NOTES ON THE BERNSTEIN PRESENTATION OF THE HECKE ALGEBRA

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ABSTRACT. These notes are based on my talk titled "The Bernstein presentation of the Hecke algebra" given in the Informal Friday Seminar organised by Anna Romanov.

Let $G \supset B \supset T$ be a split semi-simple simply-connected complex algebraic group. Let H_W be the Hecke algebra associated to the (finite) Weyl group $W \coloneqq N_G(T)/T$ of G. In this notes I introduce the affine Hecke algebra $\mathbf{H}$, this algebra was first introduced by J. Bernstein, and is isomorphic to the Iwahori-Hecke algebra of a split p-adic group with connected center. It contains H_W as a subalgebra and a large complementary corresponding to "translation part". I will show how the above isomorphism look like in the case of $G = \mathrm{SL}(2, \mathbb{C})$.

1. MOTIVATION

1.1. **The puzzle. Two different geometric incarnations of the affine Hecke algebra.** Let (R, P) be a reduced semi-simple simply-connected root datum, where P is the character lattice. Let $G_{\mathbb{Z}} \supset B_{\mathbb{Z}} \supset T_{\mathbb{Z}}$ be the split Chevalley group scheme over $\mathbb{Z}$ associated to (R, P). For any field $\Bbbk$, the extension of scalars produces an algebraic group $G_{\Bbbk} \supset B_{\Bbbk} \supset T_{\Bbbk}$ over k which is split semi-simple simply-connected and has the same root data (R, P). Let $G \coloneqq G_{\mathbb{C}}$ be the associated algebraic group over $\mathbb{C}$. The subgroup $B \coloneqq B_{\mathbb{C}}$ is a Borel subgroup of G, i.e., a maximal solvable subgroup, and $T \coloneqq T_{\mathbb{C}}$ is a maximal torus; for example, $G = \mathrm{SL}(n, \mathbb{C})$, the set of $n \times n$ matrices with determinant 1, B the subset of G of upper triangular matrices and T the subset of B of diagonal matrices.

Let $W_{\mathrm{aff}} \coloneqq W \ltimes P$ be the affine Weyl group associated to (R, P), where W is the (finite) Weyl group $W \coloneqq N_G(T)/T$ of G. The positive roots R^+ of R are the non-trivial weight spaces for the T-action on the Lie algebra of B.

Iwahori and Matsumoto introduced in [IM65] the affine Hecke algebra as the convolution algebra of some bi-invariant complex-valued functions on the p-adic points on $G_{\mathbb{Z}}$. More specifically, let p be a prime number, $\mathbb{Q}_p$ be the corresponding field of p-adic numbers with ring of integers $\mathbb{Z}_p$ and residue field $\mathbb{F}_p$. Let $G_{\mathbb{Z}}(\mathbb{Q}_p)$ be the set of p-adic points of $G_{\mathbb{Z}}$. Consider the standard diagram

$$\mathbb{Q}_p \hookleftarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p / p \cdot \mathbb{Z}_p = \mathbb{F}_p.$$

It induces the new diagram

$$G_{\mathbb{Z}}(\mathbb{Q}_p) \hookleftarrow G_{\mathbb{Z}}(\mathbb{Z}_p) \twoheadrightarrow G_{\mathbb{Z}}(\mathbb{F}_p).$$

Let I be an Iwahori subgroup of $G_{\mathbb{Z}}(\mathbb{Q}_p)$, i.e., the inclusion into $G_{\mathbb{Z}}(\mathbb{Q}_p)$ of the inverse image of $B_{\mathbb{Z}}(\mathbb{F}_p)$ via the projection above.

Denote by $\mathbb{C}[I \backslash G_{\mathbb{Z}}(\mathbb{Q}_p)/I]_c$ the vector space of all I-bi-invariant complex-valued functions on $G_{\mathbb{Z}}(\mathbb{Q}_p)$ with compact support. This set of functions has a natural algebra structure given by convolution of functions on $G_{\mathbb{Z}}(\mathbb{Q}_p)$. Iwahori and Matsumoto in [IM65] gave a presentation of this algebra in terms of generators and relations. For this purpose they first defined the "abstract" Hecke algebra H_{aff} as a q-analogue of the group algebra of W_{aff}, where q is an indeterminate. Specialising at $q = p$ it gives an algebra isomorphism between H_{aff} and $\mathbb{C}[I\backslash G_{\mathbb{Z}}(\mathbb{Q}_p)/I]_c$.

