcal{Y}$, a quasi-coherent sheaf

$$f^*M\in\mathrm{QCoh}(S)\,.$$

2. For each morphism $g: S_1\to S_2$ of affine schemes over $\mathcal{Y}$, an isomorphism

$$M(g): g^*(f_2^*M)\xrightarrow{\sim} f_1^*M\in\mathrm{QCoh}(S_1)\,.$$

3. Higher coherence data: for example, given a diagram of schemes over $\mathcal{Y}$,

$$S_1\xrightarrow{g_1} S_2\xrightarrow{g_2} S_3,$$

we obtain two isomorphisms in QCoh(S_1):

$$g_1^* \circ g_2^*(f_3^* M) \simeq (g_2 \circ g_1)^*(f_3^* M) \xrightarrow{M(g_2 \circ g_1)} f_1^* M;$$
$$g_1^* \circ g_2^*(f_3^* M) \xrightarrow{g_1^* M(g_2)} g_1^*(f_2^* M) \xrightarrow{M(g_1)} f_1^* M.$$

We have a natural isomorphism $M(g_2 \circ g_1) \Rightarrow M(g_1) \circ g_1^* M(g_2)$.

We also have higher coherence isomorphisms for repeated compositions.

Definition 2.1.2. Let $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of prestacks. Then we define the *pullback* functor of F

$$F^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$$

using the universal property of limits: it suffices to define a compatible family of functors $\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(S)$ for $S \in \mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}_1}$. Given $S \to \mathcal{Y}_1$, we compose with the morphism F to obtain $S \to \mathcal{Y}_2$, and then we take the tautological projection $\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(S)$. It is easy to see that these functors are compatible.

In terms of the description of a sheaf M on $\mathcal{Y}_2$ as the family $(f^*M, M(g))$, we can describe F^*M explicitly as well: given $f : S \to \mathcal{Y}_1$, we need to specify $f^*(F^*M) \in \mathrm{QCoh}(S)$. It is just $(F \circ f)^*M$.

We have the following three convenient properties of quasi-coherent sheaves. The first two results simplify computations when working with prestacks which are ind-schemes or locally of finite type:

Lemma 2.1.3 (2.1.2 [19])**.** *Let*

$$\mathcal{Y} \simeq \operatorname*{colim}_{I \in \mathcal{S}} Z(I)$$

be an indscheme. Then the tautological maps $Z(I) \to \mathcal{Y}$ induce functors

$$QCoh(\mathcal{Y}) \to QCoh(Z(I))$$

and hence a functor

$$QCoh(\mathcal{Y}) \to \lim_{I \in \mathcal{S}^{op}} QCoh(Z(I)).$$

This functor is an equivalence.

Lemma 2.1.4 (Lemma 1.2.7, [14])**.** *Let $\mathcal{Y}$ be a prestack locally of finite type. There is a natural map*

$$QCoh(\mathcal{Y}) \to \lim_{S \in (Sch^{Aff}_{f.t.})^{op}} QCoh(S);$$

it is an equivalence.

The third result tells us that, from the perspective of quasi-coherent sheaves, the geometry of a prestack depends only on its stackification:

Lemma 2.1.5 (Corollary 1.3.6, [14])**.** *Let $\mathcal{Y}$ be a prestack. Then the canonical map $\mathcal{Y} \to \mathcal{Y}^+$ induces an equivalence*

$$QCoh\left(\mathcal{Y}^+\right) \xrightarrow{\sim} QCoh(\mathcal{Y}).$$

We will use this result in our discussion of universal $\mathcal{D}$-modules and quasi-coherent sheaves on the stack of étale germs.

There is a natural t-structure on $\mathrm{QCoh}(\mathcal{Y})$ induced by requiring the projections

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(S)$$

to be exact. Notice that the functors $f^* : \mathrm{QCoh}(T) \to \mathrm{QCoh}(S)$ in the limit diagram are exact. It follows that the heart $\mathrm{QCoh}(\mathcal{Y})^{\heartsuit}$ can therefore be identified with the limit (over cocomplete abelian categories) of the abelian categories $\mathrm{QCoh}(S)^{\heartsuit}$.

It also follows that the functors

$$F^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$$

are also exact, and hence induce the expected functors at the level of abelian hearts. The results in Lemmas 2.1.4 and 2.1.5 hold also for the abelian categories.

2.2 Ind-coherent sheaves

In fact, it turns out that for many purposes, the category of ind-coherent sheaves is a better-behaved alternative to the category of quasi-coherent sheaves. A significant advantage of working with ind-coherent sheaves is that we can define continuous pullback functors $f^!$; we will see that especially when working with indschemes and pseudo-indschemes this gives us a much better handle on the corresponding categories of sheaves.

On the other hand, a disadvantage of the category of ind-coherent sheaves is that it cannot be defined for arbitrary prestacks, and is only well-behaved for schemes and prestacks which are (locally) of finite type. Hence in this section we work with the categories $\mathrm{Sch}_{\mathrm{f.t.}}$ and $\mathrm{PreStk}_{\mathrm{l.f.t.}}$.

Definition 2.2.1. Let $S \in \mathrm{Sch}_{\mathrm{f.t.}}$. Consider the full subcategory $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$ of coherent sheaves (that is, those $M \in \mathrm{QCoh}(S)$ which have bounded cohomology, finitely generated over $\mathcal{O}_S$). We define the category of *ind-coherent sheaves on* S to be the ind-completion of $\mathrm{Coh}(S)$:

$$\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S)).$$

More precisely, $\mathrm{IndCoh}(S)$ is a cocomplete compactly generated DG-category equipped with a fully faithful and continuous functor $\mathrm{Coh}(S) \to \mathrm{IndCoh}(S)$, which is universal for continuous functors from $\mathrm{Coh}(S)$ to cocomplete categories $\mathcal{C}$. Concretely, objects of $\mathrm{IndCoh}(S)$ are formal filtered colimits $\mathrm{colim}_{i\in I}\, x_i$ of objects in $\mathrm{Coh}(S)$. Morphisms are uniquely determined by requiring that objects of $\mathrm{Coh}(S)$ become compact in $\mathrm{IndCoh}(S)$:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{IndCoh}(S)}(\operatorname*{colim}_{i\in I} x_i, \operatorname*{colim}_{j\in J} y_j) &\simeq \lim_{i\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{IndCoh}(S)}(x_i, \operatorname*{colim}_{j\in J} y_j)\\ \simeq \lim_{i\in I^{\mathrm{op}}} \operatorname*{colim}_{j\in J} \mathrm{Hom}_{\mathrm{IndCoh}(S)}(x_i, y_j) &\simeq \lim_{i\in I^{\mathrm{op}}} \operatorname*{colim}_{j\in J} \mathrm{Hom}_{\mathrm{Coh}(S)}(x_i, y_j).\end{aligned}$$

The embedding $\mathrm{Coh}(S) \hookrightarrow \mathrm{QCoh}(S)$ gives rise to a canonical functor

$$\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S).$$

There is a canonical t-structure on $\mathrm{IndCoh}(S)$ induced by the t-structure on $\mathrm{Coh}(S)$; it follows from the fact that the t-structure on $\mathrm{QCoh}(S)$ is compatible with the t-structure on $\mathrm{Coh}(S)$ and with filtered colimits that the functor Ψ_S is t-exact. The functor Ψ_S satisfies the following additional properties:[2]

Lemma 2.2.2. 1. *(Lemma 1.1.6 and Proposition 1.6.4, [17].) The scheme S is smooth if and only if Ψ_S is an equivalence.*

2. *(Proposition 1.2.4, [17].) For every n, the induced functor*

$$\Psi_S : \mathit{IndCoh}(S)^{\geq n} \to \mathit{QCoh}(S)^{\geq n}$$

is an equivalence. In particular, we have an equivalence of the abelian hearts

$$\Psi_S : \mathit{IndCoh}(S)^{\heartsuit} \xrightarrow{\sim} \mathit{QCoh}(S)^{\heartsuit}$$

for any S of finite type.

