DIFFERENTIABILITY OF TRANSITION SEMIGROUP OF GENERALIZED ORNSTEIN-UHLENBECK PROCESS: A PROBABILISTIC APPROACH

BEN GOLDYS AND SZYMON PESZAT

ABSTRACT. Let $P_s\phi(x) = \mathbb{E}\,\phi(X^x(s))$, be the transition semigroup on the space $B_b(E)$ of bounded measurable functions on a Banach space E, of the Markov family defined by the linear equation with additive noise

$$\mathrm{d}X(s) = (AX(s) + a)\,\mathrm{d}s + B\mathrm{d}W(s), \qquad X(0) = x \in E.$$

We give a simple probabilistic proof of the fact that null-controllability of the corresponding deterministic system

$$\mathrm{d}Y(s) = (AY(s) + B\mathcal{U}(t)x)(s))\,\mathrm{d}s, \qquad Y(0) = x,$$

implies that for any $\phi \in B_b(E)$, $P_t\phi$ is infinitely many times Fréchet differentiable and that

$$D^n P_t\phi(x)[y_1, \ldots, y_n] = \mathbb{E}\,\phi(X^x(t))(-1)^n I_t^n(y_1, \ldots, y_n),$$

where $I_t^n(y_1, \ldots, y_n)$ is the symmetric n-fold Itô integral of the controls $\mathcal{U}(t)y_1, \ldots \mathcal{U}(t)y_n$.

CONTENTS

1

1. Introduction

We will consider the following linear stochastic differential equation

$$(1.1) \qquad \mathrm{d}X(s) = (AX(s) + a)\,\mathrm{d}s + B\mathrm{d}W(s), \qquad X(0) = x \in E\,,$$

where E is a Banach space and A generates a strongly continuous semigroup on E (see Section 2 for precise formulation). Under conditions specified in Section 2 this equation has, for every $x \in E$, a unique mild solution X^x known as a (generalised) Ornstein-Uhlenbeck process. Since the family $\{X^x;\ x \in E\}$ is Markov in E, we can define the corresponding transition semigroup $P_t\phi(x) = \mathbb{E}\,\phi(X^x(t))$. Investigation of differentiability of $P_t\phi$ was initiated by Kolmogorov in [16], where a simple version of equation (1.1) is considered in $\mathbb{R}^2$. If $E = \mathbb{R}^d$ and the pair (A, B) satisfies the Hörmander hipoellipticity condition (see [15]) then the function $P_t\phi$ is Fréchet differentiable for every bounded Borel function ϕ and

$$(1.2) \qquad \|DP_t\phi\|_\infty \le \frac{c}{t^{k/2}}\|\phi\|_\infty, \quad t \le 1\,,$$

where the value of k follows from the Hörmander condition as well.

It has been known for a long time that the Hörmander condition holds if and only if the deterministic controlled system

$$\frac{dY}{dt}(t) = AY(t) + Bu(t), \quad Y(0) = x$$

is null-controllable for every $x \in \mathbb{R}^d$ and if and only if the transition semigroup (P_t) is strongly Feller. It was proved in [7] that this formulation can be carried to Hilbert spaces, where the estimate (1.2) takes the form

$$(1.3) \qquad \||DP_t\phi\|_\infty \le \|\mathcal{U}(t)\|\|\phi\|_\infty,$$

with $\|\mathcal{U}(t)\|$ being the norm of the controllability operator.

In many problems it is important to estimate the gradient of the transition semigroup in function spaces such as weighted L^p spaces or

weighted Sobolev and Besov spaces, in which case (1.3) is not satisfactory. The aim of this paper is to derive a pointwise formula for $DP_t\phi$ that may be used to derive norm estimates in various function norms. Our main result, see Theorem 2.6, provides a probabilistic formula

$$(1.4) \qquad D^n P_t\phi(x)[y_1, \dots y_n] = \mathbb{E}\,\phi(X^x(t))(-1)^n I_t^n(y_1, \dots, y_n),$$

where $I_t^n(y_1, \dots, y_n)$ is the symmetric n-fold Itô integral of the controls $\mathcal{U}(t)y_1, \dots \mathcal{U}(t)y_n$ driving points $y_1, \dots, y_n$ in E to zero. This result is obtained under rather mild assumptions on (A, B). In particular, the process BW (hence the null control Bu) can take values in a Banach space bigger than E (see Section 2 for details) if the C_0-semigroup generated by A has appropriate smoothing properties. As a result, we obtain smoothing properties of P_t and the estimate (1.3) as well as the estimates for $\|D^n P_t\phi\|_\infty$. One of the main results of the paper are the formulae and estimates for $D^n P_t\phi$ in the case of boundary noise studied recently in [14]. Let us note that the case of boundary noise requires use of weighted L^p-spaces, where the pointwise formula (1.4) for the gradient of the transition semigroup has to be used.

We easily recover the known results (1.2), (1.3) in the hypoelliptic case in $\mathbb{R}^d$. Theorem 2.6 will also allow us to give rapid proofs of some known facts such as the Liouville property and absolute continuity of transition measures.

While some of our results are known, the method of proof based on the Girsanov theorem seems to be new. Let us recall that a probabilistic formula for the gradient of the transition semigroup is also provided by the famous Bismut–Elworthy–Li formula [12] extended later to Hilbert spaces in [19]. However, this formula requires the operator B in (1.1) to be invertible for $E = \mathbb{R}^d$ or the process BW close to cylindrical Wiener process if E is a Hilbert space. We do not require these assumptions.

Additionally, we will provide detailed arguments for the null controllability of equation (1.2) in the case of boundary control.

2. Formulation and main results

Let (W_k) be a finite or infinite sequence of independent standard real-valued Wiener processes defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Let $(E, \|\cdot\|_E)$ and $(V, \|\cdot\|_V)$ be Banach spaces. We assume that E is continuously and densely embedded into V. Let $(H, \langle\cdot,\cdot\rangle_H)$, be a Hilbert space endowed with an orthonormal basis (e_k). We will consider the following linear stochastic evolution equation with an additive noise

$$(2.1) \qquad \mathrm{d}X(s) = (AX(s) + a)\,\mathrm{d}s + B\mathrm{d}W(s), \qquad X(0) = x\,.$$

We assume that
$$W(s) = \sum_k W_k(s)e_k,$$
is a cylindrical Wiener process on H, B is a linear operator from H into V, $(A, \mathrm{Dom}(A))$ is the generator of a C_0-semigroup $S = (S(s); \, s \geq 0)$ on E, and $a \in V$.

Our first assumption ensures the existence of a solution to (2.1) in E.

Hypothesis 2.1. *For each $s > 0$, $S(s)$ has an extension to a bounded linear operator from V to E. Moreover, for each k and $s > 0$, the mapping*
$$[0, s) \ni r \mapsto S(s - r)Be_k$$
is integrable in E with respect to W_k, and the series
$$\sum_k \int_0^s S(s - r)Be_k \mathrm{d}W_k(r)$$
converges in probability in E. We also assume that for any $s > 0$ the integral $\int_0^s S(r)a\mathrm{d}r$ converges in E.

