A MODIFIED HEGSELMANN–KRAUSE MODEL FOR INTERACTING VOTERS AND POLITICAL PARTIES

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Abstract. The Hegselmann–Krause model is a prototypical model for opinion dynamics. It models the stochastic time evolution of an agent's or voter's opinion in response to the opinion of other like-minded agents. The Hegselmann–Krause model only considers the opinions of voters; we extend it here by incorporating the dynamics of political parties which influence and are influenced by the voters. We show in numerical simulations for 1- and 2-dimensional opinion spaces that, as for the original Hegselmann–Krause model, the modified model exhibits opinion cluster formation as well as a phase transition from disagreement to consensus. We provide an analytical sufficient condition for the formation of unanimous consensus in which voters and parties collapse to the same point in opinion space in the deterministic case. Using mean-field theory, we further derive an approximation for the critical noise strength delineating consensus from non-consensus in the stochastically driven modified Hegselmann–Krause model. We compare our analytical findings with simulations of the modified Hegselmann–Krause model.

Keywords: Hegselmann–Krause model, interacting particle systems, sociological modelling, sociophysics

1. Introduction

Voting and elections are an essential part of modern democracies. Typically, elections consist of many voters and a few political parties. While the individual behaviour of voters and parties is difficult to predict, simple behavioural rules can produce complex behaviour that resembles known political dynamics. State-based models, such as the voter model, have long been used to model the propagation of opinions across a population [42, 32]. In reality, voters rarely consider the parties for which they vote to share their opinions exactly. More complex considerations are made about which party's view is closer to their own. In addition, political parties respond to voters through political campaigns and by supporting or opposing policies. In this paper, we propose an interacting particle system, which treats parties and voters as agents that evolve and interact within an opinion space.

The collective behaviour of interacting particle systems has been extensively studied [5, 11, 7, 37, 12, 13, 20, 9]. The celebrated Hegselmann–Krause model describes the temporal evolution of opinions of interacting agents in a continuous opinion space [31, 28]. An agent's opinion evolves towards the average opinion of agents with similar opinions. To account for any, possibly irrational, individual behaviour of voters additive noise is added to the dynamics. Such dynamics may give rise to clustering dynamics and exhibits a phase transition from disagreement to consensus formation with decreasing noise strength. In recent years, several modifications to the original Hegselmann–Krause model were proposed to include grouped populations, self-belief and heterogeneity [24, 15, 27], the effect of leadership voters [46, 49, 4], inertial effects [14] and underlying social network structure [38]. For recent reviews of opinion spreading models see [11, 34]. These models only describe the mutual interaction of voters and neglect the dynamics of parties. Parties shape the opinion space of voters with a strong impact on voters' behaviour [6, 29, 30], and similarly voters correct and influence parties who are competing for votes [48, 1]. The competition for votes also leads to parties delineating themselves from other parties and carving out opinion space for themselves [33, 1]. We introduce an extended noisy Hegselmann–Krause model in which the voter dynamics are augmented by a dynamical model for parties which takes into account these interactions. Voters and parties are represented in a d-dimensional opinion space, representing their respective political orientations towards d separate political issues [17, 41, 22, 3, 30]. Being able to describe complex voter and party behaviour as dynamics in an opinion space requires the availability of information across the whole electorate, which has been made even more effective with the advent of social media [45].

The modified Hegselmann–Krause model will be shown to exhibit known voter and party dynamics such as party bases, swing voters and disaffected voters. Moreover, depending on the strength of the interactions between voters and parties, we show that there exist metastable states where voters are attracted to different parties which occupy different regions in opinion space, before eventually unanimous consensus occurs, in which voters and parties collapse to a single localised area in opinion space. A second scenario may occur in which parties and voters decouple and voters collapse to a single opinion state whereas parties arrange themselves in separate areas in opinion space. Using linear stability analysis and mean-field theory we analytically provide sufficient conditions for consensus and find an expression for the critical noise strength above which no consensus is possible.

The paper is organized as follows: In Section 2 we propose our new model combining consensus dynamics of voters with party dynamics. In Section 3 we show how the inclusion of parties recovers to known political scenarios such as party base formation, disaffected voters and swing voters. In Section 4 we numerically explore the evolution to consensus. For the deterministic case we provide a sufficient condition for consensus formation in Section 5. In Section 6 we provide analytical results on a noise-induced phase transition from random voter dynamics to consensus. We conclude in Section 7 with a discussion.

2. A modified Hegselmann–Krause model incorporating party dynamics

The original Hegselmann–Krause model is concerned with the interaction of N_v voters v_i [28, 39, 37]. Voters are represented by their position in some d-dimensional opinion space. A point in the opinion space can be thought of as representing a set of views. For $d = 2$, the opinion space is referred to as the political compass [17, 41, 22, 3] with the two coordinates of v_i representing, for example, a voter's political leaning on the spectrum from left-wing to right-wing social views and their economic preferences on the spectrum from libertarian to socialist. For $d > 2$, each dimension might represent opinions on specific issues [43]. In the noisy Hegselmann–Krause model, the dynamics of the voters $v_i(t) \in \mathbb{R}^d$ in this opinion space is governed by the following weakly interacting particle system

$$dv_i = \frac{1}{N_v} \sum_{j}^{N_v} \phi(v_i, v_j)(v_j - v_i)dt + \sigma dW_t^i. \tag{1}$$

Here W^i are d-dimensional independent standard Brownian motions representing uncertainty with strength σ. The interaction kernel ϕ depends on the Euclidean distance in opinion space between the voters and encodes how voters v_j with different opinions to a voter v_i can influence voter v_i. We choose here an isotropic kernel which is compactly supported on $[0, R_{vv}]$ with

$$\phi(v_i, v_j) = \phi\left(x = \frac{||v_i - v_j||}{R_{vv}}\right) = \begin{cases} 1 & x < 1 \\ 0 & \text{else} \end{cases}, \tag{2}$$

where R_{vv} is the interaction radius of voters. The interaction kernel (2) lets a voter i interact equally with all voters that are within a radius of R_{vv} in opinion space. Other choices of interaction kernels are used, for example, allowing for heterophilious dynamics [37]. Noting that only differences in opinion enter the dynamics, we restrict the opinion space without loss of generality to the d-dimensional unit square $[0, 1]^d$. Note that any domain size of the opinion space can be absorbed by scaling the interaction radius R_{vv}.

One of the key features of Hegselmann–Krause models is the emergence of consensus, which occurs when voters collapse in opinion space to a single opinion cluster, the size of which depends on the strength of the noise σ [37, 25, 44]. In particular, the system exhibits a phase transition [23, 47]; there exists a critical noise strength σ_c such that for $\sigma < \sigma_c$ the noisy Hegselmann–Krause model (1) asymptotically approaches consensus, whereas for $\sigma > \sigma_c$ no consensus is reached and instead voters behave as N_v independent stochastic processes [44].

The noisy Hegselmann–Krause model (1) only models the dynamics of individual voters and is not designed to model the dynamics of a political landscape involving political parties. Therefore, it cannot be used to serve as a model for elections. We will extend the classical Hegselmann–Krause model to couple the preferences and dynamics in the d-dimensional opinion space of N_v individual voters $v_i \in \mathbb{R}^d$ and a finite number N_p of parties $p_\alpha \in \mathbb{R}^d$. We make the reasonable assumption that voters will vote for political parties that share similar opinions to them and that both, voters and parties, share the same d-dimensional opinion space [10]. We make the following assumptions about the interactions between voters and parties: voters are effected by their interactions with other voters and also with parties. In particular, voters attract one another according to the right-hand side drift term of the Hegselmann–Krause model (1) which we denote as F_{vv}. The effect is that voters move towards the mean of the voter opinion within a d-dimensional sphere with radius R_{vv} centred on the voter. Voters are further attracted by parties which are sufficiently close to them in opinion space [30, 1]. The associated force can be thought of as a leadership effect and will be denoted as F_{vp} [33, 6]. Voters who have a similar opinion to a political party, will shape their beliefs to conform to the leaders with whom they are similar. Conversely, parties are affected by the voters and by other parties [48, 45]. It is reasonable to assume that political parties aim to maximise the number of votes they receive. A natural way of achieving this is to minimise the distance between them and their potential voters. However, parties also have "identities" - for example, some are considered left-wing and others centrist - which means that the voters they seek to be close to (or best represent) are within a certain distance R_{vp} in opinion space. We denote the associated attracting force by F_{pv}. Finally, it is reasonable to assume that political parties want to differentiate themselves from each other [21]. Why vote for one party if there is another party that is exactly the same? Hence, we assume a repulsive interactive force F_{pp} between political parties. Figure 1 is a schematic of these four interactions.

Summarizing the assumed interactions outlined above we propose the following coupled voter-party Hegselmann–Krause model

$$dv_i = F_{vv}(v_i, v)\, dt + F_{pv}(v_i, p)\, dt + \sigma_v dW_t^i \tag{3}$$

$$dp_\alpha = F_{vp}(p_\alpha, v)\, dt - F_{pp}(p_\alpha, p)\, dt + \sigma_p dW_t^\alpha, \tag{4}$$

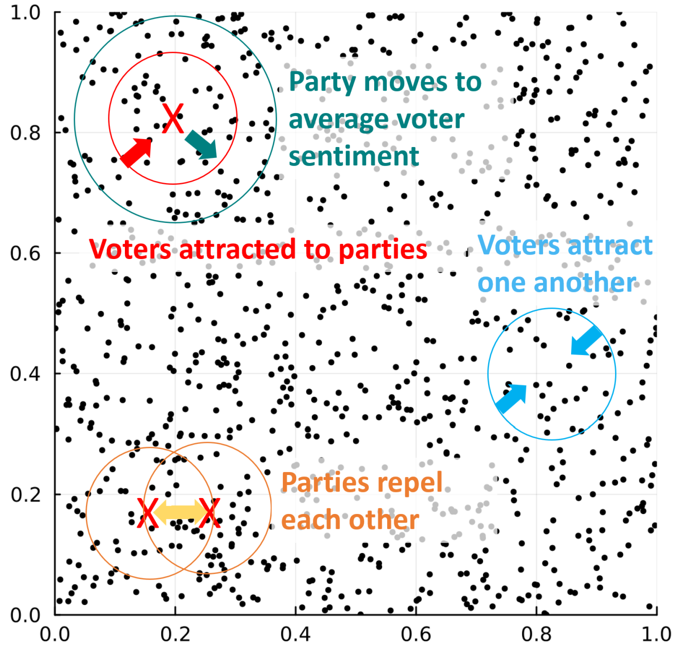


FIGURE 1. Sketch of the different interaction forces between voters (black dots) and parties (grey stars) in a 2-dimensional opinion space. The blue force corresponds to F_{vv} with (5); the red force corresponds to F_{pv} with (6); the green force corresponds to F_{vp} with (7); the orange force corresponds to F_{pp} with (8).