3. *(Proposition 1.5.3, [17].) The functor Ψ_S admits a fully faithful left adjoint Ξ_S.*

Let us now consider how maps $f : S \to T$ of schemes of finite type induce functors between the categories $\mathrm{IndCoh}(S)$ and $\mathrm{IndCoh}(T)$.

[2]Let us emphasise again that in this thesis we are working only with classical schemes, rather than DG-schemes.

2.2.1 The $(\mathbf{IndCoh}, *)$-pushforward

First, given any $S, T \in \mathrm{Sch}_{\mathrm{f.t.}}$ and any $f : S \to T$, there is a unique continuous functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}\,(S) \to \mathrm{IndCoh}\,(T)\,,$$

induced by the composition

$$\mathrm{Coh}\,(S) \xrightarrow{\Psi_S} \mathrm{QCoh}\,(S) \xrightarrow{f_*} \mathrm{QCoh}\,(T)^+ \xrightarrow{\sim} \mathrm{IndCoh}\,(T)^+\,.$$

(Here $\mathcal{C}^+$ is our notation for the full subcategory of objects living in $\mathcal{C}^{\geq n}$ for some n. The equivalence $\mathrm{IndCoh}\,(T)^+ \xrightarrow{\sim} \mathrm{QCoh}\,(T)^+$ is given by Ψ_T.)

We have the following properties, by construction:

Lemma 2.2.3 (Propositions 3.1.1 and 3.6.7, [17])**.** *The functor f_*^{IndCoh} is left t-exact, and is compatible with the pushforward functor $f_* : QCoh\,(S) \to QCoh\,(T)$ of quasi-coherent sheaves:*

$$f_* \circ \Psi_S \simeq \Psi_T \circ f_*^{IndCoh} \text{ and } \ \Xi_T \circ f_* \simeq f_*^{IndCoh} \circ \Xi_S.$$

We can also show the following:

Lemma 2.2.4 (Proposition 3.2.4, [17])**.** *The assignment*

$$S \mapsto IndCoh\,(S)$$
$$(f : S \to T) \mapsto (f_*^{IndCoh} : IndCoh\,(S) \to IndCoh\,(T))$$

gives rise to a functor

$$IndCoh : Sch_{f.t.} \to DGCat_{cont}.$$

Moreover, the assignment

$$S \mapsto \Psi_S : IndCoh\,(S) \to QCoh\,(S)$$

extends to a natural transformation

$$IndCoh\,(S) \to QCoh\,(S)\,.$$

2.2.2 The $(\mathbf{IndCoh}, *)$-pullback

Let us again consider $f : S \to T$ a morphism of schemes of finite type. Then $f^* : \mathrm{QCoh}(T) \to \mathrm{QCoh}(S)$ maps coherent sheaves to coherent sheaves, and hence there is a morphism

$$f^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(T) \to \mathrm{IndCoh}(S)$$

induced by the following composition:

$$\mathrm{Coh}(T) \xrightarrow{f^*} \mathrm{Coh}(S) \hookrightarrow \mathrm{QCoh}(S)^+ \xrightarrow{\sim} \mathrm{IndCoh}(S)^+ .$$

It satisfies the following properties:

Lemma 2.2.5. *1. (Propositions 3.5.4 and 3.5.11, [17].) The functors f^* and $f^{IndCoh,*}$ are compatible under the functors Ψ and Ξ:*

$$f^* \circ \Psi_T \simeq \Psi_S \circ f^{IndCoh,*} \text{ and } f^{IndCoh,*} \circ \Xi_T \simeq \Xi_S \circ f^*.$$

2. (Corollary 3.5.6, [17].) The assignment

$$S \mapsto IndCoh(S)$$
$$(f : S \to T) \mapsto (f^{IndCoh,*} : IndCoh(T) \to IndCoh(S))$$

upgrades to a functor

$$IndCoh^* : Sch_{f.t.}^{op} \to DGCat_{cont}.$$

Furthermore, the assignment

$$S \mapsto \Psi_S : IndCoh(S) \to QCoh(S)$$

upgrades to a natural transformation

$$\Psi : IndCoh^* \to QCoh^*_{Sch_{f.t.}}.$$

3. (Lemma 3.5.8, [17].) The functor $f^{IndCoh,*}$ is left adjoint to the functor f_*^{IndCoh}.

2.2.3 The !-pullback

We will now define the !-pullback functor for morphisms between schemes of finite type. We can define it directly for $f : S \to T$ proper; the construction for more general f is much more involved.

First assume that f is proper. We can show (see Lemma 3.3.5 and Corollary 3.3.6, [17]) that the functors

$$f_* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(T)$$
$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(T)$$

both send $\mathrm{Coh}(S)$ to $\mathrm{Coh}(T)$—so f_*^{IndCoh} sends compact objects of $\mathrm{IndCoh}(S)$ to compact objects. It follows that its right adjoint

$$f^! : \mathrm{IndCoh}(T) \to \mathrm{IndCoh}(S)$$

is continuous. (Note that this is not necessarily true of the right adjoint $f^{\mathrm{QCoh},!}$: $\mathrm{QCoh}(T) \to \mathrm{QCoh}(S)$ of f_*.)

Lemma 2.2.6 (Corollary 3.3.9, [17])**.** *The assignment*

$$S \mapsto \mathit{IndCoh}(S)$$
$$(f : S \to T) \mapsto (f^! : \mathit{IndCoh}(T) \to \mathit{IndCoh}(S))$$

upgrades to a functor

$$\mathit{IndCoh}^! : \mathit{Sch}^{op}_{f.t.,proper} \to \mathit{DGCat}_{cont}.$$

(Here $\mathrm{Sch}_{\mathrm{f.t.,proper}}$ is the category of schemes of finite type with proper morphisms between them.)

Fact 2.2.7. The functors $f^!$ and $f^{\mathrm{QCoh},!}$ are compatible under the functors Ψ when restricted to $\mathrm{IndCoh}(T)^+$, but not in general on all of $\mathrm{IndCoh}(T)$. See Lemma 3.4.4 and remark 3.4.5 of [17] for a proof and a counterexample, respectively.

In order to define the !-pullback of a more general morphism $f : S \to T$ of schemes of finite type, we introduce an $(\infty, 2)$-category $\mathcal{C}$ as follows:

- Objects of $\mathcal{C}$ are schemes S of finite type.
- 1-morphisms $S_1 \to S_2$ are diagrams of the form

$$\begin{array}{ccc} S_{1,2} & \xrightarrow{g} & S_1 \\ {\scriptstyle f}\downarrow & & \\ S_2. & & \end{array}$$

- 2-morphisms $(f, S_{1,2}, g) \to (f', S'_{1,2}, g')$ are given by proper maps $h : S_{1,2} \to S'_{1,2}$ such that the following diagram commutes:

$$\begin{array}{ccc} S_{1,2} & \xrightarrow{g} & S_1 \\ {\scriptstyle f}\downarrow & \searrow^{h} & \uparrow{\scriptstyle g'} \\ S_2 & \xleftarrow[f']{} & S'_{1,2}. \end{array}$$

  Of course, we can also view $\mathcal{C}$ as an $(\infty, 1)$-category by considering only those 2-morphisms such that h is an isomorphism.