Later, Bernstein (in an unpublished work [1]) gave a completely different presentation of the affine Hecke algebra. Firstly, he realised that the group algebra of W_{aff}—the affine Weyl group defined above—can be described (as a module) as the tensor product of $\mathbb{Z}[P]$ (the "translation part") and H_W (the "finite part"), where H_W is the Hecke algebra associated to W. He defined an algebra $\mathbf{H}$ such presentation is a q-analogue of the description above. By the work of Kazhdan and Lusztig [KL87, Theorem 3.5], and Chriss and Ginzburg [CG97, Theorem 7.2.5], we have an isomorphism between $\mathbf{H}$ and the ${}^LG \times \mathbb{C}^*$-equivariant K-group of the Steinberg variety Z.

Summarising, we have the following diagram

$$\begin{tikzcd} H_{\text{aff}} \arrow[r, leftrightarrow, "\text{Bernstein isomorphism}"] \arrow[d, leftrightarrow, "q=p"] & \mathbf{H} \arrow[d, leftrightarrow] \\ \mathbb{C}[I\backslash G_{\mathbb{Z}}(\mathbb{Q}_p)/I]_c \arrow[r, leftrightarrow, "???"] & K^{{}^LG\times\mathbb{C}^*}(Z), \end{tikzcd} \tag{1.1}$$

where the isomorphism in blue on the left side is due to Iwahori and Matsumoto, the brown isomorphism on the top is due to Bernstein and the purple isomorphism on the right is due to Kazhdan and Lusztig, and Chriss and Ginzburg. The top part of the diagram is the algebraic part of the story. The bottom part is the mystery, we have in both sides different geometric realisations. On the one hand, we have a convolution algebra of some complex-valued functions on an algebraic group, on the other hand, we have formal linear combinations of classes of vector bundles over a variety Z modulo the relation $[E \oplus F] = [E] + [F]$. Surprisingly, and for no apparent reason, both sides coincide.

2. The affine Hecke algebra of (R, P)

2.1. Let us fix an abstract torus T. Let $P := \operatorname{Hom}_{\text{alg}}(T, \mathbb{C}^*)$ be the weight lattice and let $P^\vee := \operatorname{Hom}_{\text{alg}}(\mathbb{C}^*, T)$ be the coweight lattice. Let $a \in P^\vee$ and $\alpha \in P$. The composition $\alpha \circ a$ is an algebraic map from $\mathbb{C}^*$ to itselt, hence of the form $z \mapsto z^{n(\alpha,a)}$ for some $n(\alpha, a) \in \mathbb{Z}$. This gives us a perfect pairing

$$\begin{aligned} \langle\ \ ,\ \ \rangle : P \times P^\vee &\to \mathbb{Z} \\ (\alpha, a) &\mapsto n(\alpha, a). \end{aligned}$$

Definition 2.1. $R \subset P$ is a *reduced root system* if there is a finite set $R^\vee \subset P^\vee$ and a bijection $R \leftrightarrow R^\vee$, $\alpha \leftrightarrow \alpha^\vee$, such that:

- $\langle \alpha, \alpha^\vee \rangle = 2$ for any $\alpha \in R$.
- For any $\alpha \in R$, the map $s_\alpha : P \to P$ (resp. $s_{\alpha^\vee} : P^\vee \to P^\vee$) given by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle\, \alpha$ (resp. $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle\, \alpha^\vee$) preserves $R \subset P$ (resp. $R^\vee \subset P^\vee$).
- If $\alpha \in R$ then $c\alpha \in R$ if and only if $c = \pm 1$.

[1] This construction first appeared in [Lus83] and with greater amount of detail in [Lus89].

We call (R, P) a *simply-connected root system* if $R^\vee$ generates $P^\vee$.

Remark 2.2. If G is a semi-simple simply-connected (in the topological sense) algebraic group over $\mathbb{C}$ then $\mathbb{Z}R^\vee = P^\vee$.

