Graded Nakajima Quiver Varieties in Type A

A conjecture on the SL_n representation afforded by a graded quiver variety

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Abstract

Nakajima quiver varieties give geometric realisations of representations and crystals of any symmetrisable Kac-Moody Lie algebra $\mathfrak{g}$. Graded quiver varieties are (for very generic choices of the grading) used to construct geometrically the tensor product of irreducible representations of $\mathfrak{g}$, however arbitrary choices of the grading have not been studied extensively. Arbitrary gradings arise in the context of truncated shifted Yangians, varieties quantising slices in the affine Grassmannian. For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, we conjecture which representation is afforded by an arbitrary grading, and prove this conjecture in a restricted set of gradings by an explicit bijection to the crystal of skew tableaux.

Nakajima Quiver Varieties

A smooth symplectic variety $\mathcal{M}(\lambda)$ defined by the data:

- $n \geq 2$, representing $\mathfrak{g} = \mathfrak{sl}_{n+1}$ with Dynkin quiver $Q = \ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$
- A dominant coweight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$

Example: $\lambda = (N, 0, 0, \ldots, 0) \implies \mathcal{M}(\lambda) \cong T^*(\mathcal{F}l_{n\text{-step}}(\mathbb{C}^N))$.

Theorem: The homology $H^\bullet(\mathcal{M}(\lambda))$ has a convolution product, and accepts a surjective map $U(\mathfrak{g}) \twoheadrightarrow H^\bullet(\mathcal{M}(\lambda))$, making the homology ring of $\mathcal{M}(\lambda)$ a quotient of the universal enveloping algebra of $\mathfrak{g}$ [Na1].

Graded Nakajima Quiver Varieties

A family of $\mathbb{C}^\times$-actions on $\mathcal{M}(\lambda)$ are parametrised by some *grading data* R:

- A collection of integers $R \in \mathbb{Z}^{\lambda_1} \times \cdots \times \mathbb{Z}^{\lambda_n}$.

The $\mathbb{C}^\times$-attracting sets define a Lagrangian subvariety $\mathcal{L}(\lambda, R) \subseteq \mathcal{M}(\lambda)$.

Theorem: The homology $H^{\text{top}}(\mathcal{L}(\lambda, R))$ is a $\mathfrak{g}$-module, and the $\mathfrak{g}$-crystal for this representation is given by the irreducible components of $\mathcal{L}(\lambda, R)$.

Example: $R = 0 \times \cdots \times 0 \implies H^{\text{top}}(\mathcal{L}(\lambda, R)) \cong L(\lambda)$, the irrep of $\mathfrak{g}$ with highest weight λ.

The trivial grading $R = 0 \times \cdots \times 0$ always gives an irreducible. Other choices of grading may be used to geometrically construct arbitrary tensor products of irreducibles [Na2].

Example: $\lambda = (3, 2, 2)$, $R = (M, M, N) \times (M, N) \times (N, N)$, for $|M - N| \gg 0$.
Then $H^{\text{top}}(\mathcal{L}(\lambda, R)) \cong L(2, 1, 0) \otimes L(1, 1, 2)$.

Question: What is the representation $H^{\text{top}}(\mathcal{L}(\lambda, R))$ for general R?

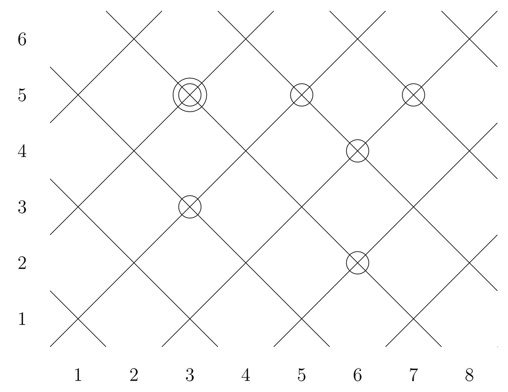
Product Monomial Crystal

An explicit combinatorial model for the crystal $\mathrm{Irr}(\mathcal{L}(\lambda, R))$ is given by the *product monomial crystal*, defined on the data n, λ, R [KTWWY]. The data defines a configuration of pegs on a lattice:

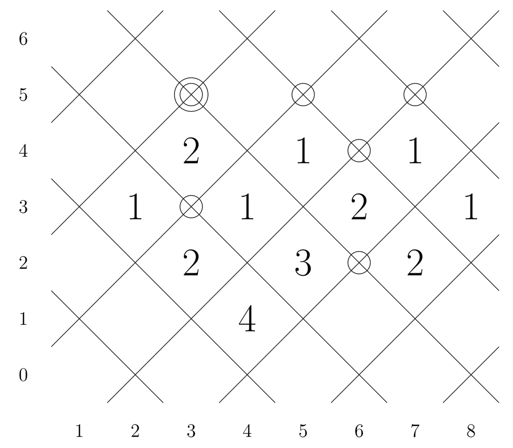
- To $\mathfrak{g} = \mathfrak{sl}_{n+1}$, associate a lattice $\mathbb{Z} \times \{1, \ldots, n\}$.
- $\lambda = (\lambda_1, \ldots, \lambda_n)$ specifies the number of pegs in each column.
- $R = R_1 \times \cdots \times R_n$ determines where to place the pegs.