with the interaction forces

$$F_{vv}(v_i, v) = \mu_{vv} \sum_{j=1}^{N_v} \phi\left(\frac{||v_j - v_i||}{R_{vv}}\right)(v_j - v_i), \tag{5}$$

$$F_{pv}(v_i, p) = \mu_{pv} \sum_{\beta=1}^{N_p} \phi\left(\frac{||p_\beta - v_i||}{R_{pv}}\right)(p_\beta - v_i), \tag{6}$$

$$F_{vp}(p_\alpha, v) = \mu_{vp} \sum_{j=1}^{N_v} \phi\left(\frac{||v_j - p_\alpha||}{R_{vp}}\right)(v_j - p_\alpha), \tag{7}$$

$$F_{pp}(p_\alpha, p) = \mu_{pp} \sum_{\beta=1}^{N_p} \phi\left(\frac{||p_\beta - p_\alpha||}{R_{pp}}\right)(p_\beta - p_\alpha). \tag{8}$$

Here $W^i(t)$ for $i = 1, \ldots, N_v$ and $W^\alpha(t)$ for $\alpha = 1, \ldots, N_p$ are independent standard Brownian motion processes which represent unpredictable changes in a particular agent's opinion with the strength given by the diffusion coefficients σ_v and σ_p. In the following, Latin alphabet sub- and superscripts refer to voters and Greek alphabet sub- and superscripts refer to political parties. The forces are assumed to have compact support in a d-dimensional sphere with positive radii R_{vv}, R_{pv}, R_{vp} and R_{pp}, respectively (cf. (2) for R_{vv}). It is reasonable to assume that $R_{vp} > R_{vv}$ as parties typically consider a much larger contingency of voters than individual voters do. Further, we assume that R_{pp} is small as parties only repel each other when they become too close in opinion space to delineate themselves from each other. The strength of the respective forces is controlled by the strength parameters $\mu_{vv}, \mu_{pv}, \mu_{vp}, \mu_{pp} \geq 0$ which are assumed to be constant in time. Note that for $\mu_{vv} = 1, \mu_{pv} = 0$ the evolution of the voters (3) reduces to the original noisy Hegselmann–Krause model (1).

3. Political scenarios in the modified Hegselmann–Krause model

We now show that the modified Hegselmann–Krause model (3)-(4) is able to reproduce several types of voter behaviour in a political landscape with competing parties. We restrict here to a 1-dimensional opinion space.

Let us start from a standard scenario where each party has their own loyal political base, sometimes referred to as "core voters" [16]. This politically stable situation occurs when each party occupies a particular region in opinion space and all parties are sufficiently far apart in opinion space from each other with

$$||p_\alpha - p_\beta|| > 2\max(R_{pv}, R_{vv}) \tag{9}$$

for all parties $\alpha \neq \beta$ so that parties do not compete for voters and interactions between voters attached to different parties is suppressed. We further require $\sigma_v \gg \sigma_p$ to ensure the voter dynamics occurs much faster than the party dynamics. In such situations clusters may form around each party. The number of voters $N_v^{(c)}$ attached to each cluster c depends on the initial political opinions of the voters. We assume here that each cluster is centred around a single party for simplicity. The size of each party base cluster δ_{base} is determined by twice the size of the interaction radius, $2\max(R_{pv}, R_{vv})$, of the cluster and the size of the voter cluster which is fully interacting with the party and every other voter in its cluster. We find

$$\delta_{\text{base}} = \min\left(2\max(R_{pv}, R_{vv}),\, 4\frac{\sigma_v}{\sqrt{2}}\left(\frac{N_v^{(c)}}{N_v}\mu_{vv} + \frac{N_p^{(c)}}{N_p}\mu_{pv}\right)^{-\frac{1}{2}}\right), \tag{10}$$

where $N_p^{(c)} = 1$ for a party base cluster centred around a single party. Equality occurs when the interaction radii R_{pv}, R_{vv} cover the entire cluster. In this case the voter dynamics (3) reduces to an Ornstein-Uhlenbeck process

$$dv_i = -\left(\frac{N_v^{(c)}}{N_v}\mu_{vv} + \frac{1}{N_p}\mu_{pv}\right) v_i + \sigma_v dW_t^i, \tag{11}$$

with asymptotic mean given by the respective party position and standard deviation

$$\text{std}_{\text{OU}} = \frac{\sigma_v}{\sqrt{2}}\left(\frac{N_v^{(c)}}{N_v}\mu_{vv} + \frac{N_p^{(c)}}{N_p}\mu_{pv}\right)^{-\frac{1}{2}}. \tag{12}$$

Voters evolving according to (11) form hence a cluster with size roughly 4std_{OU}, motivating the overall party base cluster size (10).

Figure 2a shows an example for such party base formation with $N_v = 1,000$ voters which initially are distributed uniformly across $[0, 1]$ and with 3 parties which are initially located in opinion space at $p_1(0) = 0.86$, $p_2(0) = 0.53$ and $p_3(0) = 0.2$. The strengths of the voter forces were chosen as $\mu_{vv} = \mu_{pv} = 1$ and those of the party forces as $\mu_{vp} = 0.03$ and $\mu_{pp} = 0.01$. The interaction radii are $R_{vv} = R_{pv} = R_{pp} = 0.1$ and $R_{vp} = 0.2$. The noise strengths are $\sigma_v = 0.03$ and $\sigma_p = 0.005$, which ensures that the voter dynamics occurs much faster than the party dynamics. Figure 3 shows the standard deviation of the voters in the party base cluster around party p_1 with $p_1(0) = 0.2$. We show results of the numerical simulation of the modified Hegselmann–Krause model (3)-(4) for the case in Figure 2a as well as the prediction of the cluster size δ_{base} given by (10) which is dominated here by the standard deviation (12) and where we determine $N_v^{(c)}$ as the number of voters which are within a distance of 0.15 away from party p_1. It is seen that the observed cluster size of the base cluster which forms around $t \approx 20$ is well described by (10).

Such stable political bases break down if parties increase their interaction radius R_{pv} and hence are competing with each other over voters which are affected by two or more parties, or equivalently by parties moving towards each other in opinion space, induced by their nonzero diffusion σ_p. In particular, consider two parties p_α and p_β that satisfy

$$||p_\alpha - p_\beta|| < \max(R_{pv}, R_{vv}), \tag{13}$$

and where all other parties are sufficiently far away in opinion space such that they do not interact with any voters attracted to parties p_α and p_β. In this case this will lead to the emergence of swing voters which are confined between two parties. In elections such swing voters vote for either of the two close parties, depending on small hard to predict preferences[16]. The state of swing voters is illustrated in Figure 2b. Here the parameters are as for the political base scenario but for a larger party interaction radius $R_{pv} = 0.35$. The size of swing voter clusters δ_{swing} is also given by (10) with $N_p^{(c)} = 2$ because the party above and below are both within the interaction radius of the voter cluster. We find $\delta_{\text{swing}} = 0.085$ which approximates well the observed cluster size. The state of swing voters can be disturbed by the party dynamics. This is seen in Figure 2b around $t \approx 80$. The swing voter cluster between parties p_1 and p_3 is broken up after the middle party p_2 drifted sufficiently towards the swing voter cluster which now interacts with all three parties, eventually leading to the formation of a party base cluster around p_1. It is also possible for party base clusters to break up and form swing voter clusters when parties move closer to each other. An example of this is shown in Figure 2e around $t \approx 70$ when the middle two parties p_1 and p_2 move closer together and their party bases merge to a single swing cluster.

In the previous scenario we saw that swing voters can be persuaded to commit to a single party and join their party base. This change was entirely determined by the party dynamics and required two parties to sufficiently distinguish themselves from each other. Thus, when the party dynamics is faster - or alternatively the voter dynamics is slower - we may observe more competitive behaviour between the parties. Figure 2c shows clusters of swing voters that are slowly entrained by one of the two parties. We further note the coexistence of party bases and swing voter clusters, although neither is stable. Here $\sigma_v = 0.01$ ensures the voter behaviour is slower than in Figure 2a. Swing voters decide to join a particular party base either by their individual stochastic slow exploration of the opinion space or by parties moving towards them on a faster time scale. The latter scenario may be viewed as a form of party competition to attract more voters.

Political competition can lead to intricate complex transitions between the scenarios depicted in Figures 2a-2c. In particular, if two parties p_α and p_β satisfy $\max(R_{pv}, R_{vv}) < ||p_\alpha - p_\beta|| < 2\max(R_{pv}, R_{vv})$, neither the criterion for the existence of party base clusters (9) nor the criterion of the existence of swing voter clusters (13) are met and it is possible for both states to coexist and interact. For example, Figure 2d shows a party base forming around the top party p_3 while the voters around the bottom two parties p_1 and p_2 alternate between party base and swing voter behaviour. There is further competition between the top and middle parties p_3 and p_2 over a swing voter cluster that dissolves around $t \approx 60$.

Yet another important political scenario occurs when parties leave a region in opinion space unoccupied and this region is larger than the interaction radius R_{pv} of any party. In this case voters can occupy this region, attracted by voter-voter interactions only. In this case the voter dynamics degenerates to the original noisy Hegselmann–Krause model (1) and a voter-only cluster will form of size $\delta_{\text{dis}} \approx 4\sigma_v/\sqrt{2\mu_{vv}}$ provided that $R_{vv} > \delta_{\text{dis}}/2$ [44]. Note that we have $\delta_{\text{dis}} = \delta_{\text{base}}$

for $N_p^{(c)} = 0$. In this scenario voters may be viewed as disaffected voters who feel unrepresented by political parties, and whose support the parties are not interested in earning. In trying to attract those politically far away voters parties may risk loosing voters that are more aligned with a parties current views. Figure 2e shows an example with disaffected voters. Here the same parameters are used as for the political base clusters in Figure 2a but with different initial conditions for the parties allowing for unoccupied political space. Note that the disaffected voters do not form a localized cluster here of size δbase since R_{vv} is too small.

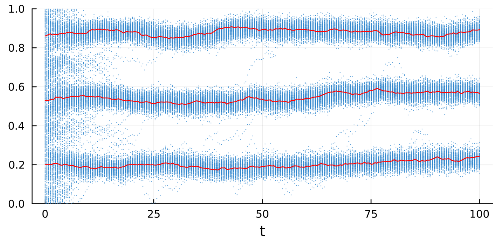
Lastly, we report on a scenario in which voters abandon their political base and aggregate to a single cluster. This occurs when $R_{vv} > R_{pv}$ and $\mu_{vv} \gtrapprox \mu_{pv}$. This scenario is akin to a political revolution in the political landscape during which a single party is able to attract all voters. Note that this is achieved with all parties having the same force strength μ_{pv} and without one party having a significantly higher attraction force than the other parties. An example of such a revolution is seen in Figure 2f where a single party base cluster forms around party p_1. We remark that if the condition for swing voters (13) is satisfied, the final state will be a swing voter cluster.