It is easy to see that we have faithful (but not full) functors

$$\mathrm{Sch}_{\mathrm{f.t.}} \hookrightarrow \mathcal{C}$$
$$\mathrm{Sch}^{\mathrm{op}}_{\mathrm{f.t.}} \hookrightarrow \mathcal{C}$$

given as the identity on objects and by sending a morphism $f : S \to T$ to the morphism (f, S, id_S) and (id_S, S, g) respectively. The important result is the following:

Theorem 2.2.8 (Theorem 5.2.2, [17]). *There exists a canonically defined functor of* $(\infty, 1)$*-categories*

$$IndCoh_{\mathcal{C}} : \mathcal{C} \to DGCat_{cont}$$

such that

1. *The restriction of* $IndCoh_{\mathcal{C}}$ *to* $Sch_{f.t.}$ *is canonically isomorphic to the functor* $IndCoh_{Sch_{f.t.}}$ *of Lemma 2.2.4.*

2. *The restriction of* $IndCoh_{\mathcal{C}}$ *to* $Sch^{op}_{f.t.,proper}$ *is canonically isomorphic to the functor* $IndCoh^{!}$ *of Lemma 2.2.6.*

3. *The restriction of* $IndCoh_{\mathcal{C}}$ *to* $Sch^{op}_{f.t.,open}$ *is canonically isomorphic to the functor* $IndCoh^{*}$ *of Lemma 2.2.5.*

In particular, the restriction of $\mathrm{IndCoh}_{\mathcal{C}}$ to ${\mathrm{Sch}_{\mathrm{f.t.}}}^{\mathrm{op}}$ gives a functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{f.t.}}} : \mathrm{Sch}^{\mathrm{op}}_{\mathrm{f.t.}} \to \mathrm{DGCat}_{\mathrm{cont}},$$

satisfying the following properties:

1. For $S \in \mathrm{Sch}_{\mathrm{f.t.}}$, $\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{f.t.}}}(S) \simeq \mathrm{IndCoh}\,(S)$.

2. For $f : S \to T$ a proper map of schemes of finite type,

   $$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{f.t.}}}(f) \simeq f^!.$$

   In particular, it is right adjoint to f^{IndCoh}_*.

3. For $j : S \to T$ an open embedding of schemes of finite type,

   $$\mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{f.t.}}}(j) \simeq j^{\mathrm{IndCoh},*}.$$

Notation 2.2.9. We shall write $f^! := \mathrm{IndCoh}^!_{\mathrm{Sch}_{\mathrm{f.t.}}}(f)$ for any morphism $f : S \to T$ of schemes of finite type.

We have base-change formulas relating the pushforward and pullback functors: let $S, T, T' \in \mathrm{Sch}_{\mathrm{f.t.}}$, and consider the Cartesian diagram *of DG-schemes.* [3]

$$\begin{array}{ccc} S' & \xrightarrow{g'} & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ T' & \xrightarrow[g]{} & T. \end{array}$$

Proposition 2.2.10. *1. (Proposition 5.2.5, [17].) There is an equivalence*

$$g^! \circ f_*^{IndCoh} \simeq (f')_*^{IndCoh} \circ (g')^!$$

2. (Lemma 3.6.9, [17].) The natural transformation

$$f^{IndCoh,*} \circ g_*^{IndCoh} \xrightarrow{\sim} (g')_*^{IndCoh} \circ (f')^{IndCoh,*}$$

is an equivalence.

3. (Proposition 7.1.6, [17].) The natural transformation

$$(f')^{IndCoh,*} \circ g^! \to (g')^! \circ f^{IndCoh,*}$$

induced by the base-change equivalence in (1) is an equivalence.

[3] This is the one occasion in these notes where we need to work with DG-schemes. However, we will in fact only use these results when working with $\mathcal{D}$-modules, and as we will see in 2.3.5 in that case it suffices to consider the ordinary fibre product of classical schemes or prestacks.

2.2.4 Monoidal structures and the dualising sheaf

Recall that QCoh (S) has a natural symmetric monoidal structure. It turns out that IndCoh (S) is naturally a module over QCoh (S): the action

$$\mathrm{QCoh}\,(S) \otimes \mathrm{IndCoh}\,(S) \to \mathrm{IndCoh}\,(S)$$

is induced by the action of QCoh $(S)^{\mathrm{perf}}$ on Coh (S), which is just given by the ordinary (derived) tensor product of sheaves. (See 1.4, [17].) We use the notation

$$(\mathscr{E}, \mathscr{F}) \in \mathrm{QCoh}\,(S) \otimes \mathrm{IndCoh}\,(S) \mapsto \mathscr{E} \otimes \mathscr{F} \in \mathrm{IndCoh}\,(S)\,.$$

We have two projection formulas:

Lemma 2.2.11. *Let $f : S \to T$ be a morphism of schemes of finite type.*

1. *(Proposition 3.1.3, [17].) Suppose that we have $\mathscr{E}_T \in QCoh(T)$ and $\mathscr{F}_S \in IndCoh(S)$. Then*

$$\mathscr{E}_T \otimes f_*^{IndCoh}(\mathscr{F}_S) \simeq f_*^{IndCoh}(f^*(\mathscr{E}_T) \otimes \mathscr{F}_S).$$

2. *(Proposition 3.6.11, [17].) Suppose that we have $\mathscr{E}_S \in QCoh(S)$ and $\mathscr{F}_T \in QCoh(T)$. Then*

$$f_*(\mathscr{E}_S) \otimes \mathscr{F}_T \simeq f_*^{IndCoh}(\mathscr{E}_S \otimes f^{IndCoh,*}\mathscr{F}_T).$$

In fact, the symmetric monoidal structure on QCoh (S) induces a symmetric monoidal structure on IndCoh (S) as well (see 5.6.7, [17]): first, we define an external tensor product

$$\mathrm{IndCoh}\,(S) \otimes \mathrm{IndCoh}\,(T) \xrightarrow{\boxtimes} \mathrm{IndCoh}\,(S \times T)\,,$$

using the observation that the composition

$$\mathrm{IndCoh}\,(S) \otimes \mathrm{IndCoh}\,(T) \xrightarrow{\Psi_S \otimes \Psi_T} \mathrm{QCoh}\,(S) \otimes \Psi T \xrightarrow{\boxtimes} \mathrm{QCoh}\,(S \times T)$$

takes compact objects in IndCoh $(S) \otimes$ IndCoh (T) to Coh $(S \times T)$. Then we have the following:

Lemma 2.2.12 (Proposition 4.6.2, [17])**.** *The external tensor product gives an equivalence*

$$IndCoh\,(S) \otimes IndCoh\,(T) \xrightarrow{\boxtimes} IndCoh\,(S \times T)\,.$$

Now the monoidal operation on IndCoh (S) is given by the composition

$$\operatorname{IndCoh}(S) \otimes \operatorname{IndCoh}(S) \xrightarrow{\boxtimes} \operatorname{IndCoh}(S \times S) \xrightarrow{\Delta^!} \operatorname{IndCoh}(S).$$

We use the notation

$$(\mathscr{E}, \mathscr{F}) \in \operatorname{IndCoh}(S) \otimes \operatorname{IndCoh}(S) \mapsto \mathscr{E} \otimes^! \mathscr{F} \in \operatorname{IndCoh}(S).$$

The unit in this symmetric monoidal category is the object

$$p_S^!(k) \in \operatorname{IndCoh}(S),$$

where $p_S : S \to \mathrm{pt} = \operatorname{Spec} k$ is the projection to the point, and

$$k \in \operatorname{IndCoh}(\mathrm{pt}) \simeq \mathrm{Vect}$$

is the complex with the field k concentrated in degree 0.

Definition 2.2.13. We denote this object by $\omega_S := p_S^!(k)$, and call it the *dualising sheaf.*

The symmetric monoidal structure on IndCoh (S) is compatible with the action of QCoh (S) in the following sense: suppose we have $\mathscr{E} \in \operatorname{QCoh}(S)$ and $\mathscr{F}_1, \mathscr{F}_2 \in \operatorname{IndCoh}(S)$. Then we have canonical isomorphisms

$$\mathscr{E} \otimes (\mathscr{F}_1 \otimes^! \mathscr{F}_2) \simeq (\mathscr{E} \otimes \mathscr{F}_1) \otimes^! \mathscr{F}_2 \simeq \mathscr{F}_1 \otimes^! (\mathscr{E} \otimes \mathscr{F}_2).$$

Lemma 2.2.14 (Corollaries 5.7.4 and 9.3.3, [17])**.** *We have a symmetric monoidal functor*

$$\Upsilon_S : \mathit{QCoh}(S) \to \mathit{IndCoh}(S)$$

given by

$$\mathscr{E} \mapsto \mathscr{E} \otimes \omega_S.$$

It intertwines the $*$-pullback functors for quasi-coherent sheaves with the $!$-pullback functors for ind-coherent sheaves:

$$f^! \circ \Upsilon_T \simeq \Upsilon_S \circ f^*.$$

It sends compact objects to compact objects and is fully faithful. It is an equivalence if S is smooth.