Example: For the data

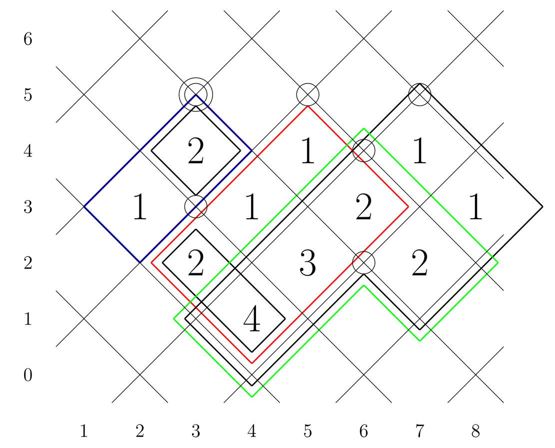
- $n = 8$, $\lambda = (0, 0, 3, 0, 1, 2, 1, 0)$.
- $R_3 = (1, 2, 2)$, $R_5 = (2)$, $R_6 = (1, 2)$, $R_7 = (2)$.



Elements of the crystal are *numberings* of the boxes inside the lattice, such that they count some covering by partitions hung from pegs.



Crystal element



A covering by partitions

In general, there may be many ways to cover a crystal element by partitions hung from pegs. Some key behaviours of this crystal:

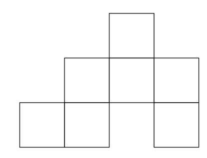
- Crystal operators f_i act either by 0, or by incrementing a box in column i.
- Far-apart collections of pegs do not interact $\implies$ tensor product behaviour ($|N - M| \gg 0$ from above).
- The crystal is always of a homogeneous degree $|\lambda| = \lambda_1 + \ldots + \lambda_n$ representation.

Conjecture: Generalised Schur Modules

Let $D \subseteq \mathbb{Z}^2$ be a finite set of boxes in the plane.

- D defines a Young symmetriser $y_D \in \mathbb{C}[S_{|\lambda|}]$.
- For $V = \mathbb{C}^{n+1}$, the *generalised Schur module* for D is the subspace $S_D := V^{\otimes|\lambda|} \cdot y_D$.

Example: S_D may also be described as the image of a composite map from a space alternating in the columns of D to a space symmetric in the columns of D, through the space $V^{\otimes|D|}$:



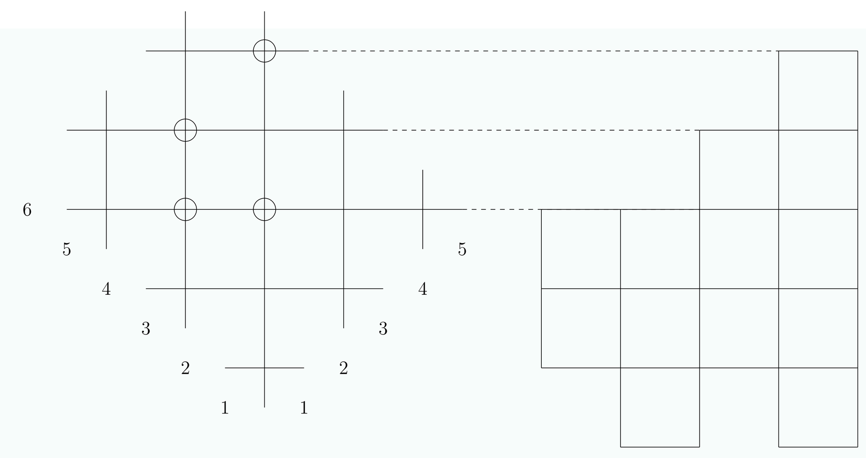
$$S_D = \mathrm{Im}\left[\ \bigwedge^1 V \otimes \bigwedge^2 V \otimes \bigwedge^2 V \otimes \bigwedge^2 V \hookrightarrow \cdots \twoheadrightarrow S^1V \otimes S^3V \otimes S^3V\ \right]$$

The modules S_D are not so well-known. Some key results are:

- Permuting rows/columns of D preserves isoclass of S_D.
- D is a skew shape $\implies$ S_D has decomposition according to the Littlewood-Richardson rule (S_D is irreducible for D a Young diagram).
- S_D may be constructed as the space of sections of a line bundle on a flag-like variety [Mag].

Building a diagram from a graded quiver variety: to the data n, λ, R, associate the diagram D:

- Rotate the peg diagram by 45°.
- For each peg over column i, draw a column of boxes of length i.
- Relative heights are given by positioning of pegs.



Grading data R — Diagram D

Conjecture: As $\mathfrak{sl}_{n+1}$-representations, $H^{\text{top}}(\mathcal{L}(\lambda, R)) \cong S_D$.

The conjecture would imply the first known combinatorial model for the crystal of S_D.

Results

Suppose there is a path through the lattice containing all pegs, moving strictly south-west or south-east at each step, then:

- D is a skew shape, so S_D has a crystal described by semistandard tableaux of shape D.
- There is a bijection of the elements of the product monomial crystal for R to the semistandard tableaux.
- It is expected that this bijection is also a crystal map, which would imply the conjecture for any R satisfying this path condition.

Forthcoming research

The proof in the case above relied on a combinatorial model for the crystal of S_D when D is a skew shape. For general D, a combinatorial model is not known, so a proof of the conjecture in general needs to take a different approach.

Say D has column lengths $c_1, \ldots, c_n$ and row lengths $r_1, \ldots, r_m$. Then S_D is the image of

$$\bigwedge^{c_1} V \otimes \cdots \otimes \bigwedge^{c_n} V \longhookrightarrow V^{\otimes|\lambda|} \xrightarrow{\sigma} V^{\otimes|\lambda|} \longtwoheadrightarrow S^{r_1}V \otimes \cdots \otimes S^{r_m}V$$

where $\sigma \in S_n$ rearranges a column ordering of D to a row ordering of D. The above maps can be categorified, using for example the Category $\mathcal{O}(\mathfrak{gl}_m)$ methods of Sartori-Stroppel [SS]. The image of the resulting functors categorifies S_D, and we hope that by studying the combinatorics of these functors, we can relate S_D to the combinatorics of the product monomial crystal.

References

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