We remark that typically the scenarios described above are transitory and the stochasticity and/or party dynamics lead to a break up of these structures. The inclusion of heterogeneous and possible time-varying equation parameters, in particular of the force strengths, would allow for an even richer set of scenarios and transitions between them. In the next Section we will see how and when our model allows for a final state of unanimous consensus, which however, unlike the transient scenarios described above, is not a likely political scenario.

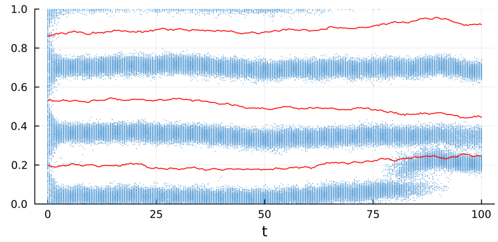
4. Consensus in the modified Hegselmann–Krause model

As we have seen in the previous Section the modified Hegselmann–Krause model (3)–(4) allows for the formation of opinion clusters. Consensus in our model refers to all voters and parties collapsing to a a cluster the size of which is determined by the noise strengths σ_v and σ_p. We will see that if σ_v, σ_p are sufficiently small and $\mu_{pp} < \mu_{vp}$ the system approaches consensus. The route from a disordered state of uniformly distributed voters and parties to a state of consensus in which voters and parties collapse to a localised region in opinion space is typically via partially ordered states in which voters and parties form several clusters. We show in Figure 4 an example for a $d = 1$-dimensional opinion space and in Figure 5 an example for a $d = 2$-dimensional opinion space. The model recreates some of the core features of the classical noisy Hegselmann–Krause model. In particular, in both cases the voters and parties form smaller clusters before eventually reaching a consensus state. Figure 4 shows that initially uniformly distributed voters begin to form clusters centred around parties. These clusters constitute what we encountered in Section 3 as political base of a party. The top two parties p_1 and p_2 develop well separated party base clusters since the distance between them is larger than the interaction-radius R_{vv}, prohibiting interactions between voters of different party bases. The bottom two parties p_3 and p_4 located near 0.25 and 0.35 are closer to one another and a cluster of swing voters forms. Around $t \approx 140$ all voter clusters have merged to form a single cluster with mean $v = 0.5$. The outer two parties p_1 and p_4 slowly move into the cluster as the attraction of the voters overcomes their repulsion of the other parties and they enter the cluster of voters.

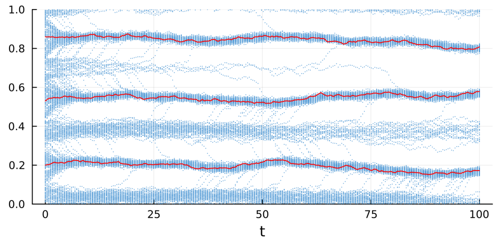
Similarly, in the two dimensional case depicted in Figure 5, we observe clusters of voters forming around parties. However, due to the increased size of the opinion space, many more voters are not part of a cluster loyal to a single party. Instead as seen at $t > 120$ there is a cluster of disaffected voters in the bottom right of the opinion space which are not centered around any party. Note that in contrast to the disaffected voters depicted in Figure 2e here R_{vv} is sufficiently large to allow for a cluster of disaffected voters of size δ_{dis}. In the model, over time these disaffected voters randomly evolve into one of the attractive areas of the parties. Parties



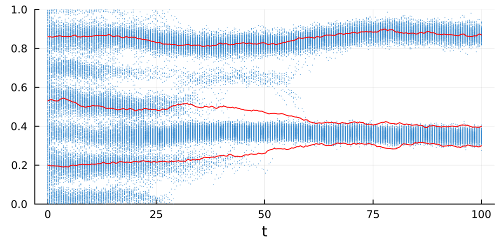
(A) *Party base*



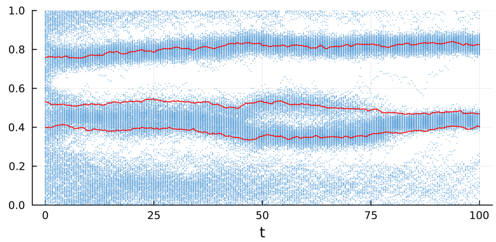
(B) *Swing voters.* Here $R_{pv} = 0.35$.



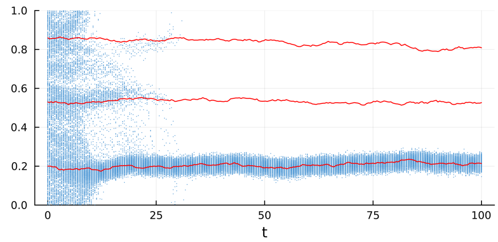
(C) *Political competition.* Here $\sigma_v = 0.01$.



(D) *Political competition*: Here $R_{pv} = 0.23$.



(E) *Disaffected voters*: Here $p_1(0) = 0.76$, $p_2(0) = 0.53$ and $p_3(0) = 0.4$.



(F) *Revolution.* Here $R_{vv} = 0.3$

FIGURE 2. Prototypical political scenarios. Unless stated otherwise, at time $t = 0$, $N_v = 1,000$ voters are distributed uniformly across $[0, 1]$ and parties are initially at $p_1(0) = 0.86$, $p_2(0) = 0.53$ and $p_3(0) = 0.2$. The strengths of the voter forces are $\mu_{vv} = \mu_{pv} = 1$. The strengths of the party forces are $\mu_{vp} = 0.03$ and $\mu_{pp} = 0.01$. The interaction radii are $R_{vv} = R_{pv} = R_{pp} = 0.1$ and $R_{vp} = 0.2$. The noise strengths are $\sigma_v = 0.03$ and $\sigma_p = 0.005$.

evolve to minimise their distance from voters around and so move towards the single largest cluster. This process of consensus formation can be slowed down by decreasing the interaction radius R_{vp}. This prohibits that parties are effected by voters which are further away, requiring slow diffusion to form consensus. Depending on the strength and interaction radii of the relative forces, parties may be attracted to voters when $\mu_{vp} > \mu_{pv}$, whereas in the case that $\mu_{pv} > \mu_{vp}$, voters will cluster around parties before the voter-voter interactions lead to global consensus. We remark that in bounded opinion spaces Brownian motion is recurrent even in high dimensions,

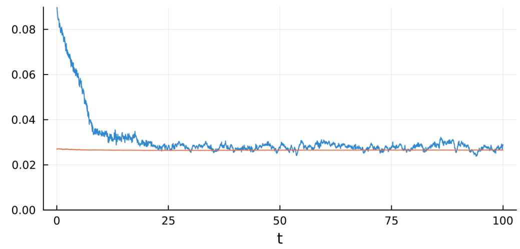


FIGURE 3. Standard deviation of the voter cluster centred around party p_1 with $p_1(0) = 0.2$ in Figure 2a. The horizontal orange predicted line denotes the analytical approximation (10).

and hence voters which are not yet entrained by the main cluster at the final time in Figures 4 and 5, will eventually join the cluster.

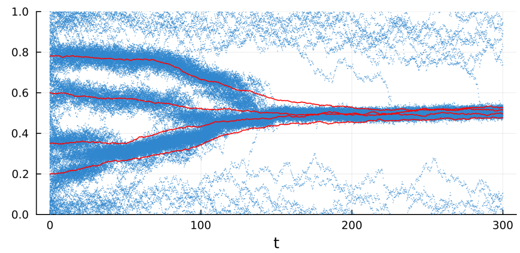
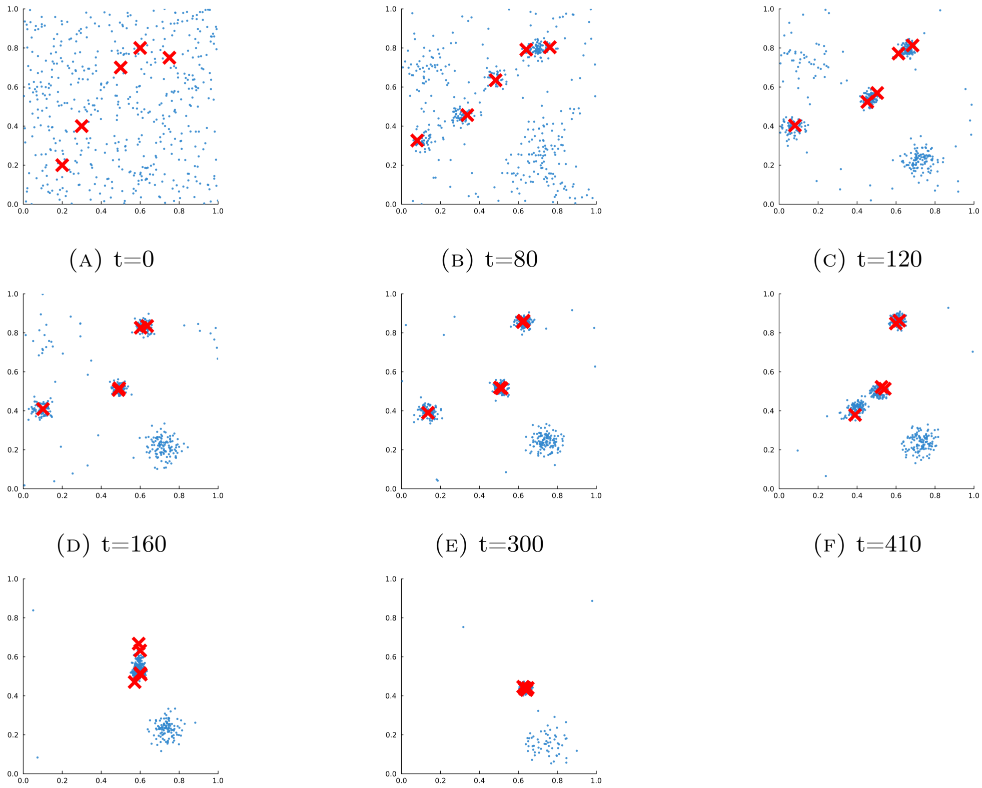


FIGURE 4. Evolution of the modified Hegselmann–Krause model (3)–(4) in a $d = 1$-dimensional opinion space for 500 voters (blue) and 4 parties (red). Initially voters are distributed uniformly across $[0, 1]$ and parties are initially at $p_1(0) = 0.78$, $p_2(0) = 0.6$, $p_3(0) = 0.35$ and $p_4(0) = 0.2$. Parameters are $R_{vv} = 0.05$, $R_{pv} = 0.1$, $R_{vp} = 0.4$, $R_{pp} = 0.05$ and $\mu_{vv} = 0.5$, $\mu_{pv} = 0.8$, $\mu_{vp} = 0.02$, $\mu_{pp} = 0.02$. The noise strengths are $\sigma_v = 0.02$ and $\sigma_p = 0.002$.