In particular, the dualising sheaf $\omega_S = \Upsilon_S(\mathcal{O}_S)$ is compact.

2.2.5 Ind-coherent sheaves on prestacks locally of finite type

We extend the definition of IndCoh from schemes of finite type to prestacks $\mathcal{Y}$ locally of finite type using the functor $\mathrm{IndCoh}^{!}_{\mathrm{Sch_{f.t.}}}$ as follows:

$$\mathrm{IndCoh}(\mathcal{Y}) := \lim_{S\in\left((\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}})_{/\mathcal{Y}}\right)^{\mathrm{op}}} \mathrm{IndCoh}(S).$$

That is, an object M of $\mathrm{IndCoh}(\mathcal{Y})$ is given by a family

$$\left\{y^! M \in \mathrm{IndCoh}(S)\right\}_{y:S\to\mathcal{Y}},$$

together with compatibility isomorphisms

$$M(f) : f^! y_2^! M \xrightarrow{\sim} y_1^! M$$

for any $f : (S_1 \xrightarrow{y_1} \mathcal{Y}) \to (S_2 \xrightarrow{y_2} \mathcal{Y})$ in $(\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}})_{/\mathcal{Y}}$. We also have higher coherence data.

Given a morphism $F : \mathcal{Y}_1 \to \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{l.f.t.}}$, there is a natural morphism $F^! : \mathrm{IndCoh}(\mathcal{Y}_2) \to \mathrm{IndCoh}(\mathcal{Y}_1)$ defined as in the construction of $f^* : \mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1)$. We cannot define the $*$-pushforward F_* in general, but it is defined for F schematic and quasi-compact; when F is schematic and proper, F_* is left adjoint to $F^!$.

In particular, we have $\omega_{\mathcal{Y}} := p_{\mathcal{Y}}^!(k)$, where $p_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{pt}$. We call this the *dualising sheaf* of $\mathcal{Y}$. The category $\mathrm{IndCoh}(\mathcal{Y})$ has a symmetric monoidal structure induced from the symmetric monoidal structures on $\mathrm{IndCoh}(S)$, and $\omega_{\mathcal{Y}}$ is the unit.

The base-change and projection formulas given in Lemmas 2.2.10 and 2.2.11 continue to hold for ind-coherent sheaves on prestacks locally of finite type.

The functors $\Upsilon_S : \mathrm{QCoh}(S) \to \mathrm{IndCoh}(S)$ give rise to a monoidal functor

$$\begin{aligned}\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) &\to \mathrm{IndCoh}(\mathcal{Y})\\ \mathscr{F} &\mapsto \mathscr{F} \otimes \omega_{\mathcal{Y}}.\end{aligned}$$

Lemma 2.2.15 (Lemma 10.3.4, [17])**.** *The functor $\Upsilon_{\mathcal{Y}}$ is fully faithful.*

Lemma 2.2.16 (Theorem 10.1.1, [19])**.** *Let $\mathcal{Y}$ be an indscheme which can be expressed as a colimit*

$$\mathcal{Y} \simeq \operatorname*{colim}_{I\in\mathcal{S}} Z(I)$$

where the index category $\mathcal{S}$ is equivalent to the poset $\mathbb{N}$. Suppose that $\mathcal{Y}$ is formally smooth and locally of finite type. Then the functor

$$\Upsilon_{\mathcal{Y}} : \mathit{QCoh}(\mathcal{Y}) \to \mathit{IndCoh}(\mathcal{Y})$$

is an equivalence.

2.3 $\mathcal{D}$-modules

We approach the study of $\mathcal{D}$-modules from the perspective of crystals, following Gaitsgory and Rozenblyum.

Definition 2.3.1. Given a prestack $\mathcal{Y}$, we define its *de Rham prestack* $\mathcal{Y}_{\mathrm{dR}}$ to be the functor

$$S \mapsto \mathcal{Y}(S_{red}),$$

where S_{red} is the underlying reduced scheme of S.

There is a natural map of prestacks

$$\begin{gathered} \mathcal{Y} \to \mathcal{Y}_{\mathrm{dR}} \\ (S \to \mathcal{Y}) \mapsto (S_{red} \hookrightarrow S \to \mathcal{Y}). \end{gathered}$$

We denote this map by $p_{\mathrm{dR},\mathcal{Y}}$.

Proposition 2.3.2 (Proposition 1.1.4, [20] III.4)**.** *If $\mathcal{Y}$ is locally of finite type, so is $\mathcal{Y}_{dR}$.*

A map $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ of prestacks induces a map $F_{\mathrm{dR}} : \mathcal{Y}_{1,\mathrm{dR}} \to \mathcal{Y}_{2,\mathrm{dR}}$ in the obvious way, and the assignment $\mathcal{Y} \mapsto \mathcal{Y}_{\mathrm{dR}}$ extends to a functor $\mathrm{dR} : \mathrm{PreStk} \to \mathrm{DGCat}_{\mathrm{cont}}$.

2.3.1 Right $\mathcal{D}$-modules

Definition 2.3.3 (Section 1.2, [20] III.4)**.** Let $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{l.f.t.}}$. The category of *right crystals* is by definition

$$\mathrm{Crys}(\mathcal{Y}_{\mathrm{dR}}) := \mathrm{IndCoh}\left(\mathcal{Y}_{\mathrm{dR}}\right).$$

Given a morphism $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ in $\mathrm{PreStk}_{\mathrm{l.f.t.}}$ we obtain a morphism

$$F_{\mathrm{dR}} : \mathcal{Y}_{1,\mathrm{dR}} \to \mathcal{Y}_{2,\mathrm{dR}},$$

and consequently can form

$$F^!_{\mathrm{dR}} : \mathrm{IndCoh}\left(\mathcal{Y}_{2,\mathrm{dR}}\right) \to \mathrm{IndCoh}\left(\mathcal{Y}_{1,\mathrm{dR}}\right).$$

We denote this functor by

$$F^{\mathrm{dR},!} : \mathrm{Crys}(\mathcal{Y}_2) \to \mathrm{Crys}(\mathcal{Y}_1),$$

and note that the assignment $\mathcal{Y} \mapsto \mathrm{Crys}(\mathcal{Y}), F \mapsto F^{\mathrm{dR},!}$ upgrades to a functor

$$\mathrm{Crys}^! : \mathrm{PreStk}_{\mathrm{l.f.t.}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

Indeed, it is just the composition of the functors dR and $\mathrm{IndCoh}^!$. There is a natural transformation $\mathbf{oblv}_{\mathrm{dR}} : \mathrm{Crys}^! \to \mathrm{IndCoh}^!$ given by

$$\mathbf{oblv}_{\mathrm{dR},\mathcal{Y}} := p^!_{\mathrm{dR},\mathcal{Y}} : \mathrm{IndCoh}\left(\mathcal{Y}_{\mathrm{dR}}\right) \to \mathrm{IndCoh}\left(\mathcal{Y}\right).$$

When we interpret objects of $\mathrm{Crys}(\mathcal{Y})$ as right $\mathcal{D}$-modules on $\mathcal{Y}$, $\mathbf{oblv}_{\mathrm{dR},\mathcal{Y}}$ is just the functor of forgetting the $\mathcal{D}$-module structure on a sheaf.

We have the following alternative presentations of the category $\mathrm{Crys}(\mathcal{Y})$:

Lemma 2.3.4 (Proposition 1.2.5, [20] III.4)**.** *For $\mathcal{Y} \in \mathit{PreStk}_{l.f.t.}$, we have an equivalence*

$$\mathit{Crys}(\mathcal{Y}) \xrightarrow{\sim} \lim_{S \in (\mathcal{C}_{/\mathcal{Y}})^{op}} \mathit{Crys}(S),$$

where $\mathcal{C}$ is the category $\mathit{Sch}^{\mathit{Aff,red}}_{f.t.}, \mathit{Sch}^{\mathit{Aff}}_{f.t.}, \mathit{Sch}^{\mathit{red}}_{f.t.},$ or $\mathit{Sch}_{f.t.}$.

