Some questions on low-dimensional topology

The references in square brackets are to questions from 2KG and ACG4M, and results from FMGK.

1. Groups:

Show that if G is a finitely presentable group and either

(1) G is an ascending HNN extension $G \cong B*_\phi$ with base B finitely generated and 1-ended; or

(2) G has an elementary amenable normal subgroup E such that $h(E) > 2$ or $h(E) = 2$ and G/E is infinite or $h(E) = 1$ and G/E has one end

then $H^2(G; \mathbb{Z}[G]) = 0$.
(Applications to Theorems 8.1, 9.1, 15.13.) [2KG-17, AC-4]

Let π be a finitely presentable group with an FP_2 normal subgroup G of infinite index. Suppose that C_* is a finite free $\mathbb{Z}[\pi]$-complex with $H_0(C_*) \cong \mathbb{Z}$ and $H_1(C_*) = 0$, and that $C_*|_G$ is chain homotopy equivalent to a finite free $\mathbb{Z}[G]$-complex. Is $Hom_{\mathbb{Z}[\pi]}(H_2(C_*), \mathbb{Z}[\pi]) = 0$? (Yes if G is FP_3 – see Theorem 3.9.) [AC-5]

If ν is amenable and $c.d.\nu < \infty$ is it virtually solvable? [AC-12] What if ν has subexponential growth?

If π is almost coherent, restrained and $c.d.\pi = 2$ must π be solvable? (Yes if π/π' is infinite – see Theorem 2.6.)

If π is a PD_4-group is $def(\pi) \leq 0$? $\chi(\pi) \geq 0$? (See Theorem 3.6. Note that if π is orientable and $\chi(\pi) > 0$ then $def(\pi) \leq 0$.)

Let $G = F(r) \rtimes \mathbb{Z}$. If $\mathbb{Z}^2 \not\leq G$ does G satisfy the volume condition?

Are amenable groups good? Can one use the Følner criterion?

If G is finitely presentable and has a nontrivial finite subgroup F is $def(G) \leq 0$? (We may assume $F = \zeta G$.)

2. 2-complexes:

Let G be a group with weight and deficiency 1. Is $\beta_1^{(2)}(G) = 0$? (Implies finite case of the Whitehead Conjecture. See also [2KG-21].)

Show that if $f : M \to X$ is a degree-1 map from a closed surface M to a PD_2-complex X and $\chi(M) < \chi(X)$ there is a non-separating simple closed curve $\gamma : S^1 \to M$ with $[f \circ \gamma] = 1$. (This would lead to a "surgery" proof that PD_2-complexes are homotopy equivalent to closed surfaces.)

3. PD_3-complexes and groups:

Let X be a PD_3-complex and $f : M \to X$ a degree-1 map, where M is a closed 3-manifold. Show that if X is aspherical we may kill $\mathrm{Ker}(f_*)$ by (Dehn) surgery (and passage to finite covers?).

Let G be the group of an aspherical closed 3-manifold. Is $\mathbb{Z}[G]$ coherent? [AC-10]

Are there any indecomposable PD_3-complexes whose groups have a graph of groups structure with one edge $Z/6Z$?

Are PD_3-groups finitely presentable? coherent? virtually representable onto $\mathbb{Z}$? Do they have PD_2 subgroups?

Let G be an FP_2 group such that $H^3(G;\mathbb{Z}[G]) \cong \mathbb{Z}$. Is G virtually a PD_3-group?

See also "pdq.tex".

4. Homotopy invariants for closed 4-manifolds:

Strongly minimal, $c.d.\pi = 2$: e.g., $\pi = \langle x, t \mid tx^m t^{-1} = x^n \rangle$. [2KG-10, AC-13] (Some progress, by [JAH] and [H-K-T].)

If X is strongly minimal is $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$?

How many homotopy types are there of 4-manifolds with $\pi \cong (Z/2Z)^2$ and $\chi = 1$? [AC-8]

Are the 4-manifolds $T_{\leq 1}(RP^2) \cup MC$ and $T_{\leq 1}(RP^2) \cup MC'$ (with $\pi \cong Z/4Z$ and $\chi = 1$) homeomorphic? diffeomorphic?

If $\chi = 0$ and $\pi \cong (Z/nZ) \rtimes_d Z$ is $k_1(M) = k_1(L(n,d))$?
(Yes, if M *non*-orientable [D-W]. [AC-17]

S^1-bundles over PD_3-complexes. [AC-6]

Is there a PD_4-complex P such that $\pi_1(P) \cong F(r) \rtimes (Z/2Z)^2$ and $\pi_2(P) = 0$?