2.2. Let W be the Weyl group of (R, P), i.e., the group generated by all s_α. We can fix a set S of elements of W called simple reflections such that (W, S) is a Coxeter system. This choice also fixes a set $\Delta \subset R$ of simple roots. Let q be an indeterminate. Let $\mathcal{L} \coloneqq \mathbb{Z}[q^{\pm}]$ be the ring of Laurent polynomials with integer coefficients over $\mathbb{Z}$. The *Hecke algebra* of W is the $\mathcal{L}$-module spanned by $\{T_w \mid w \in W\}$ such that

$$T_s^2 = (q-1)T_s + q \quad \text{for all} \quad s \in S,$$
$$T_y T_w = T_{yw} \quad \text{for all} \quad \ell(yw) = \ell(y) + \ell(w).$$

The Weyl group W naturally acts on P. The weight lattice acts on itself via translations. Let $Q = \mathbb{Z}R \subset P$. Then we can define $\tilde{W} \coloneqq W \ltimes Q$ and the affine Weyl group $W_{\text{aff}} \coloneqq W \ltimes P$, the group operation is given by

$$(w, \lambda) \cdot (v, \mu) = (wv, v^{-1}(\lambda) + \mu).$$

One can prove there is a set $\tilde{S}$ such that $(\tilde{W}, \tilde{S})$ is a Coxeter system. It has a length function $\ell \colon \tilde{W} \to \mathbb{Z}_{\geq 0}$. However, W_{aff} is not in general a Coxeter group.

Let R^+ be the set of positive roots as in the first section. We define the real vector spaces $P_{\mathbb{R}} \coloneqq P \otimes_{\mathbb{Z}} \mathbb{R}$ and $P_{\mathbb{R}}^\vee \coloneqq P^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with pairing $\langle \ , \ \rangle_{\mathbb{R}}$ induced by the perfect pairing $\langle \ , \ \rangle$ from before. Note that W_{aff} acts via isometries on $P_{\mathbb{R}}$. For each $\alpha \in R^+$ and $m \in \mathbb{Z}$, we can define the hyperplane

$$H_{\alpha,m} \coloneqq \{\lambda \in P_{\mathbb{R}} \mid \langle \alpha^\vee, \lambda \rangle_{\mathbb{R}} = m\}.$$

The fundamental alcove $\mathcal{A}^+$ of $P_{\mathbb{R}}$ is the set

$$\{\lambda \in P_{\mathbb{R}} \mid 0 \leq \langle \alpha^\vee, \lambda \rangle_{\mathbb{R}} \leq 1\}.$$

We can extend the definition of ℓ to W_{aff} by:

$$\ell \colon W_{\text{aff}} \to \mathbb{Z}_{\geq 0},$$
$$x \mapsto \sharp \begin{Bmatrix} H_{\alpha,m} \text{ separating the interior} \\ \text{of } x(\mathcal{A}^+) \text{ from the one of } \mathcal{A}^+ \end{Bmatrix}.$$

Let $\Omega \coloneqq \ell^{-1}(0)$ be the set of length zero elements of W_{aff}. It follows immediately from the definition that any element of Ω preserves $\mathcal{A}^+$, and hence acts on its walls. This action induces an action of Ω into $\tilde{S}$ and hence on $\tilde{W}$, see [IM65]. We have that $W_{\text{aff}} = \Omega \ltimes \tilde{W}$.

2.3. **Iwahori-Matsumoto's world.** The affine Hecke algebra H_{aff} of (R, P) is the $\mathcal{L}$-module with basis $\{T_w \mid w \in W_{\text{aff}}\}$ with multiplication given by

$$T_s^2 = (q-1)T_s + q \quad \text{for all} \quad s \in \tilde{S}, \tag{2.1}$$
$$T_y \cdot T_w = T_{yw} \quad \text{for all} \quad \ell(yw) = \ell(y) + \ell(w).$$

Remark 2.3. Note the second relation implies

$$T_\sigma T_\rho = T_{\sigma\rho} \quad \text{for all} \quad \sigma, \rho \in \Omega.$$

In particular, we have $\mathcal{L}[\Omega] \hookrightarrow H_{\text{aff}}$ as a $\mathcal{L}$-subalgebra. Let $H_{\tilde{W}}$ be the subalgebra of H_{aff} generated by $\{T_w \mid w \in \tilde{W}\}$, we have $H_{\text{aff}} \cong \mathcal{L}[\Omega] \otimes_{\mathcal{L}} H_{\tilde{W}}$ as left $\mathcal{L}$-modules (**Caution:** this map is *not* an isomorphism of $\mathcal{L}$-algebras.)