Remark 2.2. If E is a Hilbert space then the first part of Hypothesis 2.1 can be formulated equivalently: for each $r > 0$, $S(r)B$ is a Hilbert–Schmidt operator from H into E and
$$\int_0^s \|S(r)B\|_{L_{(HS)}(H,E)}^2 \mathrm{d}r < +\infty, \qquad s > 0,$$
where $\| \cdot \|_{L_{(HS)}(H,E)}^2$ stands for the Hilbert–Schmidt operator norm. Explicit conditions for stochastic integrability in L^p-spaces are also known, see for example Proposition A.1 from [3].

Clearly, under Hypothesis 2.1, equation (2.1) has a unique mild solution
$$X^x(s) = S(s)x + \int_0^s S(s - r)a\mathrm{d}r + \sum_k \int_0^t S(s - r)Be_k \mathrm{d}W_k(r).$$

In particular
$$X^0(s) = \int_0^t S(s - r)a\mathrm{d}r + \sum_k \int_0^t S(s - r)Be_k \mathrm{d}W_k(r)$$
is a solution starting from $x = 0 \in E$.

Moreover, (2.1) defines on the space $B_b(E)$ of bounded measurable functions on E, the transition semigroup $P_s\phi(x) = \mathbb{E}\,\phi(X^x(s))$, $\phi \in B_b(E)$, $s \geq 0$, $x \in E$.

2.1. **Some definitions.** Let $L^2(0, T; H)$ denote the Hilbert space of measurable square integrable mappings $u \colon [0, T] \mapsto H$.

Consider the following deterministic system

$$(2.2) \qquad \mathrm{d}Y(s) = (AY(s) + Bu(s)) \, \mathrm{d}s, \qquad Y(0) = x.$$

Definition 2.3. The deterministic system (2.2) is said to be *null-controllable at time T* if for any $t \geq T$ there exists a bounded linear operator $\mathcal{U}(t) \colon E \mapsto L^2(0, t; H)$, such that:

$$S(t)x + \sum_k \int_0^t S(t-s)Bu(t, x; s)\mathrm{d}s = 0,$$

where $u(t, x; s) = (\mathcal{U}(t)x)(s)$.

It is easy to see that the system (2.2) is null controllable if and only if the system

$$\mathrm{d}Y(s) = (AY(s) + a + Bu(s)) \, \mathrm{d}s, \qquad Y(0) = x$$

is null-controllable.

Connections between null-controllability and strong Feller property of linear stochastic evolution equations on Hilbert spaces with additive noise were establish in the following series of papers and books: Zabczyk [25], Da Prato and Zabczyk [10, 11], and Da Prato, Pritchard, and Zabczyk [8].

Definition 2.4. System (2.2) is *null-controllable with uniform vanishing energy*, if it is null controllable at some time T, and the controllability operator $\mathcal{U}(t) \to 0$ in the operator norm as $t \to +\infty$. It is *null-controllable with vanishing energy* if $\mathcal{U}(t)x \to 0$ in the $L^2(0, t; H)$ norm as $t \to +\infty$ for any $x \in E$.

The connections between null controllability with vanishing energy and the Liouville type property was established in [18], [21], and [22], see also our Corollary 2.8.

Definition 2.5. Let (P_s) be a transition semigroup on $B_b(E)$. A function $\psi \colon E \mapsto \mathbb{R}$ is called (P_s)-*harmonic* if $P_s\psi = \psi$ for any $s \geq 0$. We say that (P_s) has the *Liouville type property* if any bounded (P_s)-harmonic function is constant.

2.2. n-**fold symmetric Itô's integral.** Assume that the deterministic system (2.2) is null-controllable at time T. Let us fix $t \geq T$ and a control operator $\mathcal{U}(t)$. Given y and a sequence (y_l) of elements of E,

and $s \in [0, t]$, write

$$I_s(y) = \int_0^s \langle u(t, y; r), \mathrm{d}W(r) \rangle_H,$$

(2.3)

$$J_s(y_l, y_j) = \int_0^s \langle u(t, y_l; r), u(t, y_j; r) \rangle_H \mathrm{d}r,$$

$$u(t, y; r) = (\mathcal{U}(t)y)(r).$$

For $n \geq 1$ we define n-folde symmetric integrals $I_s^n(y_1, \ldots, y_n)$ putting $I_s^1(y) = I_s(y)$ and

$$I_s^n(y_1, \ldots, y_n)$$
$$= \sum_{j=1}^n \int_0^s I_r^{n-1}(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots y_n) \langle u(t, y_j; r), \mathrm{d}W(r) \rangle_H.$$

2.3. **General result.** Given $u \in L^2(0, t; H)$ define

$$M_u(t) := \exp \left\{ \int_0^t \langle u(s), \mathrm{d}W(s) \rangle_H - \frac{1}{2} \int_0^t \|u(s)\|_H^2 \mathrm{d}s \right\}.$$

The following result is a generalization of Theorem 9.26 from [10]. We denote by D the operator of Fréchet derivative in space variable.

Theorem 2.6. *Assume that the deterministic system* (2.2) *is null-controllable at time T. Then for $\phi \in B_b(E)$, $t \geq T$ and $x \in E$ we have*

(2.4) $$P_t \phi(x) = \mathbb{E}\, \phi(X^0(t)) M_{-u(t,x,\cdot)}(t).$$

Moreover, for any n and $\phi \in B_b(E)$, $P_t \phi$ is n-times differentiable, and

(2.5) $$D^n P_t \phi(x)[y_1, \ldots, y_n] = \mathbb{E}\, \phi(X^x(t))(-1)^n I_t^n(y_1, \ldots, y_n).$$

Finally, we have

$$|D^n P_t \phi(x)[y_1, \ldots, y_n]| \leq \|\mathcal{U}(t)\|_{L(E, L^2(0,t;H))}^n \|\phi\|_{B_b(E)} \prod_{l=1}^n \|y_l\|_E.$$

Proof. Let $t \geq T$. By the Girsanov theorem

$$W^*(s) = W(s) - \int_0^s u(t, x; r) \mathrm{d}r, \qquad s \in [0, t],$$

is a Wiener process under the probability measure

$$\mathrm{d}\mathbb{P}^* = M_{u(t,x;\cdot)}(t) \mathrm{d}\mathbb{P} = \mathrm{e}^{I_t(x) - \frac{1}{2} J_t(x,x)}.$$

