Unimodality of Bruhat intervals

Gaston Burrull

The University of Sydney

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- Overview about unimodality and top-heaviness

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- The Top-Heavy Conjecture

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- The affine Weyl group

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- The Top-Heavy Conjecture
- The affine Weyl group
- Towards unimodality

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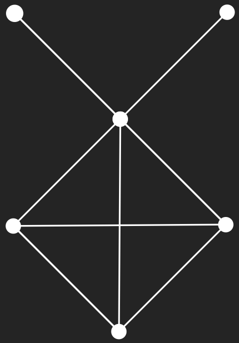

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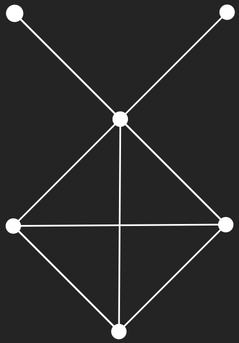

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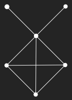

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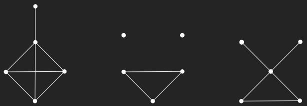

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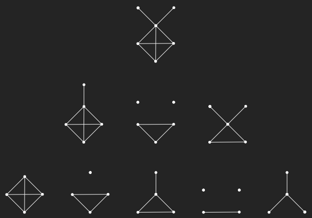

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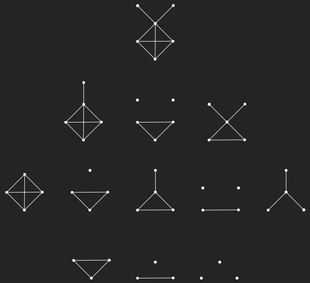

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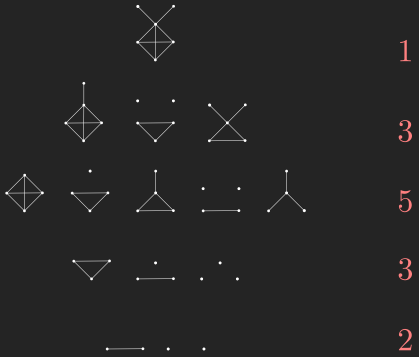

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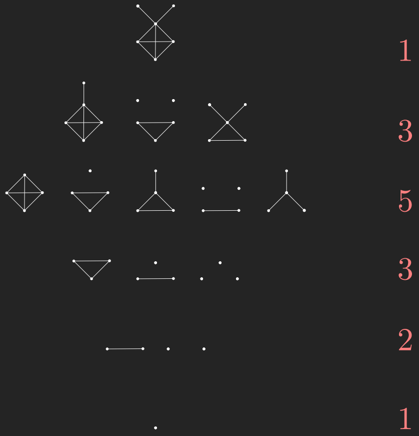

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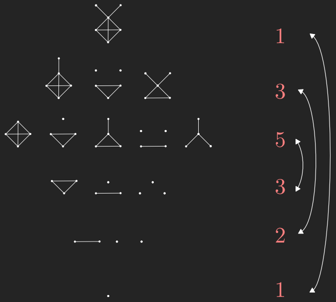
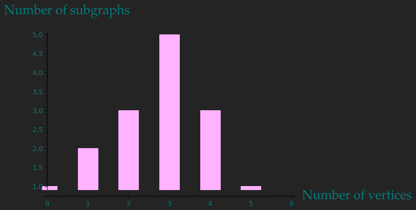
We observe a “unimodal” and “top-heavy” behaviour





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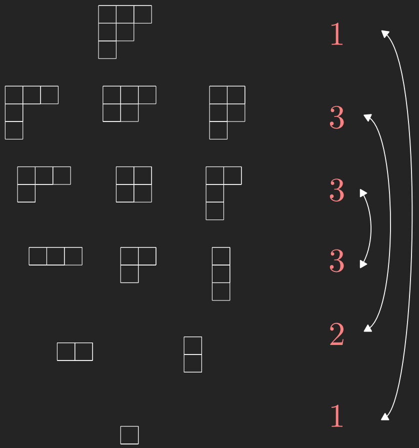
Unimodal and top-heavy behaviours also appear in partitions



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Then

$$b_i \leq b_{n-i}, \text{ for all } i \leq \frac{\dim(V)}{2}.$$

The Top-Heavy Conjecture: An example

Let $V = \mathbb{R}^3$ and

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

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This was recently proved (Braden, Huh, Matherine, Proudfood, and Wang, 2020).