2.4. **Bernstein's idea.** Let $P^+ \subset P$ be the set of dominant weights. Let $R(T)$ be the representation ring of the torus. We know that $R(T) = \mathbb{Z}[P]$. We want

$$\mathbb{Z}[P] \hookrightarrow H_{\text{aff}}. \tag{2.2}$$

Inside W_{aff} we have translations $t_\lambda \colon P \to P$ such that $t_\lambda(\mu) = \mu + \lambda$. Hence, we have the associated elements T_{t_λ} in H_{aff}. The thing to consider is that the equality

$$T_{t_\lambda} \cdot T_{t_\mu} \neq T_{t_\lambda t_\mu}$$

only holds when $\ell(t_\lambda) + \ell(t_\mu) = \ell(t_\mu t_\lambda)$. This happens for example if we consider only translations associated to dominant weights. To solve this problem, for a general weight $\lambda \in P$, we can write $\lambda = \mu - \mu'$ with $\mu, \mu' \in P^+$. We can send λ to the element $T_{t_\mu} \cdot T_{t'_\mu}^{-1}$. Note that this element is well defined no matter the choice of μ and μ'. To see this, let $\lambda = \mu - \mu' = \rho - \rho'$ with $\mu, \mu', \rho, \rho' \in P^+$. Then $\rho' + \mu = \rho + \mu'$ and $T_{t_{\rho'+\mu}} = T_{t_{\rho'}} \cdot T_{t_\mu} = T_{t_\rho} \cdot T_{t_{\mu'}}$. This implies $T_{t_\mu} = T_{t_{\rho'}}^{-1} \cdot T_{t_\rho} \cdot T_{t_{\mu'}}$ and hence

$$\begin{aligned}
T_{t_\mu} \cdot T_{t_{\mu'}}^{-1} &= T_{t'_\rho}^{-1} \cdot T_{t_\rho} \\
&= T_{t_{\rho'}}^{-1} \cdot T_{t_\rho} \cdot T_{t_{\rho'}} \cdot T_{t_{\rho'}}^{-1} \\
&= T_{t_{\rho'}}^{-1} \cdot T_{t_{\rho+\rho'}} \cdot T_{t'_\rho}^{-1} \\
&= T_{t_{\rho'}}^{-1} \cdot T_{t_{\rho'}} \cdot T_{t_\rho} \cdot T_{t_{\rho'}}^{-1} \\
&= T_{t_\rho} \cdot T_{t'_\rho}^{-1}.
\end{aligned}$$

This new family of elements together with their relations with T_s with $s \in S$, gives the ingredients for the Bernstein presentation.

2.5. **Bernstein's world.** The affine Hecke algebra $\mathbf{H}$ of (R, P) is the free $\mathcal{L}$-module with basis $\{e^\lambda \cdot T_w \mid w \in W, \lambda \in P\}$ and with multiplication given by

$$\begin{gathered}
T_s^2 = (q-1)T_s + q \quad \text{for all} \quad s \in S, \\
e^\lambda \cdot e^\mu = e^{\lambda+\mu} \quad \text{for all} \quad \lambda, \mu \in P, \\
T_{s_\alpha} \cdot e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1-q)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1 - e^{-\alpha}} \quad \text{for all} \quad \lambda \in P, \alpha \in \Delta.
\end{gathered}$$

The following theorem states the isomorphism given in the top part of the diagram (1.1).