Then

$$\mathbb{E}\, \phi(X^x(t)) = \mathbb{E}^* \phi(X^x(t)) \left(M_{u(t,x;\cdot)}(t) \right)^{-1},$$

$$\left(M_{u(t,x;\cdot)}(t)\right)^{-1}$$

$$= \exp\left\{-\int_0^t \langle u(t,x;s), \mathrm{d}W(s)\rangle_H + \frac{1}{2}\int_0^t \|u(t,x;s)\|_H^2 \mathrm{d}s\right\}$$

$$= \exp\left\{-\int_0^t \langle u(t,x;s), \mathrm{d}W^*(s)\rangle_H - \frac{1}{2}\int_0^t \|u(t,x;s)\|_H^2 \mathrm{d}s\right\},$$

and from the null-controllability assumption

$$X^x(t) = \int_0^t S(t-s)a\mathrm{d}s + \int_0^t S(t-s)B\mathrm{d}W^*(s).$$

Hence we have (2.4), and moreover,

$$D^n P_t \phi(x)[y_1, \ldots, y_n] = \mathbb{E}\, \phi(X^0(t))D^n M_{-u(t,x,\cdot)}(t)[y_1, \ldots, y_n].$$

Under the measure $\mathrm{d}\mathbb{P}^* = M_{-u(t,x;\cdot)}(t)\mathrm{d}\mathbb{P}$ the process

$$W^*(s) = W(s) + \int_0^s u(t,x;r)\mathrm{d}r$$

is a Wiener process. Moreover,

$$X^0(t) = \int_0^t S(t-s)a\mathrm{d}s + \int_0^t S(t-s)B\left(\mathrm{d}W^*(s) - u(t,x;s)\mathrm{d}s\right)$$

$$= S(t)x + \int_0^t S(t-s)a\mathrm{d}s + \int_0^t S(t-s)B\mathrm{d}W^*(s).$$

Therefore

$$D^n P_t \phi(x)[y_1, \ldots, y_n] = \mathbb{E}\, \phi(X^x(t))Y(t, y_1, \ldots, y_n),$$

where $Y(t, y_1, \ldots, y_n)$ can be obtain from

$$\frac{D^n M_{-u(t,x,\cdot)}(t)[y_1, \ldots, y_n]}{M_{-u(t,x,\cdot)}(t)}$$

$$= \mathrm{e}^{I_t(x) + \frac{1}{2}J_t(x,x)} D^n \mathrm{e}^{-I_t(x) - \frac{1}{2}J_t(x,x)}[y_1, \ldots, y_n]$$

by replacing above $\mathrm{d}W(s)$ by $\mathrm{d}W(s) - u(t,x;s)\mathrm{d}s$, that is each $I_t(y)$ by $I_t(y) - J_t(y,x)$. In particular, we have

$$Y(t, y) = -\int_0^t \langle u(t,y;s), \mathrm{d}W(s) - u(t,x;s)\mathrm{d}s\rangle_H$$

$$- \int_0^t \langle u(t,x;s), u(t,y;s)\rangle_H \mathrm{d}s$$

$$= -\int_0^t \langle u(t,y;s), \mathrm{d}W(s)\rangle_H = -I_t(y),$$

and

$$Y(t, y_1, y_2) = I_t(y_1)I_t(y_2) - J_t(y_1, y_2).$$

In order to see that for any n, $Y(t, y_1, \ldots y_n) = (-1)^n I_t^n(y_1, \ldots, y_n)$, let us fix t, x, n, $y_1, \ldots, y_n$, and let us write

$$Z_n(s) := D^n M_{-u(t,x,\cdot)}(t)[y_1, \ldots, y_n],$$
$$Z_n^j(s) := D^{n-1} M_{-u(t,x,\cdot)}(t)[y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n].$$

Then

$$\mathrm{d}M_{-u(t,x,\cdot)}(s) = -M_{-u(t,x,\cdot)}(s) \langle u(t,x;s), \mathrm{d}W(s) \rangle_H,$$
$$\mathrm{d}Z_n(s) = -Z_n(s) \langle u(t,x;s), \mathrm{d}W(s) \rangle_H$$
$$- \sum_{j=1}^n Z_n^j(s) \langle u(t,y_j;s), \mathrm{d}W(s) \rangle_H,$$
$$\mathrm{d}\frac{1}{M_{-u(t,x,\cdot)}(t)} = \frac{1}{M_{-u(t,x,\cdot)}(t)} \left[\langle u(t,x;s), \mathrm{d}W(s) \rangle_H + \|u(t,x;s)\|_H^2 \right] \mathrm{d}s,$$

and consequently, for

$$V_n(s) := \frac{Z_n(s)}{M_{-u(t,x,\cdot)}(t)} \qquad \text{and} \qquad V_n^j(s) := \frac{Z_n^j(s)}{M_{-u(t,x,\cdot)}(t)}$$

we have

$$\mathrm{d}V_n(s) = \frac{\mathrm{d}Z_n(s)}{M_{-u(t,x,\cdot)}(t)} + Z_n(s)\mathrm{d}\frac{1}{M_{-u(t,x,\cdot)}(t)}$$
$$- V_n(s)\|u(t,x;s)\|_H^2 \mathrm{d}s - \sum_{j=1}^n V_n^j(s)\langle u(t,x;s), u(t,y_j;s) \rangle_H \mathrm{d}s$$
$$= -V_n(s) \langle u(t,x;s), \mathrm{d}W(s) \rangle_H - \sum_{j=1}^n V_n^j(s) \langle u(t,y_j;s), \mathrm{d}W(s) \rangle_H$$
$$+ V_n(s) \langle u(t,x;s), \mathrm{d}W(s) \rangle_H + V_n(s)\|u(t,x;s)\|_H^2 \mathrm{d}s$$
$$- V_n(s)\|u(t,x;s)\|_H^2 \mathrm{d}s - \sum_{j=1}^n V_n^j(s)\langle u(t,x;s), u(t,y_j;s) \rangle_H \mathrm{d}s$$
$$= -\sum_{j=1}^n V_n^j(s) \left[\langle u(t,y_j;s), \mathrm{d}W(s) \rangle_H + \langle u(t,x;s), u(t,y_j;s) \rangle_H \mathrm{d}s \right].$$

Now, let $Y_n(s)$ and $Y_n^j(s)$ be obtained from $V_n(s)$ and $V_n^j(s)$ by replacing $\mathrm{d}W(s)$ by $\mathrm{d}W(s) - u(t, x; s)\mathrm{d}s$. We have

$$\mathrm{d}Y_n(s) = -\sum_{j=1}^{n} Y_n^j(s) \langle u(t, y_j; s), \mathrm{d}W(s) \rangle_H,$$

as required. $\qquad\square$

Corollary 2.7. *If the deterministic system is null-controllable with vanishing energy then for any $\phi \in B_b(E)$, n, and $y_1, \ldots, y_n \in E$,*

$$\sup_{x \in E} |D^n P_t \phi(x)[y_1, \ldots, y_n]| \to 0 \quad \text{as } t \to +\infty.$$

Corollary 2.8. *If the deterministic system is null-controllable with vanishing energy then (P_t) has the Liouville property.*

Proof. Let ψ be a bounded harmonic function. Then $P_t\psi = \psi$ for any $t \geq T$. If the system is null-controllable with vanishing energy, then by Corollary 2.7, for all $x, y \in E$ we have $DP_t\psi(x)[y] \to 0$ as $t \to +\infty$. Since $P_t\psi = \psi$, we have $D\psi(x)[y] = 0$ for $x, y \in E$, and therefore ψ is constant. $\qquad\square$