5. Geometries and surface bundles:

Are any closed $\mathbb{H}^4$-manifolds symplectic?

Which Seifert fibred 4-manifolds are symplectic?

Which mapping tori are symplectic?

If a symplectic 4-manifold is homotopy equivalent to the total space of a surface bundle is it such a bundle space? (Compare Theorem 13.7.)

Surface bundles over RP^2. [AC-8]

If G is a PD_4-group with $\chi(G) > 0$ is the number of normal subgroups K such that K and G/K are PD_2-groups bounded by $c^{\chi(G)}$ for some $c > 1$? (See Theorems 5.5, 7.3.)

Does every virtual surface bundle group have a characteristic PD_2-subgroup?

Geometric decompositions of (orbifold) bundle spaces. [some progress]

Which bundle groups are realized by algebraic surfaces? [AC-9]

If $\mathbb{S}ol_{m,n}$ has an arithmetic lattice must $m = n$?

Which mapping tori/aspherical surface bundles are Spin? [Stern]

Complete the determination of geometric S^2-fibrations over hyperbolic 2-orbifolds.

6. Classical knots and links:

Is every μ-component link L such that $\widehat{\pi L} \cong \widehat{F(\mu)}$ concordant to a sublink of an homology boundary link (cSHB)?

Is "SHB" $\equiv$ "concordant to an homology boundary link"?

If L is a boundary 1-link L such that $M(L) \cong \#^{\mu}(S^1 \times S^2)$ must it be trivial?

Is there a 1-knot K which is not a torus knot but for which $M(K)$ is Seifert fibred?

7. 2-knots and homology 4-spheres:

If K is π_1-slice is $def(\pi K) = 1$? Is $def(I(\pi K)) = 1$?
(Yes if $\pi K'$ is finitely generated.)

If π' is free is π the group of a (fibred) ribbon 2-knot? [2KG-13]

Must the centre of a 2-knot group π be finitely generated? [2KG-16]

Is every PD_4 2-knot group 1-connected at ∞? [2KG-11]

If a high dimensional knot group is a PD_4-group is it a 2-knot group?

Is there an 2-knot K such that $M(K)$ is an orbifold bundle with hyperbolic fibre?

What are the base 2-orbifolds of Seifert fibred knot manifolds $M(K)$? Do they have at most 4 cone points?

Give algebraic criteria for a 2-knot to be doubly slice. [2KG-24]

Fibered 2-knots and polynomial ISOLS $f : \mathbb{R}^5 \to \mathbb{R}^2$. [2KG-25]

Is there a surface link in S^4 whose exterior fibres over S^1 and which has at least one component of negative Euler characteristic?

If a finite group G is an homology 4-sphere group must it have deficiency 0? [2KG-30]

8. Other embeddings

Which $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds embed in $\mathbb{R}^4$?
(Conjecturally: the Seifert data must be skew-symmetric. Must there be no 2-torsion in the homology? Some progress by [JAH]. New work by Andrew Donald on smooth embeddings.)

Which $\widetilde{\mathbb{SL}}$-manifolds embed in $\mathbb{R}^4$? Does $M(0; (5,1), (5,2), (5,-2))$ embed? (Not smoothly: Budney, d-test).

Does every rational homology 3-sphere with hyperbolic linking form embed as a TOP locally flat submanifold of $\mathbb{R}^4$?

Unknotting surfaces.

Classify maps from 4-complexes to RP^2.

9. High dimensional knot theory:

Longitude, Poincaré duality and the EHP sequence for simple knots. How do the pairings ℓ and ψ determine the inclusion $\partial X \to X$?

Give intrinsic descriptions of ψ, Ψ and relate to Kearton's F-form pairing. (Assume fibred odd simple, if necessary.)

Does the Farber invariant extend additively to all $2q$-knots?

Algebraic criteria for double null concordance of simple $2q$-knots.

The classification of simple 4-knots. (Complete partial work of [HK].)

Intrinsic treatment of stable knots.

Reflexivity of 1-simple knots.

Are knot groups π with π' finitely presentable realized by fibred knots?

Relate spatial and chain-complex k-invariants.

10. Earlier lists:

2KG. 4, 8, 14, 19, [26] Yes. 10 In part. 5, 7, 18 No.
10, 11, 13, 16, 17, 21, 22, 24, 25, 30 given above (in modified form).

AC. 1, 2, 3, 7, 11, [13], 14, 15 Yes. 16 No. The rest given above (in modified form).

See also pdq.tex.