To quantify the degree of consensus of voters in the original Hegselmann–Krause model (1) Wang et al. [44] introduced the order parameter

$$Q_{vv} = \frac{1}{N_v^2} \sum_{i=1}^{N_v} \sum_{j=1}^{N_v} \phi\left(\frac{||v_j - v_i||}{R_{vv}}\right). \tag{14}$$

This voter-centric parameter is, however, not well suited to quantify consensus in the presence of parties. One may envisage a situation of no consensus in which voters form a tight cluster in opinion space due to the voter-voter force proportional to μ_{vv} but the parties evolve unaffected by the voters if their distance is larger than the respective interaction radii. This, admittedly, nonphysical situation would be classified as consensus with $Q_{vv} = 1$. A more common situation is the ordered state when voters are strongly attracted to a party by the party-voter force with strength μ_{pv}, forming a cluster around the party. This cluster forms even if the voter-voter force F_{vv} is weak due to a small interaction radius R_{vv}. The smallness of the interaction radius R_{vv} implies $Q_{vv} \ll 1$ misclassifying the ordered state as disordered. Depending on the parameter space and initial conditions, it is non-trivial to determine a single order parameter that may be



(A) t=0 (B) t=80 (C) t=120

(D) t=160 (E) t=300 (F) t=410

(G) t=450 (H) t=499

FIGURE 5. Evolution of the modified Hegselmann–Krause model (3)–(4) in a $d = 2$-dimensional opinion space for 500 voters (blue dots) and 5 parties (red crosses). At time $t = 0$ voters are distributed uniformly across $[0, 1]^2$ and parties are initially at $p_1(0) = (0.75, 0.75)$, $p_2(0) = (0.6, 0.8)$, $p_3(0) = (0.5, 0.7)$, $p_4(0) = (0.3, 0.4)$, and $p_1(0) = (0.2, 0.2)$. Parameters are $R_{vv} = 0.2$, $R_{pv} = 0.1$, $R_{vp} = 0.5$, $R_{pp} = 0.05$ and $\mu_{vv} = 0.5$, $\mu_{pv} = 1.0$, $\mu_{vp} = 0.05$, $\mu_{pp} = 0.02$. The noise strengths are $\sigma_v = 0.02$ and $\sigma_p = 0.002$.

used to quantify the order. Indeed in the scenarios outline Section 3, Q_{vv} can decrease in the party base or the swing voter scenarios, compared to a uniform distribution, even though there is clear clustering occurring.

To account for the presence of parties, we introduce an alternative consensus diagnostic which is given by the weighted average distance between, voter-voter pairs, voter-party pairs and party-party pairs, which measures a deviation from the disordered state in which voters and parties are uniformly distributed over the opinion space. In particular, we define

$$\hat{D} = 1 - \frac{k_d}{3}\left(\frac{1}{N_v^2}\sum_{v_i,v_j}||v_i - v_j|| + \frac{1}{N_v N_p}\sum_{v_i,p_\alpha}||v_i - p_\alpha|| + \frac{1}{N_p^2}\sum_{p_\alpha,p_\beta}||p_\alpha - p_\beta||\right), \tag{15}$$

where

$$k_d = 2^d \int_{\mathbf{0}}^{\mathbf{1/2}} \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}\, d\mathbf{x} \tag{16}$$

is the expected distance between two random draws from a uniform distribution on the hypercube $[0, 1]^d$ with periodic boundary conditions. The parameter $\hat{D}$ is not a strict order parameter in the sense of statistical mechanics but instead a diagnostics measuring in how far the distribution of voters and parties differs from the maximally disordered state of uniformly distributed voters. The proposed consensus diagnostic satisfies $0 \le \hat{D} \le 1$ with equality given only by uniformity and total consensus, respectively.

Figure 6a shows the evolution of Q_{vv} and $\hat{D}$ for the case of the 1-dimensional opinion space depicted in Figure 4. Near consensus is achieved at $t = 300$ with $Q_{vv} > \hat{D} \approx 0.9$. The consensus diagnostic $\hat{D}$ exhibits a monotonic increase. It initially grows slowly during the observed clustering of voters into party bases and then increases more strongly when clusters coalesce. The growth in $\hat{D}$ is fastest at $t \approx 100$ when two voter clusters merge into a single cluster. By contrast, Q_{vv} better captures the periods of coexisting clusters which are characterized by plateaus in Q_{vv}. The period when there are 4 distinct party base clusters for $10 \lessapprox t \lessapprox 30$ and the period $50 \lessapprox t \lessapprox 100$ when there are two party base clusters and one cluster of swing clusters are well captured by plateaus in Q_{vv}. The existence of nonentrained voters at $t = 300$ implies that both $\hat{D}$ and Q_{vv} are not equal to 1. The nonentrained voters will eventually diffuse into the main cluster.

Figure 6b shows the evolution of Q_{vv} and $\hat{D}$ corresponding to the dynamics in the 2-dimensional opinion space shown in Figure 5. The consensus diagnostic $\hat{D}$ exhibits a plateau for $120 \lessapprox t \lessapprox 380$ corresponding the state of four weakly interacting cluster (three party base clusters and one disaffected voter cluster). When these clusters begin to interact more strongly $\hat{D}$ increases monotonically. The order parameter Q_{vv} captures the ordered state of four weakly interacting clusters less clearly. However, it better captures the merger of two clusters (three parties) at $t \approx 420$ and a further merger of all party base clusters around $t \approx 450$. The slow increase of $\hat{D}$ for $t \gtrapprox 450$ is due to the slowed down merger of parties caused by their mutual repulsion.

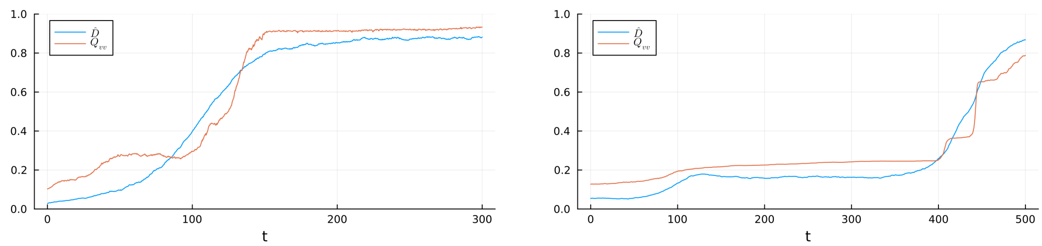


FIGURE 6. Order parameter Q_{vv} and consensus diagnostic $\hat{D}$ corresponding to the simulations in the 1-dimensional opinion space shown in Figure 4 (left) and in the 2-dimensional opinion space shown in Figure 5 (right).

A remarkable feature of the Hegselmann–Krause model (1) is the existence of a phase transition: there exists a critical noise strength σ_c such that for noise strength $\sigma > \sigma_c$ voters are not able to evolve towards consensus but instead diffuse as independent noisy agents [23, 25, 44, 18, 35, 26]. It is intuitive that the modified Hegselmann–Krause model (3)-(4) exhibits a similar phase transition. In Figure 4 we have seen an example when the system organizes in clusters before eventually reaching consensus with $Q_{vv} > 0.95$ around $t \approx 150$ when the voters collapse to a single cluster.

The remaining voters which have not yet been entrained by the cluster at $t = 200$ will eventually diffuse into the consensus cluster. If we keep all parameters the same but increase the noise strength from $\sigma_v = 0.002$ to $\sigma_v = 0.1$ final consensus cannot reached. As shown in Figure 7 the noise term dominates over the attractive interaction forces, and the system effectively behaves like a set of N_v independent Brownian motions. In this case voters do not form clusters and both $Q_{vv} \approx 2R_{vv} = 0.1$ and $\hat{D} \approx 0.028$ correspond to values when the voters are drawn from a uniform distribution. We note that with periodic boundary conditions a set of N_v independent Brownian processes are uniformly distributed.

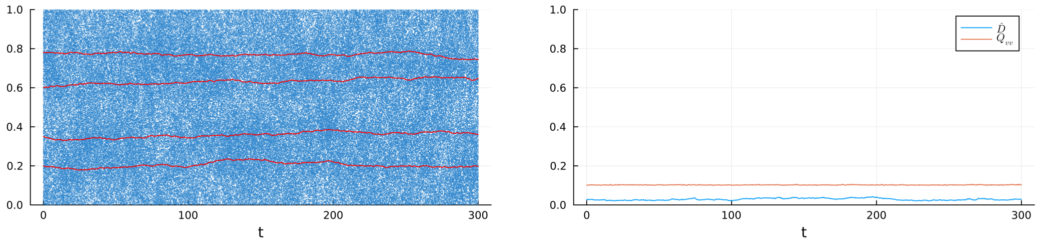


FIGURE 7. Evolution of the modified Hegselmann–Krause model (3)–(4) in a $d = 1$-dimensional opinion space for 500 voters and 4 parties for $\sigma_v = 0.1$ leading to a stable uniform distribution of voters. The initial conditions and the erst of the parameters is as in Figure 4. Left: Actual voter (blue dots) and party (red crosses) dynamics. Right: Corresponding evolution of the order parameter Q_{vv} and the consensus diagnostic $\hat{D}$.

In the following two Sections we investigate the phase transition of the modified Hegselmann–Krause model (3)–(4). In Section 5 we consider the noiseless case and provide a sufficient condition for the occurrence of consensus. In Section 6 we determine the conditions for the phase transition in the noisy case, employing the mean-field limit of the model.

5. Criterion for consensus in the noiseless modified Hegselmann–Krause model

We consider here the deterministic case with $\sigma_v = \sigma_p = 0$. In the deterministic case unanimous consensus is defined as the situation when all voters and all parties occupy the same position in opinion space. For notational convenience, we introduce the state of the system at time t as $\varphi(t) = (v(t), p(t))$ with the voter opinion profile $v(t) := (v_1(t), \dots, v_{N_v}(t)) \in \mathbb{R}^{dN_v}$ and the party opinion profile $p(t) = (p_1(t), \dots, p_{N_p}(t)) \in \mathbb{R}^{dN_p}$. If there is no repulsive force between the parties with $\mu_{pp} = 0$, global unanimous consensus occurs if all interaction radii are sufficiently large to cover the convex hull $\Omega(\varphi(0))$ of the initial distribution of voters and parties. In this case all agents, voters and parties, are mutually and attractively interacting. In the case that the smallest support of the forces does not cover the convex hull, consensus is not guaranteed and typically non-interacting clusters form.