Corollary 2.9. *Under assumptions of Theorem 2.6, for any $t \geq T$, the laws $\mathcal{L}(X^x(t))$, $x \in E$, are equivalent.*

Proof. Taking in (2.4), as ϕ the characteristic function of a Borel subset Γ of E, we obtain

$$\mathbb{P}\{X^x(t) \in \Gamma\} = \int_{\{X^0(t) \in \Gamma\}} M_{-u(t,x;\cdot)}(t)\mathrm{d}\mathbb{P}.$$

Therefore $\mathcal{L}(X^x(t))$ is absolutely continuous with respect to $\mathcal{L}(X^0(t))$ for any x. We can now repeat the arguments leading to (2.4). Namely,

$$X^x(t) = \int_0^t S(t-s)a\mathrm{d}s + \sum_k \int_0^t S(t-s)Be_k \left[\mathrm{d}W_k(s) - u_k(t, x; s)\mathrm{d}s\right]$$

$$= \int_0^t S(t-s)a\mathrm{d}s + \sum_k \int_0^t S(t-s)Be_k\mathrm{d}W_k^*(s),$$

where

$$W_k^*(s) = W(s) - \int_0^s u_k(t, -x; r)\mathrm{d}r$$

is a Wiener process with respect to $\mathrm{d}\mathbb{P}^* = M_{u(t,x;\cdot)}(t)\mathrm{d}\mathbb{P}$. Therefore

$$\int_{\{X^x(t) \in \Gamma\}} M_{u(t,x;\cdot)}(t)\mathrm{d}\mathbb{P} = \mathbb{P}^*\{X^x(t) \in \Gamma\} = \mathbb{P}\{X^0(t) \in \Gamma\},$$

hence $\mathcal{L}(X^0(t))$ is absolutely continuous with respect to $\mathcal{L}(X^x(t))$. $\qquad\square$

Corollary 2.10. *Hypothesis 2.1 and null-controllability at T of (2.2) imply that the operators $S(t)$, $t \geq T$, on E are radonifying, and in particular compact.*

Proof. By Corollary 2.9, the gaussian laws $\mathcal{L}(X^x(t))$, $x \in E$, are equivalent. Let $\mathcal{H}(t)$ be the Reproducing Kernel Hilbert Space of $\mathcal{L}(X^0(t))$. By, the Feldman–Hajek theorem, $S(t)x \in \mathcal{H}(t)$. Since the embedding $\mathcal{H}(t) \hookrightarrow E$ is radonifying, see e.g. [17], the result follows. $\square$

Example 2.11. In the case of the classical Liouville theorem for the Laplace operator Δ on $\mathbb{R}^d$, $A \equiv 0$, $B \equiv \frac{1}{2}I \in M(d \times d)$, and W is a standard Wiener process on $\mathbb{R}^d$. The corresponding deterministic system is of the form

$$\mathrm{d}Y(s) = \frac{1}{2}u(s)\mathrm{d}s, \qquad Y(0) = x.$$

Clearly, it is null-controllable with vanishing energy at any time T and $\mathcal{U}(t)x \equiv -\frac{2x}{t}$. Consequently, the classical Liouville theorem saying that any bounded harmonic function on $\mathbb{R}^d$ is constant follows.

Remark 2.12. A natural question is if in Corollary 2.8 one can replace the assumption of the boundedness of a harmonic function by its appropriate integrability. Clearly, under hypothesis of Theorem 2.6, for any $p, q > 1$ such that $1/p + 1/q = 1$, there exists a constant $c(q)$ depending only on q such that

$$|\nabla P_t \psi(x)[y]| = |\mathbb{E}\,\psi(X^x(t))I_t(y)| \leq (\mathbb{E}\,|\psi(X^x(t))|^p)^{1/p}\,(\mathbb{E}\,|I_t(y)|^q)^{1/q}$$

$$\leq c(q)\,(\mathbb{E}\,|\psi(X^x(t))|^p)^{1/p}\,\|\mathcal{U}(t)y\|_{L^2(0,t;E)}\,.$$

If ψ is harmonic, than in order to show that ψ is constant, or equivalently, that $\nabla\psi(x) \equiv 0$, we need to show that

$$(2.6) \qquad \lim_{t \to +\infty} \mathbb{E}\,|\psi(X^x(t))|^p\,\|\mathcal{U}(t)y\|_{L^2(0,t;E)} = 0.$$

If the deterministic system is null-controllable with vanishing energy, then $\|\mathcal{U}(t)y\|_{L^2(0,t;E)} \to 0$ as $t \to +\infty$. Therefore (2.6) holds if

$$(2.7) \qquad \sup_{t>0} \mathbb{E}\,|\psi(X^x(t))|^p < +\infty.$$

In the Wiener case, Lebesgue measure is invariant, but there are no finite invariant measures. Therefore

$$\mathbb{E}\,|\psi(X^x(t))|^p \to \int_{\mathbb{R}^d} |\psi(z)|^p\,\mathrm{d}z.$$

Consequently, any harmonic integrable function is constant. However, since integrable continuous functions are bounded this is not an extension of the classical property. On the other hand, if there exists a

unique finite invariant measure, say μ, that is if

$$\int_0^t S(t-s)B\mathrm{d}W(s)$$

converges weakly to μ as $t \to +\infty$, then the Liouville property can be stated as follows: if the deterministic system is null-controllable with vanishing energy than any harmonic function ψ such that $\psi \in L^p(E, \mu)$ for a certain $p > 1$ is constant.

Remark 2.13. If E is a Hilbert space, then, see e.g. [10, 11], there exists a finite invariant measure μ (say) for (2.1) if and only if

$$\int_0^{+\infty} \|S(t)B\|^2_{L_{(HS)}(H,E)}\mathrm{d}t < +\infty.$$

If there exists the spectral gap for the generator L of the transition semigroup (P_s) in $L^2(E, \mu)$ then the Liouville property holds for functions in $L^p(E, \mu)$ for every $p \in (1, +\infty)$, see [4, 5] for more explict conditions.

3. Gradient estimate implies null-controllability

Assume that for all $t \geq T$ and $x \in E$ there is a integrable $Y(t, x)\colon \Omega \mapsto E^*$ such that for any $\phi \in B_b(E)$ and any $z \in E$,

$$DP_t\phi(x)[z] = \mathbb{E}\,\phi(X^x(t))Y(t,x)[z].$$

Taking non-zero but constant ϕ we see that $\mathbb{E}\,Y(t,x)[z] = 0$.