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- The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and $-\alpha$.
- For every root $\alpha \in \Phi$, the set Φ is closed under reflection s_α through the hyperplane perpendicular to α.

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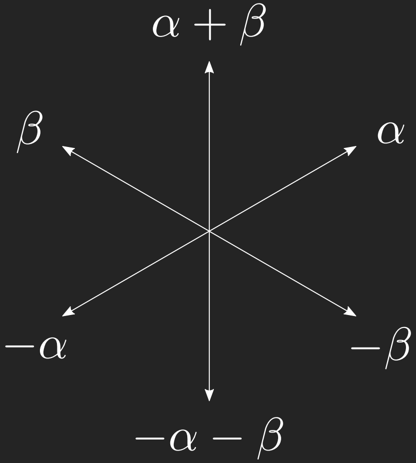

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$$C_+ = \{v \in V \mid \langle \alpha^\vee, v \rangle > 0 \text{ for every } \alpha \in \Phi_+\}.$$

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The group $\mathbb{Z}\Phi$ is a lattice and its called the root lattice.

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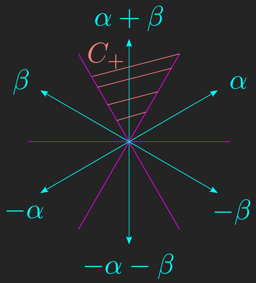

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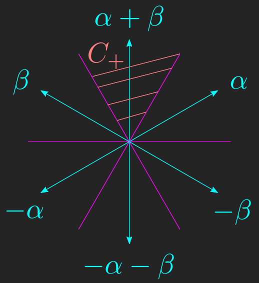

In this case the finite Weyl group

$$W_f = \langle s_\alpha, s_\beta : s_\alpha^2 = s_\beta^2 = \mathrm{id}, (s_\alpha s_\beta)^3 = \mathrm{id} \rangle$$

is isomorphic to the symmetric group $S_3 = \mathrm{Sym}(\{1, 2, 3\})$ consisting of 6 elements.

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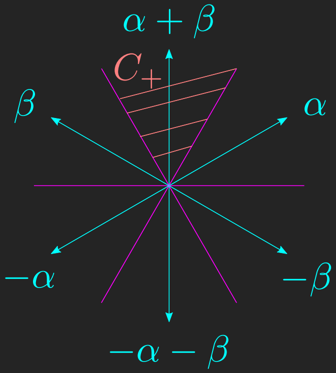

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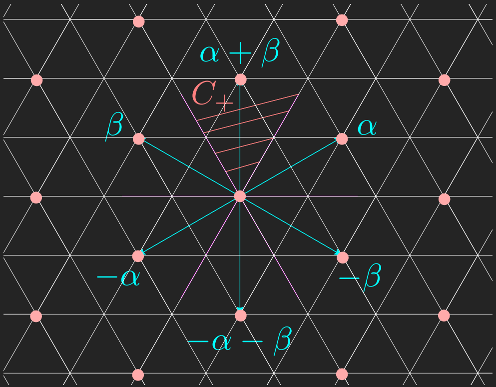

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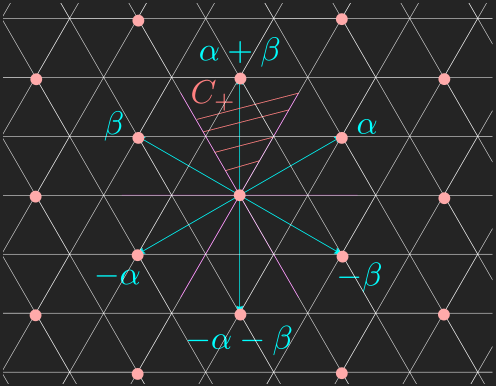

The points in the root lattice $\mathbb{Z}\Phi$ are the orange circles in the picture.

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$$\begin{aligned} W &\to \mathcal{A} \\ w &\mapsto wA_+ . \end{aligned}$$

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where t is a reflection with respect to an hyperplane in W, and the alcoves txA_+ and A_+ lie in the same side of such hyperplane.