Remark 2.3.5. In fact, this lemma holds even if we start with a *DG-prestack* $\mathcal{Y}$ (which is locally almost of finite type), i.e. a functor

$$\left(\mathrm{DG\text{-}Sch}^{\mathrm{Aff}}_{\mathrm{a.f.t.}}\right)^{\mathrm{op}} \to \infty\text{-Grpd}.$$

Then the lemma implies that $\mathrm{Crys}(\mathcal{Y})$ is determined by the underlying classical prestack ${}^{\mathrm{cl}}\mathcal{Y}$, which is the restriction of $\mathcal{Y}$ to $\mathrm{Sch}^{\mathrm{Aff}}_{\mathrm{f.t.}}$. Together with the observation that $\mathcal{Y}_{\mathrm{dR}} \simeq ({}^{\mathrm{cl}}\mathcal{Y})_{\mathrm{dR}}$, this implies that $\mathrm{Crys}(\mathcal{Y}) \simeq \mathrm{Crys}({}^{\mathrm{cl}}\mathcal{Y})$. In particular, when working with crystals, the base-change formulas from Lemma 2.2.10 hold for the classical fibre product of schemes; it is not necessary to work with the DG fibre product.

2.3.2 $\mathcal{D}$-modules on pseudo-indschemes

Suppose that $\mathcal{Y} \simeq \mathrm{colim}_{I \in \mathcal{S}}\, Z(I)$ is a pseudo-indscheme. Then it follows from Lemma 2.3.4 that

$$\mathrm{Crys}(\mathcal{Y}) \xrightarrow{\sim} \lim_{I \in \mathcal{S}^{\mathrm{op}}} \mathrm{Crys}(Z(I)),$$

where the morphisms in the diagram are given by

$$Z(\alpha)^{\mathrm{dR},!} : \mathrm{Crys}(Z(I)) \to \mathrm{Crys}(Z(J)).$$

However, the properness assumptions on the maps $Z(\alpha)$ imply that the functors $Z(\alpha)^{\mathrm{dR},!} = (Z(\alpha)_{\mathrm{dR}})^!$ have left adjoints

$$(Z(\alpha)_{\mathrm{dR}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}\left(Z(J)\right) \to \mathrm{IndCoh}\left(Z(I)\right).$$

Hence we can form the colimit

$$\operatorname*{colim}_{I \in \mathcal{S}} \mathrm{Crys}(Z(I)).$$

The following is a particular case of Lemma 1.3.3, [13]:

Lemma 2.3.6. *For any $I \in \mathcal{S}$, the tautological functor Crys($\mathcal{Y}$) $\to$ Crys($Z(I)$) admits a left adjoint Crys($Z(I)$) $\to$ Crys($\mathcal{Y}$). These induce a functor*

$$\operatorname*{colim}_{I \in \mathcal{S}} \mathit{Crys}(Z(I)) \to \mathit{Crys}(\mathcal{Y}),$$

and this is an equivalence of DG-categories.

2.3.3 De Rham cohomology of prestacks

Given an arbitrary map $F : \mathcal{Y}_1 \to \mathcal{Y}_2$, the functor

$$F^{\mathrm{dR},!} : \mathrm{Crys}(\mathcal{Y}_2) \to \mathrm{Crys}(\mathcal{Y}_1)$$

does not admit a left adjoint in general. However (as in 1.5 of [16]) we can define a full subcategory

$$\mathrm{Crys}(\mathcal{Y}_1)_{\text{good for } F} \subset \mathrm{Crys}(\mathcal{Y}_1)$$

whose objects are those $\mathscr{F} \in \mathrm{Crys}(\mathcal{Y}_1)$ for which the functor

$$\begin{aligned} \mathrm{Crys}(\mathcal{Y}_2) &\to \infty\text{-Grpd} \\ \mathscr{G} &\mapsto \mathrm{Hom}_{\mathrm{Crys}(\mathcal{Y}_1)}(\mathscr{F}, F^{\mathrm{dR},!}\mathscr{G}) \end{aligned}$$

is co-representable. That is, there exists some $\mathscr{F}' \in \mathrm{Crys}(\mathcal{Y}_1)$ such that

$$\mathrm{Hom}_{\mathrm{Crys}(\mathcal{Y}_1)}(\mathscr{F}, F^{\mathrm{dR},!}\mathscr{G}) \simeq \mathrm{Hom}_{\mathrm{Crys}(\mathcal{Y}_2)}(\mathscr{F}', \mathscr{G})$$

for every $\mathscr{G} \in \mathrm{Crys}(\mathcal{Y}_2)$. Then we define

$$\begin{aligned} F_! : \mathrm{Crys}(\mathcal{Y}_1)_{\text{good for } F} &\to \mathrm{Crys}(\mathcal{Y}_2) \\ \mathscr{F} &\mapsto \mathscr{F}'. \end{aligned}$$

Remark that if F satisfies sufficient properness conditions (for example, if it is an ind-proper map of indschemes) then $\mathrm{Crys}(\mathcal{Y}_1)_{\text{good for } F}$ is all of $\mathrm{Crys}(\mathcal{Y}_1)$, and $F_!$ is just $(F_{\mathrm{dR}})_*^{\mathrm{IndCoh}}$.

Definition 2.3.7. We call $F_!$ the *partially-defined left adjoint* of $F^{\mathrm{dR},!}$.

We allow ourselves the following abuse of notation: the dualising sheaf of $\mathcal{Y}_{\mathrm{dR}}$ will be denoted by $\omega_{\mathcal{Y}}$ rather than $\omega_{\mathcal{Y}_{\mathrm{dR}}}$, and will still be called the dualising sheaf of $\mathcal{Y}$. Note that its image under the forgetful functor $\mathrm{Crys}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$ is the dualising sheaf in $\mathrm{IndCoh}(\mathcal{Y})$ as originally defined.

Definition 2.3.8. Let $\mathcal{Y}$ be a prestack such that

$$\omega_{\mathcal{Y}} := p_{\mathcal{Y}}^{\mathrm{dR},!}(k) \in \mathrm{Crys}(\mathcal{Y})_{\text{good for } p_{\mathcal{Y}}}.$$

Then we set

$$\mathrm{H}_\bullet(\mathcal{Y}) := (p_{\mathcal{Y}})_! p_{\mathcal{Y}}^!(k) \in \mathrm{Vect}.$$

This is the *de Rham cohomology* of $\mathcal{Y}$.

If $F : \mathcal{Y}_1 \to \mathcal{Y}_2$, then we have a canonical isomorphism

$$\omega_{\mathcal{Y}_1} \simeq F^{\mathrm{dR},!}\omega_{\mathcal{Y}_2}.$$

If $\omega_{\mathcal{Y}_1} \in \mathrm{Crys}(\mathcal{Y}_1)_{\text{good for } F}$, then this induces a map

$$F_!(\omega_{\mathcal{Y}_1}) \to \omega_{\mathcal{Y}_2}.$$

In particular, we obtain a map

$$\mathrm{Tr}_{\mathrm{H}_\bullet}(F) : \mathrm{H}_\bullet(\mathcal{Y}_1) \to \mathrm{H}_\bullet(\mathcal{Y}_2),$$

provided that both sides are defined.

Suppose that $\mathcal{Y} \simeq \mathrm{colim}_{I \in \mathcal{S}} Z(I)$ and $\mathcal{Y}' \simeq \mathrm{colim}_{I' \in \mathcal{S}'} Z'(I')$ are two pseudo-indschemes. Suppose further that we have a functor $\phi : \mathcal{S} \to \mathcal{S}'$ and a natural transformation

$$F_Z : Z \Rightarrow Z' \circ \phi$$
$$F(I) : Z(I) \to Z'(\phi(I)).$$

By the universal property of colimits, this induces a morphism

$$F : \mathcal{Y} \to \mathcal{Y}',$$

and we can show (1.5.5–6, [16]) that $\omega_{\mathcal{Y}} \in \mathrm{Crys}(\mathcal{Y})_{\text{good for } F}$.