Proposition 3.1. *Assume that $Y(t,x)[z]$ are square integrable. Then the deterministic system is null–controllable at time T. Moreover, for any $x \in E$, $\mathcal{U}(t)z = -\mathbb{E}\,V(t,x;\cdot)[z]$, $z \in E$, where V is such that*

$$\mathbb{E}\,(Y(t,x)[z]|\sigma(W(s)\colon s \leq t)) = \int_0^t \langle V(t,x;s)[z], \mathrm{d}W(s)\rangle_H.$$

Proof. In the first step note that the gradient formula holds for $\phi(x) = e[x]$ where e is an arbitrary element of E^*, that is

$$DP_t\,e[X^x(t)][z] = \mathbb{E}\,e[X^x(t)]Y(t,x)[z].$$

Let (e_k) be an orthonormal basis of H. Given $e \in E^*$ let $\phi(x) = e[x]$. We have

$$DP_t\phi(x)[z] = De\left[S(t)x\right][z] = e\left[S(t)z\right]$$

$$= \mathbb{E}\, e\left[S(t)x + \int_0^t S(t-s)B\mathrm{d}W(s)\right]\int_0^t \langle V(t,x;s)[z], \mathrm{d}W(s)\rangle_H$$

$$= \mathbb{E}\sum_k \int_0^t e\left[S(t-s)Be_k\right]\mathrm{d}W_k(s)\int_0^t \langle V(t,x;s)[z], e_k\rangle_H \mathrm{d}W_k(s)$$

$$= \sum_k \int_0^t e\left[S(t-s)Be_k\right]\langle \mathbb{E}\, V(t,x;s)[z], e_k\rangle_H \mathrm{d}s.$$

Set $v(t,x;s)[z] = \mathbb{E}\, V(t,x;s)[z]$. Then

$$e\left[S(t)z\right] = \sum_k \int_0^t e\left[S(t-s)Be_k\right]\langle v(t,x;s)[z], e_k\rangle_H \mathrm{d}s.$$

Since the above identity holds for any $e \in E^*$, and

$$v(t,x;s)[z] = \sum_k \langle v(t,x;s)[z], e_k\rangle_H e_k,$$

we have

$$S(t)z = \sum_k \int_0^t S(t-s)Be_k\langle v(t,x;s)[z], e_k\rangle_H \mathrm{d}s$$

$$= \int_0^t S(t-s)Bv(t,x;s)[z]\mathrm{d}s.$$

$\square$

4. Conditions for null-controllability

Assume that E is reflexive, which is true in the case of L^p spaces. Let

$$\mathcal{Q}(t)\colon L^2(0,t;H) \ni u \mapsto \sum_k \int_0^t S(t-s)Be_k\langle u(s), e_k\rangle_H \mathrm{d}s \in E.$$

Then the deterministic system (2.2) is null-controllable at time T if and only if range $S(t) \subset$ range $\mathcal{Q}(t)$ for $t \geq T$.

A simple proof of the following useful observation is left to the reader.

Lemma 4.1. *For any $0 \leq t \leq t'$, range $\mathcal{Q}(t) \subset$ range $\mathcal{Q}(t')$. Consequently, if for given $T_0 \geq 0$,*

$$(4.1) \qquad range\, S(t+T_0) \subset range\, \mathcal{Q}(t) \quad for\ t > 0,$$

then for any $T > T_0$, the deterministic system (2.2) is null-controllable at time T.

The following lemma follows directly from more general result from [10, 11],

Lemma 4.2. *The following conditions are equivalent:*

$$\exists C: \quad \{S(t)x \colon \|x\|_E \leq 1\} \subset \{\mathcal{Q}(t)u \colon \|u\|_{L^2(0,t;H)} \leq C\},$$

and

(4.2) $\qquad \exists C: \quad \|S^*(t)z\|_{E^*} \leq C\|\mathcal{Q}(t)^*z\|_{L^2(0,t;H)} \quad for\ z \in E^*.$

Corollary 4.3. *System (2.2) is null-controllable at time T if and only if (4.2) holds true for $t \geq T$. Moreover, $\|\mathcal{U}(t)\|_{L(E,L^2(0,t;H))} \leq C$, where C is a constant in (4.2). If for $t > 0$,*

(4.3) $\qquad \exists C: \quad \|S^*(t+T_0)z\|_{E^*} \leq C\|\mathcal{Q}(t)^*z\|_{L^2(0,t;H)} \quad for\ z \in E^*,$

then (2.2) is null-controllable at any time $T > T_0$.

Note that

$$[\mathcal{Q}(t)^*z](s) = \sum_k \langle [S(t-s)B]^* z, e_k \rangle_H \, e_k.$$

Therefore (4.2) and (4.3) have the forms

(4.4) $\qquad \|S^*(t)z\|_{E^*}^2 \leq C^2 \sum_k \int_0^t \langle [S(s)B]^* z, e_k \rangle_H^2 \, \mathrm{d}s, \quad z \in E^*,$

and

(4.5) $\quad \|S^*(t+T_0)z\|_{E^*}^2 \leq C^2 \sum_k \int_0^t \langle [S(s)B]^* z, e_k \rangle_H^2 \, \mathrm{d}s, \quad z \in E^*,$

5. The case of invertible B

In this section we apply our results to a well understood case, when when $E = V = H$ is a Hilbert space and B is a bounded invertible linear operator on E, see [7], [10] and references therein. In this case Hypothesis 2.1 and the null-controllability of (2.2) can be easily verified. Then, see Remark 2.2, Hypothesis 2.1 holds if and only if

(5.1) $\qquad \int_0^t \|S(s)\|_{(HS)}^2 \mathrm{d}s < +\infty, \qquad \forall\, t > 0.$

Clearly the system (2.2) is null-controllable at any time $T > 0$, as one can takie

$$(\mathcal{U}(t)x)(s) = u(t,x;s) = -\frac{1}{t} B^{-1} S(s)x.$$

For, we have

$$\int_0^t S(t-s)B\frac{1}{t}B^{-1}S(s)x\mathrm{d}s = \frac{1}{t}\int_0^t S(t)x\mathrm{d}s = S(t)x.$$

Note that (2.2) is null-controllable with uniform vanishing energy if

$$(5.2) \qquad\qquad \sup_{t\geq 0}\|S(t)\|_{L(E,E)} < +\infty.$$

Finally, note that there is a finite invariant measure for (2.1) if S is exponentially stable (and in order to have a solution in E, (5.1) holds true).

Consequently, by Theorem 2.6 we have the following result. The theorem provides a formula for the gradient of the transition semigroup corresponding to linear equation. It is a special case of the Bismut–Elworthy–Li formula valid for non-linear diffusions, see e.g. [19].