If the repulsive party-party interaction force is included with $\mu_{pp} > 0$, more complex interactions are possible. Unanimous consensus can still occur provided the strength of the forces exerted by voters on parties dominates over the repulsion of the party-party interaction. This is formulated in the following Proposition which provides a sufficient condition for consensus.

Proposition 5.1. *The state $\varphi(t)$ approaches unanimous consensus if*

$$\mu_{pp} < \mu_{vp} \tag{17}$$

and if

$$R > E(0), \tag{18}$$

where $R = \min(R_{vv}, R_{vp}, R_{pv}, R_{pp})$ *is the smallest of all interaction radii and* $E(0)$ *is the initial value of*

$$E(t) := \Omega\left(\varphi(t) \cup \{\psi(t)\}\right) \tag{19}$$

which is the convex hull covering the voters and parties with

$$\psi(t) = \frac{\mu_{vp}\langle v(t)\rangle - \mu_{pp}\langle p(t)\rangle}{\mu_{vp} - \mu_{pp}}. \tag{20}$$

Here the angular brackets denote averages over voters and parties. Additionally, for $t_2 > t_1 \geq 0$, *we have*

$$E(t_2) \subset E(t_1).$$

Proof. To see that (17) and (18) are sufficient conditions to guarantee convergence, we present an adapted argument from [37] for the classical Hegselmann–Krause model. Note that all interaction forces are positive, in particular $\mu_{pp} > 0$. By construction, all attractive interactions are initially nonzero with

$$\phi\left(\frac{||v_i - v_j||}{R_{vv}}\right) = \phi\left(\frac{||v_j - p_\alpha||}{R_{vp}}\right) = \phi\left(\frac{||p_\alpha - v_i||}{R_{pv}}\right) = 1$$

for all $i, j \leq N_v$ and for all $\alpha \leq N_p$. Note that there is no condition on the interaction radius for the repulsive party-party interaction kernel. Hence, at $t = 0$ (3) may be written as,

$$\begin{aligned}
\frac{dv_i}{dt} &= \frac{\mu_{vv}}{N_v}\sum_j (v_j - v_i) + \frac{\mu_{pv}}{N_p}\sum_\beta (p_\beta - v_i)\\
&= \mu_{vv}\left(\langle v\rangle - v_i\right) + \mu_{pv}\left(\langle p\rangle - v_i\right)\\
&= (\mu_{vv} + \mu_{pv})\left(\frac{\mu_{vv}\langle v\rangle + \mu_{pv}\langle p\rangle}{\mu_{vv} + \mu_{pv}} - v_i\right).
\end{aligned} \tag{21}$$

Thus, v_i exponentially decays to the weighted voter and party average, where the weights are determined by the strengths of the forces acting on voters. It is clear that $\frac{\mu_{vv}\langle v\rangle + \mu_{pv}\langle p\rangle}{\mu_{vv}+\mu_{pv}} \in \Omega(\varphi(t)) \subseteq E(t)$. The evolution of the parties, (4), can be written for $t = 0$ as

$$\frac{dp_\alpha}{dt} = (\mu_{vp} - \mu_{pp})\left(\frac{\mu_{vp}\langle v\rangle - \mu_{pp}\langle p\rangle}{\mu_{vp} - \mu_{pp}} - p_\alpha\right). \tag{22}$$

Hence, if $\mu_{vp} > \mu_{pp}$, p_α decays exponentially to $\psi(t) = \frac{\mu_{vp}\langle v\rangle - \mu_{pp}\langle p\rangle}{\mu_{vp}-\mu_{pp}} \in E(t)$ by construction. Hence, $E(t)$ is a boundary on the party dynamics, as well as the voter dynamics for all $t > 0$. Next, observe that $\langle v(t)\rangle, \langle p(t)\rangle \in E(t)$. The evolution of the mean positions of the voters and parties are given by

$$\begin{aligned}
\frac{d\langle v\rangle}{dt} &= \mu_{pv}\left(\langle p\rangle - \langle v\rangle\right),\\
\frac{d\langle p\rangle}{dt} &= \mu_{vp}\left(\langle v\rangle - \langle p\rangle\right),
\end{aligned}$$

implying

$$\frac{d\left(\langle v\rangle - \langle p\rangle\right)}{dt} = -(\mu_{pv} + \mu_{vp})\left(\langle v\rangle - \langle p\rangle\right),$$

which is readily solved to yield

$$\langle p(t)\rangle = \langle v(t)\rangle + e^{-(\mu_{pv}+\mu_{vp})t}A,$$

where $A = \langle v(0)\rangle - \langle p(0)\rangle$ is constant. This implies that $\langle p(t)\rangle \to \langle v(t)\rangle$ for $t \to \infty$ with exponential rate of convergence $\mu_{pv} + \mu_{vp}$. As $\langle p(t)\rangle \to \langle v(t)\rangle$ this implies

$$\frac{\mu_{vp}\langle v(t)\rangle - \mu_{pp}\langle p(t)\rangle}{\mu_{vp} - \mu_{pp}} \to \langle v(t)\rangle \qquad \text{and} \qquad \frac{\mu_{vv}\langle v(t)\rangle + \mu_{pv}\langle p(t)\rangle}{\mu_{vv} + \mu_{pv}} \to \langle v(t)\rangle,$$

which implies according to (21) and (22) that the system approaches consensus, as desired.

Additionally, (21) and (22) imply that $\Omega(\varphi(t_2)) \subset E(t_1)$ for some $t_2 > t_1 \geq 0$. We may write $v(t)$ explicitly as

$$v(t) = \langle v(t)\rangle - \frac{\mu_{pp}}{\mu_{vp} - \mu_{pp}} e^{-(\mu_{pv}+\mu_{vp})t}A,$$

which implies that $v(t)$ monotonically converges to $\langle v(t)\rangle$ and hence we have,

$$E(t_2) \subset E(t_1).$$

□

The result of Proposition 5.1 may be interpreted in an intuitive way. For a given set of initial conditions with parameters that lead to consensus, increasing the repulsive force strength μ_{pp} or decreasing the interaction force strength μ_{pv} will lead to a lack of consensus. We illustrate the result in Figure 8 where we show how the two conditions (17) and (18) affect the formation of unanimous consensus. The top subfigure shows 100 voters uniformly distributed across $[0, 1]$ with 4 parties placed at $p_1(0) = 0.1$, $p_2(0) = 0.3$, $p_3(0) = 0.5$ and $p_4(0) = 0.7$. The parameters are chosen such that they conform with both conditions of the Proposition. We observe that voters and parties reach unanimous consensus, with the parties converging at a slower rate due to their repulsive forces $\mu_{pp} = 0.4$ being smaller than the attractive party-voter force with $\mu_{vp} = 0.5$. In the middle subfigure of Figure 8 we show the effect of breaking condition (17) in preventing convergence to unanimous consensus. The initial conditions and all parameters are kept the same as above except now $\mu_{pp} = \mu_{vp} = 0.5$, violating condition (17). Here voters still converge to a single cluster under their attractive forces since the interaction radii cover all the voters initially, i.e. condition (18) is satisfied. However, the parties do not converge but instead form two well-separated stationary clusters. Note that here two parties merge despite the repulsive party-party force. This is possible because the interaction radius encompasses the whole opinion space and hence a single party feels the effect of all parties. This can cause a party to be repelled by two other parties towards a fourth party. This is seen in the middle subfigure at times $t \approx 12$ and at $t \approx 25$. Furthermore, because of the heterophilic nature of the interaction, the force F_{vp} on party p_4 due to the voter cluster at $v = 0.5$ is larger than the force on party p_3 despite being closer to the voter cluster in opinion space. This further contributes to the merger of parties p_3 and p_4. This result may be seen as an extension of the result in [37] which showed that heterophilious interactions drive consensus in the classical Hegselmann–Krause model without parties. Finally, unanimous consensus can also be prohibited by reducing the interaction radii, i.e. violating the second condition (18). This is shown in the bottom subfigure of Figure 8. Here condition (17) is satisfied, however, the interaction radii are all 0.1 violating condition (18). This leads to the formation of distinct non-interacting clusters and a lack of unanimous consensus.

Note that the result of Proposition 5.1 is independent of the dimension d. Also note that in [37] it is shown that the classical Hegselmann–Krause model (1) satisfies $\Omega(t_2) \subset \Omega(t_1)$. This is not the case in our model because it is possible for parties near the boundary of the convex hull to be repelled beyond the boundary before being attracted back to consensus. $E(t)$ can be thought of as the area containing the convex hull with a buffer region by which the voters and parties are bounded. Indeed, (19) is in practice a weak condition for the emergence of consensus, and consensus can occur with smaller interaction radii R so long as the support of the strongest force covers a sufficiently large portion of the initial distribution. In an aside, we note that in the middle subfigure of Figure 8, $Q_{vv} = 1$ throughout the whole time period which is not representative of the behaviour of the system. Alternatively, $\hat{D}$ increases from 0 to 0.45 before stabilising and remaining there - which correctly detects that the parties have not converged to uniform consensus (not shown).

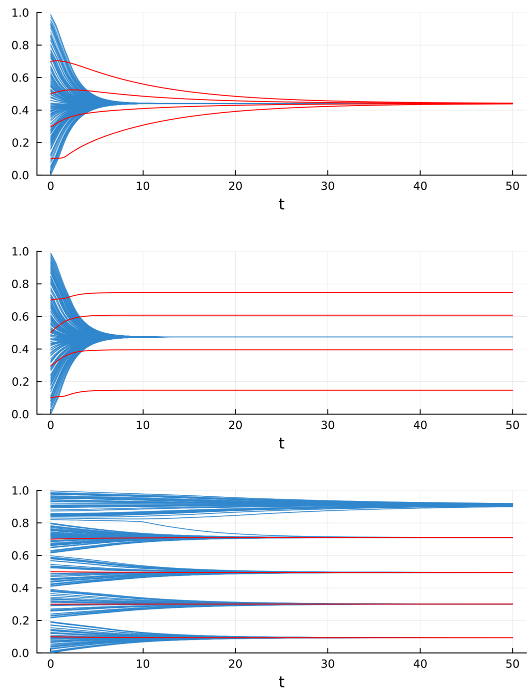


FIGURE 8. Evolution of the noise-less modified Hegselmann–Krause model (3)–(4) in a $d = 1$-dimensional opinion space with $\sigma_v = \sigma_p = 0$ for 100 voters (blue) and 4 parties (red). Initially voters are distributed uniformly across $[0, 1]$ and parties are initially at $p_1(0) = 0.1$, $p_2(0) = 0.3$, $p_3(0) = 0.5$ and $p_4(0) = 0.7$. Parameters are $\mu_{vv} = \mu_{pv} = 0.3$ and $\mu_{vp} = 0.5$. Top: unanimous consensus when both conditions (17) and (18) are satisfied with $\mu_{pp} = 0.4$ and $R_{vv} = R_{pv} = R_{vp} = R_{pp} = 0.6$. Middle: no unanimous consensus when condition (17) is not satisfied with $\mu_{pp} = 0.5$ and condition (18) is satisfied with $R_{vv} = R_{pv} = R_{vp} = R_{pp} = 0.6$. Bottom: no unanimous consensus when condition (18) is not satisfied with $R_{vv} = R_{pv} = R_{vp} = R_{pp} = 0.1$, and condition (18) is satisfied with $\mu_{pp} = 0.4$.

