Constraining overestimation of error covariances in ensemble Kalman filters

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We consider the problem of an ensemble Kalman filter when only partial observations are available. For small ensemble sizes this may lead to an overestimation of the error covariances. We show that by incorporating climatic information of the unobserved variables the variance can be controlled and superior analysis skill is obtained. We then apply this Variance Controlling Kalman Filter to

- sparse observational networks
- balance
- model error
- controlling catastrophic filter divergence

We assume that we have access to proper (noisy) observations are available for some variables (*observables*) but not for other unresolved variables, for which only their statistical climatic behaviour such as their variance and their mean is available (*pseudo-observables*). Observations $\mathbf{x}_{\rm obs}(t_i) = \mathbf{H}\mathbf{z}(t_i) + \mathbf{r}_{\rm obs}(t_i)$ are taken at equidistant observation times $t_n = n\Delta t_{\rm obs}$. Here $\mathbf{H} : \mathbb{R}^N \to \mathbb{R}^n$ is the observation operator, and the the observational noise is assumed to be Gaussian with error covariance matrix $\mathbf{R}_{\rm obs}$. For the pseudo-observables we assume climatic knowledge about the mean $\mathbf{a}_{\rm target}$ and variance $\mathbf{A}_{\rm target}$. We introduce the pseudo-observation operator $\mathbf{h} : \mathbb{R}^N \to \mathbb{R}^m$ and the error covariance matrix $\mathbf{R}_{\rm w}$ associated with the pseudo-observables. By requiring that the projected analysis error covariance assumes climatology, we can determine $\mathbf{R}_{\rm w}$.

The algorithm for the filter is as follows:

Step 1: Forecast step

$$\mathbf{Z}_f = F(\mathbf{Z}_b)$$

$$\mathbf{P}_f = \frac{1}{k-1}\mathbf{Z}'_f(t)[\mathbf{Z}'_f(t)]^T$$

Step 2: Analysis step

$$\bar{\mathbf{z}}_a = \bar{\mathbf{z}}_f - \mathbf{K}_{\rm obs}(\mathbf{H}\bar{\mathbf{z}}_f - \mathbf{x}_{\rm obs}) - \mathbf{K}_{\rm w}(\mathbf{h}\bar{\mathbf{z}}_f - \mathbf{a}_{\rm target})$$

$$\mathbf{K}_{\rm obs} = \mathbf{P}_f\mathbf{H}^T(\mathbf{H}\mathbf{P}_f\mathbf{H} + \mathbf{R}_{\rm obs})^{-1}\ , \quad \mathbf{K}_{\rm w} = \mathbf{P}_f\mathbf{h}^T(\mathbf{h}\mathbf{P}_f\mathbf{h} + \mathbf{R}_{\rm w})^{-1}$$
$$\mathbf{R}_{\rm w}^{-1} = \mathbf{A}_{\rm target}^{-1} - (\mathbf{h}\mathcal{P}_{\mathbf{a}}\mathbf{h}^T)^{-1} \qquad .$$

To assure that $\mathbf{R}_{\rm w}$ is positive definite, we diagonalize and project onto the overestimating subspace.

Step 3: Update of the ensemble
The ensemble needs to be consistent with

$$\mathbf{P}_a = [\mathbf{I} - \mathbf{K}_{\mathrm{obs}}\mathbf{H} - \mathbf{K}_{\mathrm{w}}\mathbf{h}]\ \mathbf{P}_f \ = \ \frac{1}{k-1}\mathbf{Z}'_a\,[\mathbf{Z}'_a]^T$$

using ensemble square root filters.

Step 4: Update of the forecast
Set $\mathbf{Z}_b = \mathbf{Z}_a$ to propagate the ensemble forward again with the full dynamics to the next observation time.

We have shown in simulations of the Lorenz-96 model that using this filter better skill in the data assimilation procedure is obtained compared to the classical ETKF in sparse observational networks for small observation intervals $\Delta t_{\mathrm{ons}} \leq 6$hrs, and for sufficiently large observational noise. For large observational intervals the ensemble of ETKF will have acquired climatological covariance and our constraint is not needed.

Furthermore we have presented results on a genesis for catastrophic filter divergence in the situation of sparse observations with small observational noise. Machine-infinity blow-up of the forecast model was explained by the finite-size sampling effect of large cross-covariances pushing the analysis off the attractor with subsequent rapid attraction back to the attractor. The associated stiffness causes the numerical integrator to blow-up.

References

[1] G. A. Gottwald, L. Mitchell and S. Reich, *Controlling overestimation of error covariance in ensemble Kalman filters with sparse observations: A variance limiting Kalman filter*, Monthly Weather Review **139** (2011), 2650–2667.

[2] G. A. Gottwald and L. Mitchell *Controlling model error of underdamped forecast models in sparse observational networks using a variance limiting Kalman filter*, QJRMS (in press).