Theorem 5.1. *Let $\phi \in B_b(E)$, $t > 0$ and $x \in E$. Then $P_t\phi$ has derivatives of all orders and for any $y, z \in E$,*

$$DP_t\phi(x)[y] = \mathbb{E}\left[\phi(X^x(t))\left(I_t(u)\right)\right],$$

$$D^2 P_t\phi(x)[y_1, y_2] = \mathbb{E}\left[\phi(X^x(t))\left(I_t(y_1)I_t(y_2) - J_t(y_1, y_2)\right)\right],$$

where

$$I_t(y) = -\frac{1}{t}\sum_k \int_0^t \langle B^{-1}S(s)y, e_k\rangle_E \mathrm{d}W_k(s),$$

$$J_t(y_1, y_2) = \frac{1}{t^2}\sum_k \int_0^t \langle B^{-1}S(s)y_1, B^{-1}S(s)y_2\rangle_E \mathrm{d}s.$$

Finally we have

$$\sup_{\phi\in B_b(E):\ \|\phi\|_{B_b(E)}\leq 1}\sup_{x\in E}\|D^n P_t\phi(x)\|_{L(E\times E\ldots\times E)} = O\left(t^{-\frac{n}{2}}\right) \qquad as\ t\downarrow 0.$$

Example 5.2. Consider the one-dimensional Ornstein–Uhlenbeck equation

$$\mathrm{d}X = (-\gamma X + a)\,\mathrm{d}t + b\mathrm{d}W,$$

where $\gamma > 0$, $a \in \mathbb{R}$, and $b \neq 0$. Then there is a unique invariant measure $\mu = \mathcal{N}(a/\gamma, b^2/(2\gamma))$. Moreover, the corresponding deterministic system is null-controllable with vanishing energy. Therefore, see Remark 2.12, the following Liouville property holds true: if $\psi \in C^2$ is such that

$$\frac{b^2}{2}\psi''(x) + (-\gamma x + a)\,\psi'(x) = 0, \qquad x \in \mathbb{R},$$

and for a certain $p > 1$,

$$\int_{\mathbb{R}} |\psi(x)|^p e^{-\gamma(x - \frac{a}{\gamma})^2/b^2} \mathrm{d}x < +\infty,$$

then ψ is constant.

5.1. **Analytic semigroup on a Hilbert space and degenerate noise.** Assume that E is a Hilbert space, the semigroup S is analytic, and λ is in the resolvent set of its generator $(A, \mathrm{Dom}\, A)$. Then, the fractional powers $(\lambda - A)^r$, $r \in \mathbb{R}$, are well defined. Given $r > 0$, let us equip

$$H^r := (\lambda - A)^{-\frac{r}{2}} E = \mathrm{Dom}\left((\lambda - A)^{\frac{r}{2}}\right),$$

with the scalar product and norm inherited form E by the mapping $(\lambda - A)^{-\frac{r}{2}}$.

Let $r > 0$ and let W be a cylindrical Wiener process in $H = H^r$, and let $B \colon H^r \to H^r$ be invertible. Consider the following linear equation

$$(5.3) \qquad \mathrm{d}X = (AX + a)\,\mathrm{d}t + \mathrm{d}W = (AX + a)\,\mathrm{d}t + \sum_k Be_k \mathrm{d}W_k$$

Given $t > 0$ and $x \in E$ write

$$(5.4) \qquad u(t, x; s) = \begin{cases} 0 & \text{for } s < \frac{t}{2}, \\ -\frac{2}{t} B^{-1} S(s)x & \text{for } s \geq \frac{t}{2}. \end{cases}$$

Theorem 5.3. *Assume that $a \in E$, and that for some or equivalently for any $t > 0$,*

$$(5.5) \qquad \int_0^t \left\| S(s)(\lambda - A)^{-\frac{r}{2}} \right\|_{L_{(HS)}(E,E)}^2 \mathrm{d}s < +\infty.$$

Then (5.3) defines Markov family $(X^x \colon x \in E)$ on E. Moreover, for the corresponding semigroup (P_t);

$$DP_t\phi(x)[y] = \frac{2}{t}\mathbb{E}\left\{ \phi(X^x(t)) \int_{\frac{t}{2}}^t \langle B^{-1}S(s)y, \mathrm{d}W(s)\rangle_{H^r} \right\}$$

and there is a constant C such that for all $y \in E$ and $t \leq 1$,

$$(5.6) \qquad \mathbb{E}\left| \int_{\frac{t}{2}}^t \langle S(s)y, \mathrm{d}W(s)\rangle_{H^r} \right| = \mathbb{E}\left| \sum_k \int_{\frac{t}{2}}^t \langle B^{-1}S(s)y, e_k\rangle_{H^r} \mathrm{d}W_k(s) \right|$$

$$\leq C\|y\|_E t^{\frac{1-r}{2}}.$$

Eventually,

$$|DP_t\phi(x)[y]| \leq C' t^{-\frac{1+r}{2}} \|y\|_E \|\phi\|_{B_b(E)}.$$

Finally, (5.2) guarantees that the corresponding deterministic problem is null-controllable with uniform vanishing energy.

Proof. (5.5) is an if and only if condition for the existence of a solution X^x to (5.3) for some or equivalently any initial value $x \in E$. Therefore Hypothesis 2.1 is fulfilled.

We will show that the deterministic system (2.2) is null-controllable at any time. To do this, $t > 0$, and $x \in E$. Then

$$S(t)x = -\int_0^t S(t-s)Bu(t,x;s)\mathrm{d}s,$$

where $u(t,x;\cdot)$ is given by(5.4). We need to show that $u(t,x;\cdot) \in L^2(0,t;H^r)$. Since S is analytic $S(s)x = S(s-t/2)S(t/2)x \in H^r$ for $s \geq \frac{t}{2}$, and there is a constant C such that

$$\|S(s)x\|_{H^r} \leq Ct^{-\frac{r}{2}}\|x\|_E, \qquad \frac{t}{2} \leq s \leq t \leq 1.$$

In the same way we show (5.6). $\square$

6. EXAMPLES

Example 6.1. Consider the following stochastic heat equation on $\mathbb{R}$,

$$\mathrm{d}X = \Delta X\mathrm{d}t + \mathrm{d}W,$$

driven by space-time white noise W. Thus $H = L^2(\mathbb{R})$. Since

$$\int_0^t S(t-s)\mathrm{d}W(s)$$

is stationary in space, it cannot live in $L^2(\mathbb{R})$. It is well-defined, however in weighted space $L^2(\mathbb{R}, \rho(x)\mathrm{d}x)$, where $\rho(x) = (1+x^2)^{-1}$. Therefore in our case $V = E = L^2(\mathbb{R}, \rho(x)\mathrm{d}x)$, $H = L^2(\mathbb{R})$, and B is the embedding operator. It is known, see eg. [20], that Δ generates C_0-semigroup on $L^2(\mathbb{R}, \rho(x)\mathrm{d}x)$, and Hypothesis 2.1 holds true. Unfortunately, the null-controllability hypothesis is not satisfied. For, assume that the initial condition x is the constant 1 function. Then $S(t)x = x \notin L^2(\mathbb{R})$. On the other hand, for any $u \in L^2(0,t;L^2(\mathbb{R}))$,

$$\int_0^t S(t-s)u(s)\mathrm{d}s \in L^2(\mathbb{R}).$$

Example 6.2. Consider the infinite system of independent scalar linear equations

$$\mathrm{d}X_j = -\alpha_j X_j\mathrm{d}t + \sigma_j\mathrm{d}W_j,$$

where (α_j) and (σ_j) are sequences of strictly positive numbers. Assume that $\sup_j \sigma_j < +\infty$. Then $B\colon l^2 \to l^2$, $B(x_j) = (\sigma_j x_j)$ is a bounded

linear operator. Moreover, the diagonal operator $A(x_j) = (-\alpha_j x_j)$ generates C_0-semigroup S on l^2, and $S(s)(x_j) = (e^{-\alpha_j s} x_j)$. Note that

$$\sum_j \mathbb{E} \left[\int_0^t e^{-\alpha_j(t-s)} \sigma_j dW_j(s) \right]^2 = \sum_j \int_0^t e^{-2\alpha_j s} \sigma_j^2 ds.$$