In particular, for any pseudo-indscheme $\mathcal{Y}$, the map $\mathcal{Y} \to \mathrm{pt}$ fits into the above set-up, and so $\mathrm{H}_\bullet(\mathcal{Y})$ is always defined, and there is a canonical map

$$\mathrm{Tr}_{\mathrm{H}_\bullet} : \mathrm{H}_\bullet(\mathcal{Y}) \to \mathrm{H}_\bullet(\mathrm{pt}) = k \in \mathrm{Vect}.$$

We can also show that

$$\mathrm{H}_\bullet(\mathcal{Y}) \simeq \operatorname*{colim}_{I \in \mathcal{S}} \mathrm{H}_\bullet(Z(I)).$$

It follows that $\mathrm{H}_\bullet(\mathcal{Y})$ always lives in non-positive cohomological degree. Moreover, the degree zero part of the trace map

$$\mathrm{H}_0(\mathcal{Y}) \to k$$

is non-zero whenever $\mathcal{Y}$ is non-empty, and is an isomorphism if all $Z(I)$ are connected.

Definition 2.3.9. If $\mathcal{Y}$ is a prestack such that $\mathrm{Tr}_\mathcal{Y}$ is an equivalence $\mathrm{H}_\bullet(\mathcal{Y}) \xrightarrow{\sim} k$, we say that $\mathcal{Y}$ is *homologically contractible.*

Remark 2.3.10. Note that $\mathcal{Y}$ is homologically contractible if and only if the pullback $p_\mathcal{Y}^!$ is fully-faithful.

2.3.4 Left $\mathcal{D}$-modules

We can also consider the category of left crystals on a prestack $\mathcal{Y}$:

Definition 2.3.11 (2.1.1, [18])**.** We set

$$\mathrm{Crys}^l(\mathcal{Y}) \coloneqq \mathrm{QCoh}\left(\mathcal{Y}_{\mathrm{dR}}\right) = \lim_{S \in (\mathrm{Sch}^{\mathrm{Aff}}_{/\mathcal{Y}_{\mathrm{dR}}})^{\mathrm{op}}} \mathrm{QCoh}\left(S\right).$$

Note that $\mathcal{Y}$ does not need to be locally of finite type for this definition. If S is a smooth scheme of finite type, one can show that $\mathrm{Crys}^l(S)$ is equivalent to the category of left modules over the algebra $\mathcal{D}_S$ of differential operators on S. In other words, this definition is indeed an extension of notion of left $\mathcal{D}$-modules. See section 5 of [18] for a detailed discussion.

Given any map $F : \mathcal{Y}_1 \to \mathcal{Y}_2$ of prestacks, we have the functor

$$(F_{\mathrm{dR}})^* : \mathrm{QCoh}\left(\mathcal{Y}_{2,\mathrm{dR}}\right) \to \mathrm{QCoh}\left(\mathcal{Y}_{1,\mathrm{dR}}\right).$$

We adopt the notation

$$F^{\dagger,l} : \mathrm{Crys}^l(\mathcal{Y}_2) \to \mathrm{Crys}^l(\mathcal{Y}_1)$$

for this functor. These assignments extend to a functor

$$\mathrm{Crys}^l : \mathrm{PreStk} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

We have a canonical natural transformation

$$\mathbf{oblv}^l : \mathrm{Crys}^l \Rightarrow \mathrm{QCoh}^*_{\mathrm{PreStk}}$$

which is given by the forgetful functors

$$\mathbf{oblv}^l_{\mathcal{Y}} := p^*_{\mathrm{dR},\mathcal{Y}} : \mathrm{QCoh}\left(\mathcal{Y}_{\mathrm{dR}}\right) \to \mathrm{QCoh}\left(\mathcal{Y}\right).$$

Analogously to Lemma 2.3.4, we have:

Lemma 2.3.12 (Corollaries 2.1.4 and 2.2.4, [18]). *For $\mathcal{Y} \in PreStk_{l.f.t.}$, we have an equivalence*

$$Crys^l(\mathcal{Y}) \xrightarrow{\sim} \lim_{S \in (\mathcal{C}_{/\mathcal{Y}})^{op}} Crys^l(S),$$

where $\mathcal{C}$ is any of the categories

$$Sch^{Aff,red}_{f.t.}, Sch^{Aff}_{f.t.}, Sch^{red}_{f.t.}, Sch_{f.t.}, Sch^{Aff,red}, Sch^{Aff}, \text{ or } Sch^{red}.$$

Recall that for any $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{l.f.t.}}$ we have a functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}\left(\mathcal{Y}\right) \to \mathrm{IndCoh}\left(\mathcal{Y}\right).$$

Applying this to $\mathcal{Y}_{\mathrm{dR}}$, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}) & \xrightarrow{\Upsilon_{\mathcal{Y}_{\mathrm{dR}}}} & \mathrm{Crys}(\mathcal{Y}) \\ \downarrow{\scriptstyle \mathbf{oblv}^l_{\mathrm{dR},\mathcal{Y}}} & & \downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{dR},\mathcal{Y}}} \\ \mathrm{QCoh}\left(\mathcal{Y}\right) & \xrightarrow[\Upsilon_{\mathcal{Y}}]{} & \mathrm{IndCoh}\left(\mathcal{Y}\right) \end{array}$$

Lemma 2.3.13 (Proposition 2.4.4, [18]). *The functor $\Upsilon_{\mathcal{Y}_{dR}}$ is an equivalence.*

Since the Υ functors intertwine the $*$-pullback functors for quasi-coherent sheaves with the !-functors for ind-coherent sheaves (see Lemma 2.2.14), we also have the following commutative diagram for any $F : \mathcal{Y}_1 \to \mathcal{Y}_2 \in \mathrm{PreStk}_{\mathrm{l.f.t.}}$:

$$\begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}_2) & \xrightarrow{\Upsilon_{\mathcal{Y}_{2,\mathrm{dR}}}} & \mathrm{Crys}(\mathcal{Y}_2) \\ {\scriptstyle F^{\dagger,l}}\big\downarrow & & \big\downarrow{\scriptstyle F^{\mathrm{dR},!}} \\ \mathrm{Crys}^l(\mathcal{Y}_1) & \xrightarrow[\Upsilon_{\mathcal{Y}_{1,\mathrm{dR}}}]{} & \mathrm{Crys}(\mathcal{Y}_1) \end{array}$$

By Lemma 2.3.13, for any $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{l.f.t.}}$, we can consider a single category

$$\mathcal{D}(\mathcal{Y})$$

of $\mathcal{D}$-*modules* on $\mathcal{Y}$, with realisations as either the category of left crystals or right crystals on $\mathcal{Y}$. We hence have two forgetful functors, to the categories $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$:

$$\begin{array}{ccc} & \mathcal{D}(\mathcal{Y}) & \\ {\scriptstyle \mathbf{oblv}^l_{\mathrm{dR},\mathcal{Y}}}\swarrow & & \searrow{\scriptstyle \mathbf{oblv}_{\mathrm{dR},\mathcal{Y}}} \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow[\Upsilon_{\mathcal{Y}_{\mathrm{dR}}}]{} & \mathrm{IndCoh}(\mathcal{Y}). \end{array}$$

We denote the functors $F^{\mathrm{dR},!}$ or $F^{\dagger,l}$ simply by

$$F^! : \mathcal{D}(\mathcal{Y}_2) \to \mathcal{D}(\mathcal{Y}_1)$$

when this will not be ambiguous.