Theorem 2.4 (Bernstein)**.** *We have the following isomorphism of $\mathbb{Z}[q^{\pm 1}]$-algebras*

$$\mathbf{H} \cong H_{\text{aff}}.$$

Now we have the inclusion (2.2) as desired, just define it as $\lambda \mapsto e^\lambda$. We also have $H_W \hookrightarrow \mathbf{H}$, where H_W is the Hecke algebra associate to the (finite) Weyl group W. Furthermore, we have the following proposition.

Proposition 2.5. *We have the following isomorphism of $\mathbb{Z}[q^{\pm 1}]$-modules*

$$\mathbf{H} \cong R(T) \otimes_{\mathbb{Z}[q,q^{-1}]} H_W.$$

Remark 2.6. The above isomorphism is not an isomorphism of $\mathbb{Z}[q^{\pm 1}]$-algebras.

This gives a construction of the algebra analogue to the construction as a semidirect product of the affine Weyl group, i.e., $W_{\text{aff}} = W \ltimes P$, where P represents the "translation part" and W represents the "finite part".

Proposition 2.7. *Consider $R(T) \subset \mathbb{Z}[W_{\text{aff}}]$ via the natural inclusion. Let $R(T)^W$ be the subset of $R(T)$ of W-invariant elements. Then $R(T)^W[q, q^{-1}]$ is the center of $\mathbf{H}$.*

2.6. **Example on** $\mathrm{SL}(2, \mathbb{C})$**.** Let $G = \mathrm{SL}(2, \mathbb{C})$. Let us fix the following choice of maximal torus

$$T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}.$$

The weight lattice is $P = \mathrm{Hom}_{\text{alg}}(T, \mathbb{C}^*) \cong \mathbb{Z}$ via the map

$$\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mapsto z^n \right) \mapsto n.$$

We can identify the set $R = \{\pm\alpha\}$ with the subset $\{\pm 2\} \subset \mathbb{Z}$. Let

$$P^\vee = \mathrm{Hom}_{\text{alg}}(\mathbb{C}^*, T) \cong \mathbb{Z}$$

be the coweight lattice. We can identify the set $R^\vee$ with the subset $\{\pm 1\} \subset \mathbb{Z}$. In particular, (R, P) is simply-connected. The finite Weyl group W is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and is generated by the reflection s such that $s(\lambda) = -\lambda$, for all $\lambda \in P$, i.e., the reflection around 0. Let u be the reflection around 1 and σ be the reflection around $1/2$—even though $1/2$ is not part of P, σ is a well defined involution on P—and $\tilde{W}$ is the Universal Coxeter group of rank 2 in generators s, u. The group $\Omega \cong \mathbb{Z}/2\mathbb{Z}$ is generated by σ, and $W_{\text{aff}} = W \ltimes P = \Omega \ltimes \tilde{W}$ is generated by $\{s, u, \sigma\}$.

The (Iwahori-Matsumoto) affine Hecke algebra H_{aff} of (R, P) is the $\mathcal{L}$-algebra with basis $\{T_s, T_u, T_\sigma\}$ and with multiplication given by

$$T_s^2 = (q-1)T_s + q \tag{2.3}$$

$$T_u^2 = (q-1)T_u + q \tag{2.4}$$

$$T_\sigma^2 = 1. \tag{2.5}$$

The affine Hecke algebra $\mathbf{H}$ of (R, P) is the free $\mathcal{L}$-algebra with basis $\{T_s, e^{\alpha/2}, e^{-\alpha/2}\}$ and multiplication given by

$$T_s^2 = (q-1)T_s + q,$$
$$e^{\alpha/2} \cdot e^{-\alpha/2} = 1,$$
$$T_s \cdot e^{-\alpha/2} - e^{-\alpha/2} T_s = (1-q)e^{-\alpha/2}.$$

We can deduce from this the very useful formula

$$T_s \cdot e^{-\lambda} - e^{\lambda} T_s = (1-q)\frac{e^{\lambda} - e^{-\lambda}}{1 - e^{-\alpha}} \quad \text{for all} \quad \lambda \in P. \tag{2.6}$$