Therefore Hypothesis 2.1 holds true in the case of $V = E = H = l^2$, provided that

$$\sum_j \frac{\sigma_j^2}{\alpha_j} < +\infty.$$

Let us check the null-controllability hypothesis. Given $x = (x_j) \in l^2$ and $t > 0$ we are looking for $u = (u_j) \in L^2(0,t;l^2)$ such that

$$e^{-\alpha_j t} x_j = -\sigma_j \int_0^t e^{-\alpha_j(t-s)} u_j(s) ds.$$

Clearly,

$$u_j(s) = e^{-\alpha_j s} \frac{-x_j}{t\sigma_j}$$

solves the equation. We need to verify whether $u = (u_j) \in L^2(0,t;l^2)$. We have

$$\sum_j \int_0^t u_j^2(s) ds = \sum_j \frac{x_j^2}{2t^2\sigma_j^2\alpha_j} \left[1 - e^{-2\alpha_j t} \right].$$

Therefore the null–controllability hypothesis holds true if $\inf_j \sigma_j^2 \alpha_j > 0$. Moreover, in this case the system is null-controllable with vanishing energy. For more information on such systems we refer the reader to [10, 11].

7. Finite dimensional degenerate linear system

Consider now a finite dimensional case $E = V = \mathbb{R}^d$ and $H = \mathbb{R}^m$. Then (2.1) has the form

$$dX = (AX + a)\, dt + B dW, \qquad X(0) = x \in \mathbb{R}^d,$$

where A and B are $d \times d$ and $d \times m$ matrices, W is a standard Wiener process in $\mathbb{R}^m$, and $a \in \mathbb{R}^d$. Obviously, Hypothesis 2.1 is satisfied. Moreover, the corresponding deterministic system is null-controllable if and only if

(7.1) $$\operatorname{rank} \left[B, AB, \ldots, A^{d-1}B \right] = d.$$

For the proof of the following result we refer the reader to the work of Seidman [24].

Theorem 7.1. *Assume* (7.1). *Let K be the minimal exponent such that*

$$\operatorname{rank}\left[B, AB, \ldots, A^K B\right] = d.$$

Then the deterministic system is null-controllable at any time $T > 0$, and

$$\|\mathcal{U}(t)\|_{L(\mathbb{R}^d, L^2(0,t;\mathbb{R}^m))} = O\left(t^{-(K+1/2)}\right) \qquad \text{as } t \to 0.$$

As a direct consequence of Theorems 2.6 and 7.1 we have the following result.

Theorem 7.2. *Assume* (5.1) *and let K be as in Theorem 5.1. Then for any $t > 0$ and $\phi \in B_b(\mathbb{R}^n)$, $P_t\phi$ has derivatives of all orders and*

$$\sup_{x \in \mathbb{R}^d} \|D^n P_t\phi(x)\|_{\mathbb{R}^{nd}} \leq \|\phi\|_{B_b(\mathbb{R}^d)} O\left(t^{-n(K+\frac{1}{2})}\right) \quad \text{as } t \to 0.$$

7.1. **Kolmogorov's diffusion.** Consider the so-called Kolmogorov diffusion introduced in [16]

$$dX_1 = X_2 dt, \qquad dX_2 = dW.$$

We have

$$dX = AX dt + B dW,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus

$$\operatorname{rank}\left[B, AB\right] = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2.$$

Therefore the rank condition holds with $K = 1$, and by Theorem 7.1, the deterministic system is null-controllable and the linear mapping $\mathcal{U}(t) \colon \mathbb{R}^2 \mapsto L^2(0,t;\mathbb{R})$ is such that

$$\|\mathcal{U}(t)\|_{L(\mathbb{R}^2; L^2(0,t;\mathbb{R}))} = O(t^{-3/2}) \qquad \text{as } t \downarrow 0.$$

By direct calculation on can show that

$$u(t, x; s) = a(t, x)s + b(t, x), \quad s \in [0, t],$$

$$a(t, x) = \frac{x_1}{12t^3} - \frac{35x_2}{18t^2},$$

$$b(t, x) = -\frac{1}{12}\left(\frac{x_1}{2t^2} + \frac{x_2}{3t}\right).$$

has the desired properties. Moreover, the system is null-controllable with vanishing energy as for the above controls

$$\|\mathcal{U}(t)\|_{L(\mathbb{R}^2, L^2(0,t;\mathbb{R}^2))} = O(t^{-3/2}) \qquad \text{as } t \to +\infty.$$

Remark 7.3. It is easy to check that in the case of Kolmogorov diffusion there are no finite invariant measures. Note that in the case of Kolmogorov diffusion, $\psi \in C^2(\mathbb{R}^2)$ is harmonic if

$$\frac{1}{2}\frac{\partial^2 \psi}{\partial x_2^2}(x_1, x_2) + x_2\frac{\partial \psi}{\partial x_1}(x_1, x_2) = 0, \qquad (x_1, x_2) \in \mathbb{R}^2.$$

8. The case of boundary noise

8.1. **Well posedness.** Let $\mathcal{O}$ be a bounded region in $\mathbb{R}^d$. We assume that $\mathcal{O}$ satisfies the conditions from [14]. In particular it is enough to assume that $d = 1$ and $\mathcal{O}$ is an interval, or $d > 1$ and $\mathcal{O}$ is a bounded region with boundary of class $C^{1+\alpha}$. Given $p \in (1, +\infty)$ and $\theta \in [0, 2p - 1)$ set $L_\theta^p := L^p(\mathcal{O}, \text{dist}\,(\xi, \partial\mathcal{O})^\theta\,\mathrm{d}\xi)$.