6. THE MEAN FIELD LIMIT AND PHASE TRANSITIONS

To study the stability of the consensus state when noise is added and to study the phase transition discussed in Section!4, we consider the mean-field limit $N_v \to \infty$ and derive an equation for the voter density $\rho(v, t)$. For convenience we consider periodic boundary conditions with

$v_i = \mathrm{mod}(v_i, 1)$. The classical Hegselmann–Krause model (1), which does not contain interactions with political parties, allows for a straight forward derivation of the evolution equation for the voter density $\rho(v,t)$. The voter density is the limit for $N_v \to \infty$ of the empirical measure

$$\rho^{(N_v)}(t, dv) = \frac{1}{N_v} \sum_{j=1}^{N_v} \delta_{v_j(t)}(dv).$$

The derivation of the corresponding equation for the evolution of the limiting density $\rho(v,t)$ is achieved by the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, and hinges on the exchangeability of voters. For the modified Hegselmann–Krause model (3)-(4), which includes party dynamics, the thermodynamic limit $N_v \to \infty$ can still be taken, however, one typically only has a small number of political parties and hence the thermodynamic limit $N_p \to \infty$ is not physical.

To derive an equation for the voter density associated with our modified Hegselmann–Krause model we make the following assumption about the dynamics of the political parties p_α. It is reasonable to assume that voters are much more willing to change their opinion on certain issues than parties. Political parties have a higher inertia than voters and therefore change in opinion space more slowly than individual voters. Indeed, [8] showed that as of the 2022 Australian Federal election only 37% of voters have supported the same party at every election, which suggests that individual voters are quite willing to change their opinions. Whereas political parties attempt to reinforce the issues they are traditionally seen as strong in, as well as engage in ideological politics [2]. Hence we assume that the party dynamics is slow compared to the voter dynamics and we assume $\mu_{vp}, \mu_{pp} \ll \mu_{vv}, \mu_{pv}$ and $\sigma_p \ll \sigma_v$. This allows us to treat for some time $0 \le t < T$ the parties as constant parameters in the voter dynamics (3) leading to an evolution equation for voters with frozen parties with

$$dv_i = \mu_{vv} \sum_j \phi\left(\frac{||v_j - v_i||}{R_{vv}}\right)(v_j - v_i)\, dt + \mu_{pv} F_{pv}(v_i; p)\, dt + \sigma_v dW^i(t), \tag{23}$$

where $F_{pv}(v_i; p) = \sum_\beta \phi\Big(\frac{||p_\beta - v_i||}{R_{pv}}\Big)(p_\beta - v_i)$ is the effect of the parties on the voter, and $p = \begin{pmatrix} p_1 & \dots & p_{N_p} \end{pmatrix} \in \mathbb{R}^{dN_p}$ is frozen. The voter dynamics (23) allows for an application of the BBKKY hierarchy to derive the following equation for the one-voter density $\rho(v, t; p)$, conditioned on the party parameters p,

$$\partial_t \rho(v, t; p) = -\nabla \cdot \left[\rho(v, t; p)\left(\mu_{vv} K \star \rho + \mu_{pv} F_{pv}(v; p)\right)\right] + \frac{\sigma_v^2}{2} \Delta \rho(v, t; p) \tag{24}$$

with initial data

$$\rho(v, 0; p) = \rho_0(v; p)$$

and interaction kernel

$$K \star \rho = \int \phi\left(\frac{||w - v||}{R_{vv}}\right)(w - v)\rho(w) dw.$$

To study the phase transition from global consensus to a near-uniform distribution of voters upon increasing the noise strength σ_v we follow the pipeline used by Garnier et al. [25] and Wang et al. [44] for the classical Hegselmann–Krause model (1).

Equation (24) for the stationary-party Hegselmann–Krause model allows for the stationary uniform voter distribution $\rho_0(v; p) = 1$ for $v \in [0, 1]^d$ provided that the parties have a mean zero with $\sum_\beta p_\beta = 0$. For non-zero mean party configurations ρ_0 still may serve as a good approximation for sufficiently small values of μ_{pv}.

To identify the critical noise strength σ_c below which consensus occurs, we perform a linear stability analysis around the stationary uniform state. We consider an expansion in small μ_{pv}

$$\rho(v,t) = 1 + \mu_{pv}\rho_1(v,t) + \mathcal{O}(\mu_{pv}^2). \tag{25}$$

Substituting into (24) yields

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot [(\mu_{vv} K \star \rho_1 + \rho_1(v,t)\mu_{pv} F_{pv}(v;p))] + \frac{\sigma_v^2}{2}\Delta \rho_1(v,t), \tag{26}$$

where we used $K \star \rho_0 = 0$. We solve this linear equation for ρ_1 by a Fourier ansatz $\rho_1(v,t) = T(t)e^{2\pi i \mathrm{k}\cdot v}$ for $\mathrm{k} \in \mathbb{Z}^d$ with $k := ||\mathrm{k}||_2 \neq 0$ to ensure that ρ is a probability density. The convolution term can be simplified using

$$\begin{aligned} -\nabla \cdot (K \star e^{2\pi i \mathrm{k}\cdot v}) &= -\nabla \cdot \int_{||\mathrm{z}|| \le R_{vv}} \mathrm{z} e^{2\pi i \mathrm{k}\cdot(\mathrm{z}+v)} d\mathrm{z} \\ &= s R_{vv} e^{2\pi i \mathrm{k}\cdot v} \int_{||\mathrm{z}|| \le 1} z\left(\sin(sz) - i\cos(sz)\right) d\mathrm{z}, \end{aligned} \tag{27}$$

where we introduced the scalar $z = \frac{\mathrm{k}\cdot \mathrm{z}}{k}$ and the scaled wave number modulus $s = s(k) = 2\pi R_{vv} k$. The linearized equation (26) can then be written as

$$T'(t) = A(k)T(t), \tag{28}$$

where

$$\begin{aligned} A(k) = &-\mu_{pv}\left[\nabla \cdot F_{pv}(v;p) + 2\pi i \mathrm{k} \cdot F_{pv}(v;p)\right] - \frac{s^2}{2R_{vv}^2}\sigma_v^2 \\ &+ s R_{vv} \int_{||\mathrm{z}|| \le 1} z\left(\sin(sz) - i\cos(sz)\right) d\mathrm{z}. \end{aligned} \tag{29}$$

Note that since $A(k)$ depends on v the differential equation for T has non-constant coefficients. However, we can define the largest possible growth rate as $\gamma(k) := \max_v \mathrm{Re} A(v)$, and utilize that $|| - \nabla \cdot F_{pv}(v;p)|| \le d$ with equality when voters are effected by all N_p parties at all times. We now set out to determine $\gamma(k)$, which will allows us to determine the critical noise strength σ_c as the critical noise strength for which $\gamma(k) = 0$ for all k. For convenience we will estimate the growth rate in the rescaled variable s and compute $\gamma(s)$. We first present results for a 1-dimensional opinion space and then for higher-dimensional opinion spaces.

6.1. **One-dimensional case $d = 1$.** For $d = 1$ we have that $z = \mathrm{z}$ and the real-part of the convolution term can be explicitly calculated as

$$s R_{vv} \int_{-1}^{1} z \sin(sz) dz = 2R_{vv}\left(\frac{\sin(s)}{s} - \cos(s)\right). \tag{30}$$

This yields for the maximal growth rate

$$\gamma(s) = \mu_{pv} - \frac{s^2}{2R_{vv}^2}\sigma_v^2 + 2\mu_{vv}R_{vv}\left(\frac{\sin(s)}{s} - \cos(s)\right). \tag{31}$$

Figure 9 shows the growth rate $\gamma(s)$ for various values of the noise strength σ_v. It is seen that for small values of σ_v the growth rate $\gamma(s)$ is positive for a range of values of s. For larger values of s, the growth rate can again increase obtaining positive values (not shown).

Since the smallest wave number is $k = 1$, we have $s \ge 2\pi R_{vv}$. Hence the uniform state is linearly unstable if for any $s \ge 2\pi R_{vv}$ the growth rate is positive. To determine the critical σ_c for

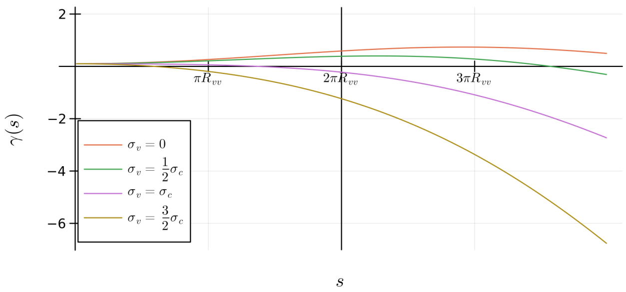


FIGURE 9. Growth rate $\gamma(s)$ for different values of the noise strength σ_v. The vertical black line denotes the smallest occurring wave number $s_{\min} = 2\pi R_{vv}$ for $k = 1$. The requirement for linear stability is that $\gamma(s) < 0$ for all $s \geq_{\min}.R_{vv} = 0.3, \mu_{vv} = 1, \mu_{pv} = 0.1$ which corresponds to $\sigma_c = 0.2$.

which the uniform state becomes unstable we assume that $R_{vv} \ll 1$. A Taylor expansion of the growth rate yields

$$\gamma(s) = \mu_{pv} + s^2 \left(\frac{2}{3}\mu_{vv}R_{vv} - \frac{\sigma_v^2}{2R_{vv}^2} \right).$$

Noticing $s \geq 2\pi R_{vv}$, finally leads to

$$\sigma_c^2 = \frac{\mu_{pv}}{2\pi^2} + \mu_{vv}\frac{4}{3}R_{vv}^3. \tag{32}$$

Note that for the standard noisy Hegselmann–Krause model (1) with $\mu_{pv} = 0$ and $\mu_{vv} = 1$, we recover the known critical noise strength $\sigma_c^2 = \frac{4}{3}R_{vv}^3$ [44, 25]. When $\mu_{pv} > 0$, the inclusion of party dynamics shifts the phase transition to higher values of the noise strength. This can be understood heuristically as parties provide additional stability to a large group of like-minded voters. We remark that the stationary party model (23) is unable to recover unanimous consensus. However, there is a clear phase transition, as we will see below, from an ordered state of party base clusters or revolution clusters to a disordered state of uniformly distributed voters.