We can compute explicitly the isomorphism given in 2.4 after extending scalars (we need the square root of q.) It is given by

$$\begin{aligned}\phi\colon H_{\text{aff}} \otimes \mathbb{Z}[q^{\pm 1/2}] &\to \mathbf{H} \otimes \mathbb{Z}[q^{\pm 1/2}]\\ T_s &\mapsto T_s\\ T_u &\mapsto qe^{\alpha} \cdot T_s^{-1}\\ T_\sigma &\mapsto q^{1/2}e^{\alpha/2} \cdot T_s^{-1}.\end{aligned}$$

We going to prove that this is in fact an algebra isomorphism. For this is enough to prove the relations (2.3), (2.4) and (2.5) in the image. The first one is immediate. For the second one, we use the fact that $T_s^{-1} = q^{-1}(T_s - (q-1))$, then

$$\begin{aligned}\phi(T_u)^2 &= e^{\alpha}(T_s - (q-1))e^{\alpha}(T_s - (q-1))\\ &= e^{\alpha}T_s e^{\alpha}T_s - (q-1)e^{\alpha}T_s e^{\alpha} - (q-1)e^{2\alpha}T_s + (q-1)^2e^{2\alpha}. \qquad (2.7)\end{aligned}$$

Now we use Equation (2.6) (with $\lambda = -\alpha$) to compute $e^{\alpha}T_s e^{\alpha}$:

$$\begin{aligned}e^{\alpha}T_s e^{\alpha} &= e^{\alpha}\left(e^{-\alpha}T_s + (1-q)\frac{e^{-\alpha} - e^{\alpha}}{1 - e^{-\alpha}}\right)\\ &= e^{\alpha}\left(e^{-\alpha}T_s + (q-1)(e^{\alpha}+1)\right)\\ &= T_s + (q-1)e^{2\alpha} + (q-1)e^{\alpha}. \qquad (2.8)\end{aligned}$$

Furthermore,

$$\begin{aligned}e^{\alpha}T_s e^{\alpha}T_s &= T_s^2 + (q-1)e^{2\alpha}T_s + (q-1)e^{\alpha}T_s\\ &= q + (q-1)T_s + (q-1)e^{2\alpha}T_s + (q-1)e^{\alpha}T_s. \qquad (2.9)\end{aligned}$$

Replacing (2.8) and (2.9) in (2.7)—and after cancelations—we get

$$\begin{aligned}\phi(T_u)^2 &= q + (q-1)e^{\alpha}T_s - (q-1)^2e^{\alpha}\\ &= q + (q-1)e^{\alpha}\left(T_s - (q-1)\right)\\ &= q + (q-1)qe^{\alpha}T_s^{-1}\\ &= q + (q-1)\phi(T_u).\end{aligned}$$

For the third one,

$$\begin{aligned}\phi(T_\sigma)^2 &= q^{-1}e^{\alpha/2}(T_s - (q-1))e^{\alpha/2}(T_s - (q-1))\\ &= q^{-1}e^{\alpha/2}T_s e^{\alpha/2}T_s - q^{-1}(q-1)e^{\alpha/2}T_s e^{\alpha/2} - q^{-1}(q-1)e^{\alpha}T_s + q^{-1}(q-1)^2e^{\alpha}. \qquad (2.10)\end{aligned}$$

Now we use Equation (2.6) (with $\lambda = -\alpha/2$) to compute $e^{\alpha/2}T_s e^{\alpha/2}$:

$$\begin{aligned}e^{\alpha/2}T_s e^{\alpha/2} &= e^{\alpha/2}\left(e^{-\alpha/2}T_s + (1-q)\frac{e^{-\alpha/2} - e^{\alpha/2}}{1 - e^{-\alpha}}\right)\\ &= e^{\alpha/2}\left(e^{-\alpha/2}T_s + (q-1)e^{\alpha/2}\right)\\ &= T_s + (q-1)e^{\alpha}. \qquad (2.11)\end{aligned}$$

Furthermore,

$$\begin{aligned}e^{\alpha/2}T_s e^{\alpha/2}T_s &= T_s^2 + (q-1)e^{\alpha}T_s\\ &= q + (q-1)T_s + (q-1)e^{\alpha}T_s. \qquad (2.12)\end{aligned}$$

Replacing (2.11) and (2.12) in (2.10)—and after cancelations—we get $\phi(T_\sigma)^2 = 1$.

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