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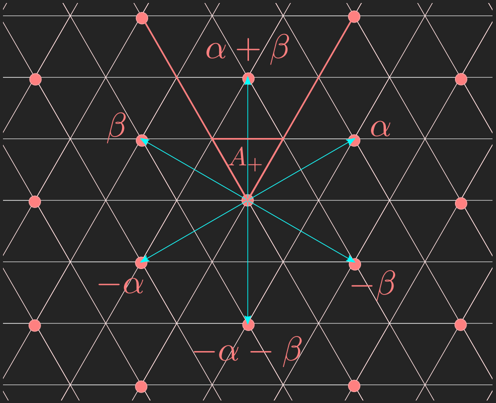

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A similar question for the whole group W is false in type A. A counterexample comes from an element in the associated Schubert variety $X = \mathrm{Gr}_4(\mathbb{C}^{12})$, where the corresponding Betti numbers $b_{2i} = f_i$ (which count the number of cells of dimension $2i$ in X) are

$$1,1,2,3,5,6,9,11,15,17,21,23,27,28,31,30,31,27,24,18,14,8,5,2,1.$$

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Theorem (Björner and Ekhedal, 2009). The Betti numbers b_{2i} for a "good stratified" variety X satisfy:

$$b_{2\ell(w)-i} \leq b_{2\ell(w)+i}, \text{ for } i \leq n,$$
$$b_i \leq b_{i+2j}, \text{ for } 0 \leq j \leq n-i.$$

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A direct corollary of the previous theorem (by taking $f_i = b_{2i}$) is:

$$f_i \leq f_j, \text{ for } i \leq j \leq \ell(w)/2,$$
$$f_i \leq f_{\ell(w)-i}, \text{ for } i < \ell(w)/2.$$

Example: Type A_2

The Poincare polynomial for $[\mathrm{id}, w]$ in the picture

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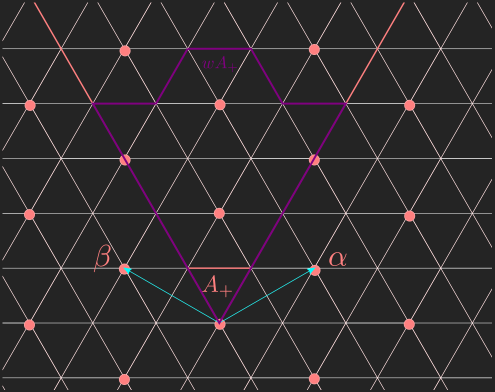
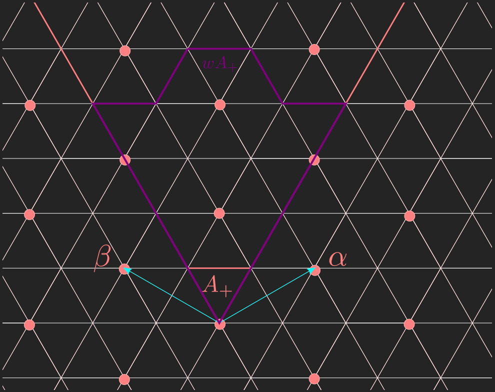
The Poincare polynomial for $[\mathrm{id}, w]$ in the picture



Example: Type A_2

The Poincare polynomial for $[\mathrm{id}, w]$ in the picture



is

$$1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 2q^7 + q^8$$

Example: Type $A2$

In bar graphics.

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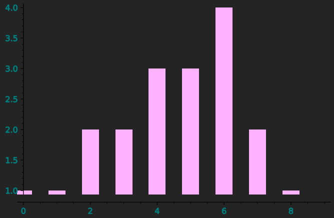

$$1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 4q^6 + 2q^7 + q^8$$

Example: Type $F4$

The Poincare polynomial for the dominant lattice interval $[0, 2\rho]$.

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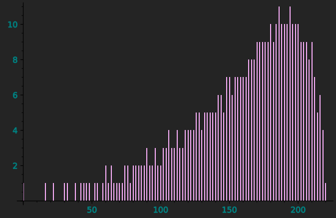
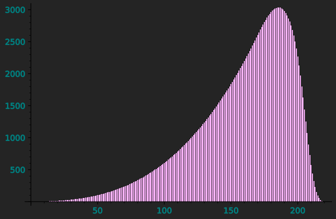


Example: Type $F4$

The Poincare polynomial for the interval $[\mathrm{id}, (2\rho, \mathrm{id})] \subset {}^{f}W$.

Example: Type $F4$

The Poincare polynomial for the interval $[\mathrm{id}, (2\rho, \mathrm{id})] \subset {}^f W$.



The End

Thank you!