Figure 10 shows a phase diagram obtained from a long simulation of the stationary-party model (23) in (σ_v, R_{vv})-space. We consider $2,000$ initially uniformly on $[0,1]$ distributed voters and stationary parties, $p_1 = 0.86, p_2 = 0.53$ and $p_3 = 0.2$. We simulated until time $t = 500$ with $\Delta t = 0.1$. We tested for statistical equilibrium of the voter dynamics using an Augmented Dickey-Fuller (ADF) test, testing for stationarity over the last 20% of the simulation. A phase transition is clearly seen, quantified by the consensus diagnostic $\hat{D}$. To best visualize the phase transition and the departure from uniformity, which for $2,000$ voters uniformly sampled across $[0,1]$ yields $\hat{D} = 0.037$, we employ a colour map with the colour transition occurring at 0.039. The phase transition is well approximated by our approximation (32) for small values of $R_{vv} \lessapprox 0.15$, consistent with the approximation of $R_{vv} \ll 1$ we made to derive (32). The orange region of high level consensus in the phase diagram Figure 10 occurring for $R_{vv} > 0.12$ and $\sigma > 0.03$ is reached via a transition from a state of three party base clusters to a single swing voter cluster of smaller size akin to the revolution scenario discussed in Section 3 (not shown).

Figure 11 shows the corresponding phase diagram in (μ_{pv}, σ_v)-space. The critical noise strength σ_c exhibits a $\sqrt{\mu_p v}$ dependency, consistent with our approximation (32), illustrating the effect of the parties on the phase transition.

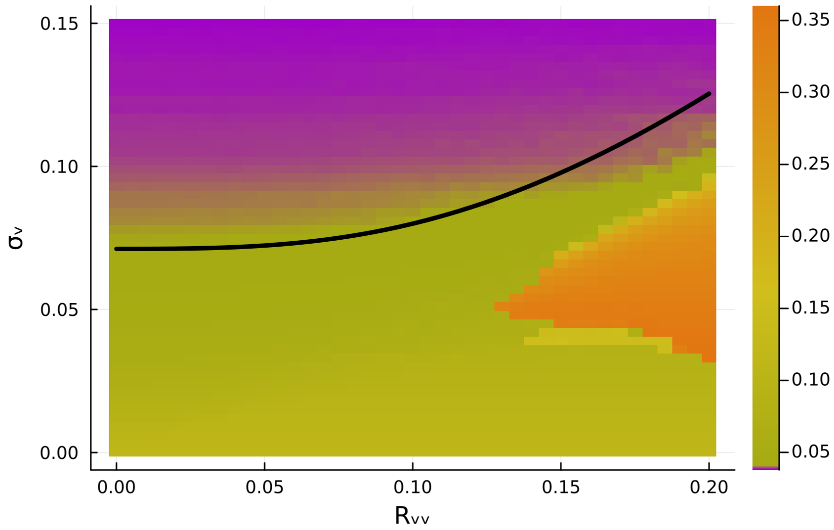


FIGURE 10. Phase diagram for the stationary-party-model (23) for 2000 voters randomly distributed across $[0,1]$ and with three parties, $p_1 = 0.86, p_2 = 0.53$ and $p_3 = 0.2$. Parameters are $\mu_{pv} = 0.1$, $\mu_{vv} = 1$ and $R_{pv} = 0.5$ with R_{vv} varying from 0 to 0.2. Colours show the value of the consensus diagnostic $\hat{D}$, averaged from $t = 400$ to $t = 500$. The black line shows the analytical critical curve (32). Eleven out of 2091 simulations did not pass the ADF.

6.2. **Higher dimensional cases.** For $d \geq 2$, the maximal growth rate can be found from (29) as

$$\gamma(s) = d\mu_{pv} + sR_{vv} \int_{||\mathrm{z}|| \leq 1} z \sin(sz) d\mathrm{z} - \frac{s^2}{2R_{vv}^2}\sigma_v^2, \tag{33}$$

where we assumed again that $R_{pv} > 1/2$. Recall $z = \hat{\mathrm{k}} \cdot \mathrm{z}$ with unit vector $\hat{\mathrm{k}} = \mathrm{k}/k$. For $s \ll 1$ we Taylor expand the integral term to obtain

$$\begin{aligned} sR_{vv} \int_{||z'|| \leq 1} z'' \sin(sz'') dz' &\approx \frac{R_{vv}}{d} s^2 \int_{||z'|| \leq 1} ||z'||^2 dz' \\ &= \frac{R_{vv}}{d} s^2 \frac{1}{d+2} S_{d-1}, \end{aligned} \tag{34}$$

where $S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the d-dimensional unit sphere with $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$. Hence, for $d \geq 2$, we obtain

$$\gamma(s) = d\mu_{pv} + \frac{2\pi^{d/2}}{d(d+2)\Gamma(d/2)} \mu_{vv} R_{vv} s^2 - \frac{\sigma_v^2}{2R_{vv}^2} s^2. \tag{35}$$

Applying a further approximation for $R_{vv} \ll 1$, we obtain the critical noise strength

$$\sigma_c^2 = d\frac{\mu_{pv}}{2\pi^2} + \frac{4\pi^{d/2}}{d(d+2)\Gamma(d/2)} \mu_{vv} R_{vv}^3. \tag{36}$$

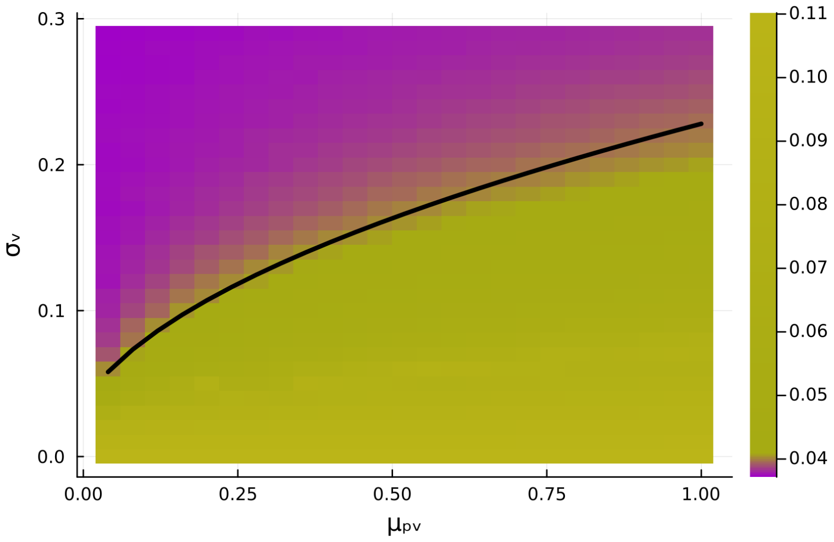


FIGURE 11. Phase diagram for the stationary-party-model (23) for 2000 voters randomly distributed across $[0, 1]$ and with three parties, $p_1 = 0.86, p_2 = 0.53$ and $p_3 = 0.2$. Parameters are $R_{vv} = 0.1$, $\mu_{vv} = 1$ and $R_{pv} = 0.5$ with μ_{pv} varying from 0 to 1. Colours show the value of the consensus diagnostic $\hat{D}$, averaged from $t = 400$ to $t = 500$. The black line shows the analytical critical curve (32). One out of 937 simulations did not pass the ADF.

For the classical noisy Hegselmann–Krause model (1) with $\mu_{vv} = 1$ and $\mu_{pv} = 0$, (36) reduces to the approximation obtained by Wang et al. [44]. Note that in the classical noisy Hegselmann–Krause model (1) the critical noise strength σ_c decreases as the dimension decreases. The inclusion of political parties, reflected in the linear contribution $d\mu_{pv}$, will be dominant for sufficiently large dimension d, consistent with our premise that parties have a stabilizing effect on the opinion dynamics of voters.

7. Discussion

We introduced and analyzed a modified Hegselmann–Krause model which describes the interactions of voters and parties in a d-dimensional opinion space. The model exhibits cluster formation and a phase transition from unstructured dynamics to unanimous consensus when all voters and parties collapse into the same region in opinion space. The model exhibits rich dynamical behaviour depending on the interaction radii of the voters and parties and the strength of the mutual interactions.

We established a sufficient condition for consensus in the deterministic version which states that consensus is guaranteed if the interaction radii are sufficiently large allowing for the interaction of all voters and parties and if the attractive forces dominate over the repulsive forces exerted by parties to delineate themselves from each other. We further employed mean-field theory to find the critical noise strength below which consensus occurs. Our analytical formula reflects a stabilizing effect of parties on consensus formation.

The proposed model recreates important and complex political dynamics such as party base clusters, swing voters, disaffected voters, revolutionary collapse and transitions between those states.

The model typically exhibits clusters of voters around individual parties. These clusters form what is known as the political base of a party. The political base represents an important feature of politics as it is a core group of voters who consistently support a specific political party [36]. The model further demonstrates the emergence of swing voters as a cluster of voters situated in opinion space between two parties. The parties then compete for the preference of the voters mediated by the repulsive force between them. The distances in opinion space between a voter from the swing cluster and each of the two parties is similar, so small changes in relative party positions can change which party is closest to a particular voter, and hence determines what party they would vote for. The model further supports clusters of disaffected voters that are not aligned with any political party. Such clusters of disaffected voters are generated when political parties evolve into regions in opinion space with large voter mass potentially leaving behind disaffected voters which do not experience any attracting force to the party if their distance is sufficiently large. This latter phenomenon is of increasing importance in modern political science [19]. More extreme political scenarios such as a sudden collapse of voters to a single party can be found in the modified Hegselmann–Krause model.

While the model assumes that the evolution of voter opinion is due to the relative positions in opinion space, the reality is far more complex. Media consumption and lack of information play an important role in contributing to voters' decisions [40], which is not covered by the model. Another limitation of the model is the assumption that all parties and all voters have the same interaction radii R_{pp} and R_{vv}, respectively, and exert the same force on the other agents. Political parties are clearly not all equal - it is conceivable that some parties exert stronger attractive forces on voters for example due to political charisma or effective advertising campaigns. Similarly, some voters may be more open to the thoughts and opinions of others so their interaction radii might be larger than those of other voters.

Data availability statement

All code and data used to produce the figures are available from the GitHub repository: `https://github.com/PatrickhCahill/ModifiedHegselmannKrauseModel`.