Notation 2.3.14. If $\mathcal{Y} = \mathrm{colim}_{I \in \mathcal{S}} Z(I)$ is a pseudo-indscheme, we have for each I a pair of adjoint functors

$$\lambda^I_! : \mathcal{D}(Z(I)) \rightleftarrows \mathcal{D}(\mathcal{Y}) : (\lambda^I)^!.$$

Notation 2.3.15. The category $\mathcal{D}(\mathcal{Y})$ of $\mathcal{D}$-modules has a symmetric monoidal structure, coming from the symmetric monoidal structures on $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$. When we are thinking of the left realisation of $\mathcal{D}$-modules, we will use the notation $\otimes$, but when we are thinking right realisation, we will typically denote the tensor operation by $\otimes^!$, just as we did for ind-coherent sheaves.

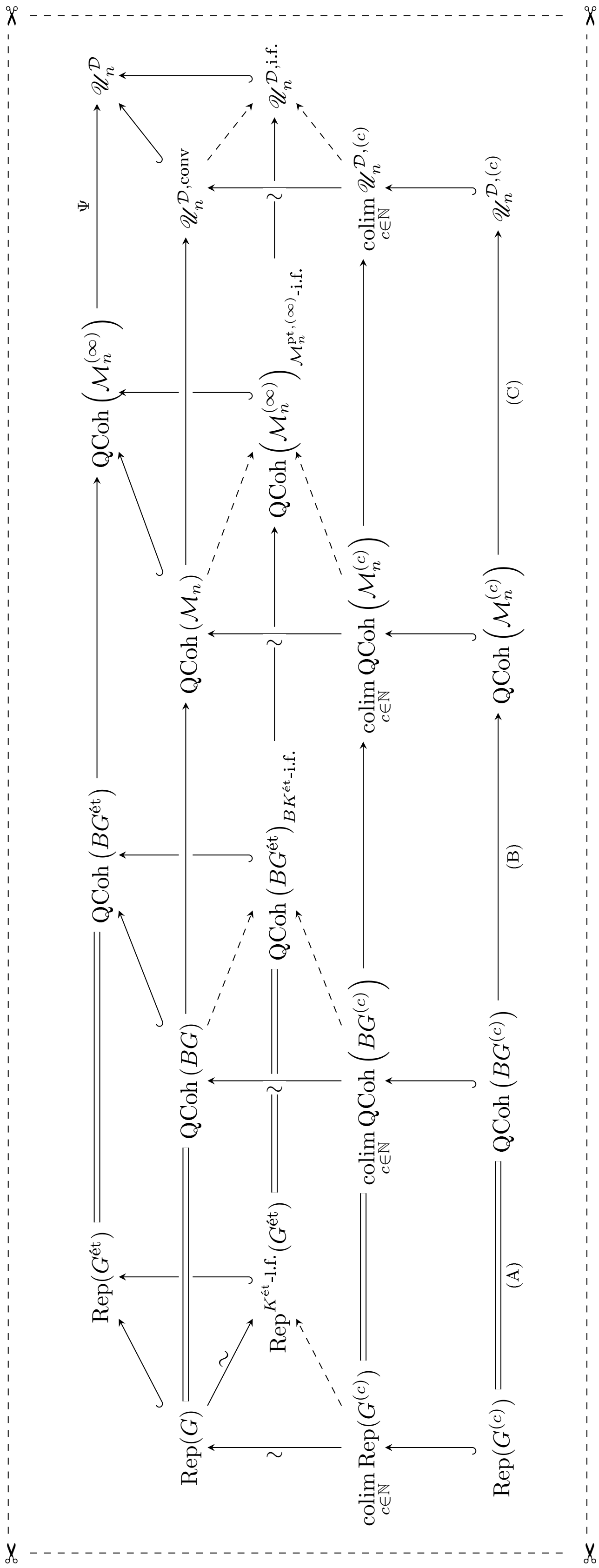
The compatibility between the two realisations is the following: suppose we have $\mathcal{D}$-modules on a scheme X given by quasi-coherent sheaves $\mathscr{F}, \mathscr{G} \in \mathrm{QCoh}(X_{\mathrm{dR}})$. If we view them instead as right $\mathcal{D}$-modules, they correspond to the ind-coherent sheaves given by $\mathscr{F} \otimes \omega_X$ and $\mathscr{G} \otimes \omega_X$. The tensor product of the quasi-coherent sheaves (or left $\mathcal{D}$-modules) is simply $\mathscr{F} \otimes \mathscr{G}$, which corresponds to the ind-coherent sheaf $(\mathscr{F} \otimes \mathscr{G}) \otimes \omega_X$ under the equivalence $\Upsilon_{X_{\mathrm{dR}}}$. On the other hand, the compatibility of the action of $\mathrm{QCoh}\,(X_{\mathrm{dR}})$ on $\mathrm{IndCoh}\,(X_{\mathrm{dR}})$ with the monoidal structures of the two categories implies that

$$(\mathscr{F} \otimes \mathscr{G}) \otimes \omega_X \simeq (\mathscr{F} \otimes \omega_X) \otimes^! (\mathscr{G} \otimes \omega_X).$$

Appendix B

The main diagram

We include on the next page an extra copy of the main diagram. The reader may wish to cut it out for ease of reference while reading Chapter II.



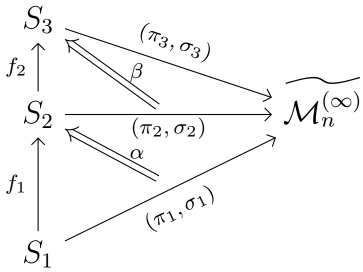
$\mathrm{Rep}(G^{\text{ét}})$
$\mathrm{QCoh}\left(BG^{\text{ét}}\right)$
$\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)$
Ψ
$\mathscr{U}_n^{\mathcal{D}}$
$\mathrm{Rep}(G)$
$\mathrm{QCoh}\left(BG\right)$
$\mathrm{QCoh}\left(\mathcal{M}_n\right)$
$\mathscr{U}_n^{\mathcal{D},\mathrm{conv}}$
$\mathrm{Rep}^{K^{\text{ét}}\text{-l.f.}}(G^{\text{ét}})$
$\mathrm{QCoh}\left(BG^{\text{ét}}\right)_{BK^{\text{ét}}\text{-i.f.}}$
$\mathrm{QCoh}\left(\mathcal{M}_n^{(\infty)}\right)_{\mathcal{M}_n^{\mathrm{pt},(\infty)}\text{-i.f.}}$
$\mathscr{U}_n^{\mathcal{D},\mathrm{i.f.}}$
$\operatorname{colim}_{c\in\mathbb{N}} \mathrm{Rep}(G^{(c)})$
$\operatorname{colim}_{c\in\mathbb{N}} \mathrm{QCoh}\left(BG^{(c)}\right)$
$\operatorname{colim}_{c\in\mathbb{N}} \mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right)$
$\operatorname{colim}_{c\in\mathbb{N}} \mathscr{U}_n^{\mathcal{D},(c)}$
$\mathrm{Rep}(G^{(c)})$
$\mathrm{QCoh}\left(BG^{(c)}\right)$
$\mathrm{QCoh}\left(\mathcal{M}_n^{(c)}\right)$
$\mathscr{U}_n^{\mathcal{D},(c)}$
(A)
(B)
(C)

Appendix C

Proof of compatibility of θ with composition

In this section, we show that for $\mathscr{F} \in \mathscr{U}_n^{\mathcal{D}}$ the assignment $(f, \alpha) \mapsto \theta(\mathscr{F})(f, \alpha)$ is compatible with composition.

Suppose that we have a commutative diagram as follows



with the commutativity of the diagram given by morphisms α and β represented by common étale neighbourhoods $(V_\alpha, \phi_\alpha, \psi_\alpha)$ between X_1/S_1 and $(S_1 \times_{S_2} X_2)/S_1$, and $(W_\beta, \phi_\beta, \psi_\beta)$ between X_2/S_2 and $(S_2 \times_{S_3} X_3)/S_2$. Then the commutativity of the large triangle in the diagram is given by the morphism $f_1^*\beta \circ \alpha$ represented by the pullback of the common étale neighbourhoods:

$$\Big(V_\alpha \times_{(S_1 \times_{S_2} X_2)} (S_1 \times_{S_2} W_\beta), \phi_\alpha \circ \mathrm{pr}_{V_\alpha}, f_1^*\psi_\beta \circ \mathrm{pr}_{S_1 \times_{S_2} W_\beta}\Big)$$
$$= \big(V_\alpha \times_{X_2} W_\beta, \phi_\alpha \circ \mathrm{pr}_{V_\alpha}, \rho_\alpha \times \psi_\beta\big).$$

We wish to show that

$$\theta(\mathscr{F})(f_2 \circ f_1, f_1^*\beta \circ \alpha) = \theta(\mathscr{F})(f_1, \alpha) \circ f_1^*\theta(\mathscr{F})(f_2, \beta). \tag{C.1}$$

From the definition of $f_1^*\beta \circ \alpha$, we have that

$$\theta(\mathscr{F})(f_2 \circ f_1, f_1^*\beta \circ \alpha)$$
$$= \overline{\tau_{f_1^*\beta \circ \alpha}}^* \Big(\mathscr{F}(\phi_\alpha \circ \mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{X_3} \circ \psi_\beta \circ \mathrm{pr}_{W_\beta}, f_2 \circ f_1)^{-1}\Big). \tag{C.2}$$

Applying the compatibility of $\mathscr{F}(\bullet)$ with composition of fibrewise morphisms, we can rewrite (C.2) as follows:

$$\overline{\tau_{f_1^*\beta\circ\alpha}}^* \Big((\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1})^*_{X/S} \mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{W_\beta}, f_1)^{-1} \\ \circ (\mathrm{pr}_{W_\beta}, f_1)^*_{X/S} \mathscr{F}(\psi_\beta, \mathrm{id}_{S_2})^{-1} \circ (\mathrm{pr}_{W_\beta}, f_1)^*_{X/S} (\psi_\beta, \mathrm{id}_{S_2})^*_{X/S} \mathscr{F}(\mathrm{pr}_{X_3}, f_2)^{-1} \Big). \quad \text{(C.3)}$$

Next we note the following equalities:

(a) $(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1})_{X/S} \circ \overline{\tau_{f_1^*\beta\circ\alpha}} = \overline{\tau_\alpha}$;

(b) $(\mathrm{pr}_{W_\beta}, f_1)_{X/S} \circ \overline{\tau_{f_1^*\beta\circ\alpha}} = \overline{\tau_\beta} \circ f_1$;

(c) $(\psi_\beta, \mathrm{id}_{S_2})_{X/S} \circ \overline{\tau_\beta} = \overline{f_2^*\sigma_3}$;

(d) $(\psi_\alpha, \mathrm{id}_{S_1})_{X/S} \circ \overline{\tau_\alpha} = \overline{f_1^*\sigma_2}$.

We will use these repeatedly in the remainder of this section. For example, using (a),(b),(c), we can rewrite (C.3) in the following way:

$$\overline{\tau_\alpha}^* \mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1}) \circ \overline{\tau_{f_1^*\beta\circ\alpha}}^* \mathscr{F}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1}) \\ \circ \overline{\tau_{f_1^*\beta\circ\alpha}}^* \mathscr{F}(\mathrm{pr}_{W_\beta}, f_1)^{-1} \circ f_1^* \overline{\tau_\beta}^* \mathscr{F}(\psi_\beta, \mathrm{id}_{S_2})^{-1} \circ f_1^* \overline{f_2^*\sigma_3}^* \mathscr{F}(\mathrm{pr}_{X_3}, f_2)^{-1}. \quad \text{(C.4)}$$

On the other hand, the right hand side of (C.1) is given by

$$\theta(\mathscr{F})(f_1, \alpha) \circ f_1^* \theta(\mathscr{F})(f_2, \beta) \\ = \left(\overline{\tau_\alpha}^* \left(\mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{X_2} \circ \psi_\alpha, f_1)^{-1} \right) \right) \\ \circ f_1^* \overline{\tau_\beta}^* \left(\mathscr{F}(\phi_\beta, \mathrm{id}_{S_2}) \circ \mathscr{F}(\mathrm{pr}_{X_3} \circ \psi_\beta, f_2)^{-1} \right). \quad \text{(C.5)}$$

We expand this using the compatibility of $\mathscr{F}(\bullet)$ with composition, and use the equalities (c) and (d) to obtain

$$\overline{\tau_\alpha}^* \mathscr{F}(\phi_\alpha, \mathrm{id}_{S_1}) \circ \overline{\tau_\alpha}^* \mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})^{-1} \circ \overline{f_1^*\sigma_2}^* \mathscr{F}(\mathrm{pr}_{X_2}, f_1)^{-1} \\ \circ f_1^* \overline{\tau_\beta}^* \mathscr{F}(\phi_\beta, \mathrm{id}_{S_2}) \circ f_1^* \overline{\tau_\beta}^* \mathscr{F}(\psi_\beta, \mathrm{id}_{S_2})^{-1} \circ f_1^* \overline{f_2^*\sigma_3}^* \mathscr{F}(\mathrm{pr}_{X_3}, f_2)^{-1}. \quad \text{(C.6)}$$

Comparing (C.4) and (C.6) we see that to prove the desired equality (C.1), it suffices to show that

$$\overline{\tau_\alpha}^* \mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1})^{-1} \circ \overline{f_1^*\sigma_2}^* \mathscr{F}(\mathrm{pr}_{X_2}, f_1)^{-1} \circ f_1^* \overline{\tau_\beta}^* \mathscr{F}(\phi_\beta, \mathrm{id}_{S_2}) \\ = \overline{\tau_{f_1^*\beta\circ\alpha}}^* \mathscr{F}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1}) \circ \overline{\tau_{f_1^*\beta\circ\alpha}}^* \mathscr{F}(\mathrm{pr}_{W_\beta}, f_1)^{-1},$$

or equivalently that

$$\overline{f_1^*\sigma_2}^*\mathscr{F}(\mathrm{pr}_{X_2}, f_1) \circ \overline{\tau_\alpha}^*\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1}) \circ \overline{\tau_{f_1^*\beta\circ\alpha}}^*\mathscr{F}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1}) \\ = f_1^*\overline{\tau_\beta}^*\mathscr{F}(\phi_\beta, \mathrm{id}_{S_2}) \circ \overline{\tau_{f_1^*\beta\circ\alpha}}^*\mathscr{F}(\mathrm{pr}_{W_\beta}, f_1). \quad \text{(C.7)}$$

Using (b), the right hand side of (C.7) becomes

$$\overline{\tau_{f_1^*\beta\circ\alpha}}^*\Big((\mathrm{pr}_{W_\beta}, f_1)^*_{X/S}\mathscr{F}(\phi_\beta, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{W_\beta}, f_1)\Big) \\ = \overline{\tau_{f_1^*\beta\circ\alpha}}^*\mathscr{F}(\phi_\beta \circ \mathrm{pr}_{W_\beta}, f_1). \quad \text{(C.8)}$$

Meanwhile, using (d) and (a), the left hand side of (C.7) can be written as

$$\overline{\tau_{f_1^*\beta\circ\alpha}}^*\big((\psi_\alpha, id_{S_1})^*_{X/S}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1})^*\mathscr{F}(\mathrm{pr}_{X_2}, f_1) \\ \circ(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1})^*\mathscr{F}(\psi_\alpha, \mathrm{id}_{S_1}) \circ \mathscr{F}(\mathrm{pr}_{V_\alpha}, \mathrm{id}_{S_1})\big). \quad \text{(C.9)}$$

Using once again the compatibility of $\mathscr{F}(\bullet)$ with composition, we see that this is equal to

$$\overline{\tau_{f_1^*\beta\circ\alpha}}^*\mathscr{F}(\mathrm{pr}_{X_2} \circ\psi_\alpha \circ \mathrm{pr}_{V_\alpha}, f_1). \quad \text{(C.10)}$$

Since $\mathrm{pr}_{X_2} \circ\psi_\alpha\circ\mathrm{pr}_{V_\alpha} = \phi_\beta\circ\mathrm{pr}_{W_\beta}$, the expression in (C.8) is equal to the expression in (C.10) and so the proof is complete.

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