References

[1] J. Adams. Causes and electoral consequences of party policy shifts in multiparty elections: Theoretical results and empirical evidence. *Annual Review of Political Science*, 15(1):401–419, 2012.

[2] B. Ames. Electoral strategy under open-list proportional representation. In *Political Parties*, pages 2–29. Routledge, 2001.

[3] A. Arian, A. Krouwel, M. Pol, and R. Ventura. The election compass: party profiling and voter attitudes. In *The Elections in Israel 2009*, pages 275–298. Routledge, 2017.

[4] F. Ataş, A. Demirci, and C. Özemir. Bifurcation analysis of Friedkin-Johnsen and Hegselmann-Krause models with a nonlinear interaction potential. *Mathematics and Computers in Simulation*, 185:676–686, 2021. ISSN 0378-4754. doi: https://doi.org/10.1016/j.matcom.2021.01.012. URL `https://www.sciencedirect.com/science/article/pii/S0378475421000276`.

[5] R. Axelrod and W. D. Hamilton. The evolution of cooperation. *Science*, 211(4489):1390–1396, 1981. doi: 10.1126/science.7466396. URL `https://www.science.org/doi/abs/10.1126/science.7466396`.

[6] C. Bean and A. Mughan. Leadership effects in parliamentary elections in Australia and Britain. *The American political science review*, 83(4):1165–1179, 1989. ISSN 0003-0554.

[7] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis. On Krause's multi-agent consensus model with state-dependent connectivity. *IEEE Transactions on Automatic Control*, 54(11): 2586–2597, 2009. doi: 10.1109/TAC.2009.2031211.

[8] S. Cameron, I. McAllister, S. Jackman, and J. Sheppard. The 2022 Australian federal election: Results from the Australian election study, 2022. URL `https://australianelectionstudy.org/charts/The-2022-Australian-Federal-Election-Results-from-the-Australian-Election-Study.pdf`.

[9] J. A. Carrillo, K. Craig, and Y. Yao. Aggregation-diffusion equations: Dynamics, asymptotics, and singular limits. In N. Bellomo, P. Degond, and E. Tadmor, editors, *Active Particles, Volume 2: Advances in Theory, Models, and Applications*, pages 65–108, Cham, 2019. Springer International Publishing. doi: 10.1007/978-3-030-20297-2_3. URL `https://doi.org/10.1007/978-3-030-20297-2_3`.

[10] A. Carson, Y. Dufresne, and A. Martin. Wedge politics: Mapping voter attitudes to asylum seekers using large-scale data during the Australian 2013 federal election campaign. *Policy & Internet*, 8(4):478–498, 2016.

[11] C. Castellano, S. Fortunato, and V. Loreto. Statistical physics of social dynamics. *Rev. Mod. Phys.*, 81:591–646, 2009. doi: 10.1103/RevModPhys.81.591. URL `https://link.aps.org/doi/10.1103/RevModPhys.81.591`.

[12] B. Chazelle. An algorithmic approach to collective behavior. *Journal of Statistical Physics*, 158:514–548, 2015.

[13] B. Chazelle. Diffusive influence systems. *SIAM Journal on Computing*, 44(5):1403–1442, 2015.

[14] B. Chazelle and C. Wang. Inertial Hegselmann-Krause systems. In *2016 American Control Conference (ACC)*, pages 1936–1941, 2016. doi: 10.1109/ACC.2016.7525202.

[15] X. Chen, X. Zhang, Y. Xie, and W. Li. Opinion dynamics of social-similarity-based Hegselmann-Krause model. *Complexity*, 2017:1820257, 2017.

[16] G. W. Cox. 13 swing voters, core voters, and distributive politics. *Political representation*, 342, 2009.

[17] O. A. Davis, M. J. Hinich, and P. C. Ordeshook. An expository development of a mathematical model of the electoral process. *American political science review*, 64(2):426–448, 1970.

[18] M. G. Delgadino, R. S. Gvalani, G. A. Pavliotis, and S. A. Smith. Phase transitions, logarithmic Sobolev inequalities, and uniform-in-time propagation of chaos for weakly interacting diffusions. *Communications in Mathematical Physics*, pages 1–49, 2023.

[19] B. E. Pinkleton, Erica Weintraub Austin. Individual motivations, perceived media importance, and political disaffection. *Political Communication*, 18(3):321–334, 2001.

[20] D. Easley, J. Kleinberg, et al. *Networks, crowds, and markets: Reasoning about a highly connected world*, volume 1. Cambridge University Press Cambridge, 2010.

[21] A. Elias, E. Szöcsik, and C. I. Zuber. Position, selective emphasis and framing: How parties deal with a second dimension in competition. *Party Politics*, 21(6):839–850, 2015.

[22] F. Falck, J. Marstaller, N. Stoehr, S. Maucher, J. Ren, A. Thalhammer, A. Rettinger, and R. Studer. Measuring proximity between newspapers and political parties: the sentiment political compass. *Policy & internet*, 12(3):367–399, 2020.

[23] S. Fortunato. On the consensus threshold for the opinion dynamics of Krause-Hegselmann. *International Journal of Modern Physics C*, 16(02):259–270, 2005. doi: 10.1142/S0129183105007078. URL `https://doi.org/10.1142/S0129183105007078`.

[24] G. Fu, W. Zhang, and Z. Li. Opinion dynamics of modified Hegselmann-Krause model in

a group-based population with heterogeneous bounded confidence. *Physica A: Statistical Mechanics and its Applications*, 419:558–565, 2015. ISSN 0378-4371. doi: https://doi.org/10.1016/j.physa.2014.10.045. URL `https://www.sciencedirect.com/science/article/pii/S0378437114008838`.

[25] J. Garnier, G. Papanicolaou, and T.-W. Yang. Consensus convergence with stochastic effects. *Vietnam Journal of Mathematics*, 45:51–75, 2017.

[26] B. D. Goddard, B. Gooding, H. Short, and G. A. Pavliotis. Noisy bounded confidence models for opinion dynamics: the effect of boundary conditions on phase transitions. *IMA Journal of Applied Mathematics*, 87(1):80–110, 11 2021. ISSN 0272-4960. doi: 10.1093/imamat/hxab044. URL `https://doi.org/10.1093/imamat/hxab044`.

[27] W. Han, C. Huang, and J. Yang. Opinion clusters in a modified Hegselmann-Krause model with heterogeneous bounded confidences and stubborness. *Physica A: Statistical Mechanics and its Applications*, 531:121791, 2019. ISSN 0378-4371. doi: https://doi.org/10.1016/j.physa.2019.121791. URL `https://www.sciencedirect.com/science/article/pii/S0378437119310441`.

[28] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: Models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3):675–686, 2002.

[29] W. G. Jacoby. Ideology and vote choice in the 2004 election. *Electoral studies*, 28(4):584–594, 2009. ISSN 0261-3794.

[30] J. H. Kim and N. Schofield. *Spatial Model of U.S. Presidential Election in 2012*, pages 233–241. Springer International Publishing, Cham, 2016. ISBN 978-3-319-40118-8. doi: 10.1007/978-3-319-40118-8_10. URL `https://doi.org/10.1007/978-3-319-40118-8_10`.

[31] U. Krause. Soziale Dynamiken mit vielen Interakteuren. eine Problemskizze. *Modellierung und Simulation von Dynamiken mit vielen interagierenden Akteuren*, 3751(2), 1997.

[32] N. Lanchier and H.-L. Li. Consensus in the Hegselmann–Krause model. *Journal of Statistical Physics*, 187(3):20, 2022.

[33] M. Laver. Policy and the dynamics of political competition. *American Political Science Review*, 99(2):263–281, 2005.

[34] P. Liu and X. Chen. An overview on opinion spreading mode. *Journal of Applied Mathematics and Physics*, 3:449–454, 2015. doi: doi:10.4236/jamp.2015.34057.

[35] V. Lucarini, G. A. Pavliotis, and N. Zagli. Response theory and phase transitions for the thermodynamic limit of interacting identical systems. *Proceedings of the Royal Society A*, 476(2244):20200688, 2020.

[36] J. Miller. The 2020 US presidential race: Mobilize the base or persuade swing voters? 2020.

[37] S. Motsch and E. Tadmor. Heterophilious dynamics enhances consensus. *SIAM review*, 56 (4):577–621, 2014.

[38] A. Nugent, S. N. Gomes, and M. T. Wolfram. Steering opinion dynamics through control of social networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 34(7):073109, 07 2024. ISSN 1054-1500. doi: 10.1063/5.0211026. URL `https://doi.org/10.1063/5.0211026`.

[39] M. Pineda, R. Toral, and E. Hernández-García. Noisy continuous-opinion dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2009(8):P08001, 2009. doi: 10.1088/1742-5468/2009/08/P08001. URL `https://dx.doi.org/10.1088/1742-5468/2009/08/P08001`.

[40] M. Prior. Media and political polarization. *Annual review of political science*, 16(1):101–127, 2013.

[41] P. H. Ray. The new political compass. *Retrieved April*, 24:2010, 2002.

[42] S. Redner. Reality-inspired voter models: A mini-review. *Comptes Rendus Physique*, 20(4): 275–292, 2019.

[43] L. F. Stoetzer and S. Zittlau. Multidimensional spatial voting with non-separable preferences. *Political Analysis*, 23(3):415–428, 2015.

[44] C. Wang, Q. Li, W. E, and B. Chazelle. Noisy Hegselmann-Krause systems: Phase transition and the 2 R-conjecture. *Journal of Statistical Physics*, 166:1209–1225, 2017.

[45] M. R. Welch, D. C. Leege, K. D. Wald, and L. A. Kellstedt. Are the sheep hearing the shepherds? Cue perceptions, congregational responses, and political communication processes. In *Rediscovering the religious factor in American politics*, pages 235–254. Routledge, 2016.

[46] S. Wongkaew, M. Caponigro, and A. Borzì. On the control through leadership of the Hegselmann-Krause opinion formation model. *Mathematical Models and Methods in Applied Sciences*, 25(03):565–585, 2015. doi: 10.1142/S0218202515400060. URL `https://doi.org/10.1142/S0218202515400060`.

[47] N. Zagli, G. A. Pavliotis, V. Lucarini, and A. Alecio. Dimension reduction of noisy interacting systems. *Physical Review Research*, 5(1):013078, 2023.

[48] E. J. Zechmeister. *Sheep or shepherds? Voter behavior in new democratic contexts*. Duke University, 2003.

[49] M. Zhu and G. Xie. Leader's opinion priority bounded confidence model for network opinion evolution. *AIP Conference Proceedings*, 1864(1):020060, 08 2017. ISSN 0094-243X. doi: 10.1063/1.4992877. URL `https://doi.org/10.1063/1.4992877`.

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