Let us denote by S the semigroup on $E = L_\theta^p$ generated by the Laplace operator Δ on $\mathcal{O}$ with homogeneous Dirichlet boundary conditions, see [14]. The semigroup is given by the Geeen kernel G;

$$S(t)x(\xi) = \int_{\mathcal{O}} G(t, \xi, \eta)x(\eta)\mathrm{d}\eta.$$

Given $\lambda > 0$ let D_λ be the Dirichlet map. Let us recall that given $\lambda \geq 0$ and a function γ on $\partial\mathcal{O}$, $u = D_\lambda\gamma$ is, the possibly weak, unique solution to the Poisson equation

$$\Delta u(\xi) = \lambda u(\xi), \quad \xi \in \mathcal{O}, \qquad u(\xi) = \gamma(\xi), \quad \xi \in \partial\mathcal{O}.$$

Let $H \hookrightarrow L^2(\partial\mathcal{O})$ be a Hilbert space, and let (e_k) be an orthonormal basis of H. Finally let (W_k) be a sequence of independent real valued standard Wiener processes defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

Let us recall, see [14], that the boundary problem

$$\frac{\partial X}{\partial s}(s, \xi) = \Delta X(s, \xi), \qquad s > 0,\ \xi \in \mathcal{O},$$

(8.1)
$$X(0, \xi) = x(\xi), \qquad \xi \in \mathcal{O},$$

$$X(s, \xi) = \sum_k e_k(\xi)\frac{\mathrm{d}W_k}{\mathrm{d}s}(s), \qquad s > 0,\ \xi \in \partial\mathcal{O},$$

can be written in form (2.1). Moreover, it is well posed on the state space L_θ^p if and only if, for a certain or equivalently for every $T > 0$,

$$\mathcal{J}_T(\{e_k\}, p, \theta)$$

(8.2)
$$= \int_{\mathcal{O}} \left[\sum_k \int_0^T ((\lambda - A)S(s)D_\lambda e_k)^2 (\xi)\mathrm{d}s\right]^{p/2} \text{dist}\,(\xi, \partial\mathcal{O})^\theta\,\mathrm{d}\xi$$

is finite. Here A is the Laplace operator with homogeneous Dirichlet boundary conditions. Moreover, it solution X^x is given by the formula

$$X^x(s,\cdot) = S(s)x(\cdot) + \sum_k \int_0^s (\lambda - A)S(s-r)D_\lambda e_k(\cdot)\mathrm{d}W_k(r)$$

$$= \int_{\mathcal{O}} G(s,\cdot,\eta)x(\eta)\mathrm{d}\eta$$

$$- \sum_k \int_0^s \int_{\partial\mathcal{O}} \frac{\partial}{\partial\mathbf{n}^a(\eta)} G(s-r,\cdot,\eta)e_k(\eta)\mathrm{d}\mathbf{s}(\eta)\mathrm{d}W_k(r),$$

where $\mathbf{n} = (\mathbf{n}^1, \ldots, \mathbf{n}^d)$ is the outward pointing unit normal vector to the boundary $\partial\mathcal{O}$, and s is the surface measure.

For more information on the deterministic and stochastic boundary problems see e.g. [2, 9, 13, 14].

Hypothesis 2.1 can be written now in the following form:

Hypothesis 8.1. *For any $T > 0$, $\mathcal{J}_T(\{e_k\}, p, \theta)$ given by (8.2) is finite.*

We define the transition semigroup on $B_b(E) = B_b(L_\theta^p)$; $P_s\phi(x) = \mathbb{E}\,\phi(X^x(s))$. In the case of the boundary noise problem the null-controllability criterion of the corresponding deterministic problem can be formulated as follows.

Hypothesis 8.2. *Assume that for each $t > 0$ there is a bounded linear operator $\mathcal{U}(t) \colon L_\theta^p \mapsto L^2(0, t; H)$, $(\mathcal{U}(t)x)(s) = u(t, x; s)$ such that for any $x \in L_\theta^p$,*

$$\int_{\mathcal{O}} G(t,\cdot,\eta)x(\eta)\mathrm{d}\eta$$

$$= \sum_k \int_0^t \int_{\partial\mathcal{O}} \frac{\partial}{\partial\mathbf{n}^a(\eta)} G(t-s,\cdot,\eta)e_k(\eta)\mathrm{d}\mathbf{s}(\eta)\langle u(t,x;s), e_k\rangle_H \mathrm{d}s.$$

Remark 8.3. Let us note that if $\mathcal{O}$ is unbounded then it is not possible that Hypotheses 8.1 and 8.2 hold simultaneously. For, by Corollary 2.10 the semigroup S need to be compact.

Remark 8.4. Note that

$$G(t,\xi,\eta) = \sum_l \mathrm{e}^{-\lambda_l t} f_l(\xi)f_l(\eta),$$

where (f_l) is the orthonormal basis of $L^2(\mathcal{O})$ of eigenvectors of the Laplace operator and $(-\lambda_l)$ are the corresponding eigenvalues. Therefore the null-controllability condition has the form

$$\langle x, f_l\rangle_{L^2(\mathcal{O})} = \sum_k \int_0^t \mathrm{e}^{\lambda_l s}\langle u(t,x;s), e_k\rangle_H \mathrm{d}s \int_{\partial\mathcal{O}} \frac{\partial f_l}{\partial\mathbf{n}^a(\eta)}(\eta)e_k(\eta)\mathrm{d}\mathbf{s}(\eta).$$

and it is enough to verify this identity for any $x \in L^2(\mathcal{O})$ and of course any l, see Lemmas 8.6 and 8.7 below.

8.2. **The case of white noise.** The following result is proved in [23]. Its formulation is adapted to notations of this paper.

Theorem 8.5. *Then there is a bounded linear control operator $\mathcal{U}(t)$ acting from $L^2(\mathcal{O})$ into $L^2(0, t; L^2(\partial\mathcal{O}))$, and*

$$(8.3) \qquad \log \|\mathcal{U}(t)\|_{L(L^2(\mathcal{O}), L^2(0,t;L^2(\partial\mathcal{O})))} = \mathcal{O}(1/t) \quad \text{as } t \downarrow 0.$$

The following result was established in [14][Lemma 5.1].

Lemma 8.6. *Let $p > 1$, $\theta \in [0, 2p - 1)$. For any $t > 0$, $S(t)$ is a bounded linear operator from L^p_θ into $L^p_0 = L^p(\mathcal{O})$. Moreover, the operator norm is of order $t^{-\frac{\theta}{2p}}$, that is there exists a $C > 0$ such that $\|S(t)\|_{L(L^p_\theta, L^p_0)} \leq C t^{-\frac{\theta}{2p}}$ for $t \leq 1$.*

Since $\mathcal{O}$ is bounded we have also the following result.

Lemma 8.7. *Let $p > 1$. Then for any $t > 0$, $S(t)$ is a bounded linear operator from $L^p_0 = L^p(\mathcal{O})$ into $L^2_0 = L^2(\mathcal{O})$. Moreover, there exists a $C > 0$ and $\alpha(p) \in [0, +\infty)$ such that $\|S(t)\|_{L(L^p_0, L^2_0)} \leq C t^{-\alpha(p)}$ for $t \leq 1$.*

The main result of this section is the following theorem

Theorem 8.8. *(i) Assume that $\mathcal{O} = (a, b)$ is a bounded open interval in $\mathbb{R}$, and that W_1, W_2 are independent Wiener processes. Let $p > 1$ and $\theta \in (p-1, 2p-1)$. Then the problem (8.1) with boundary conditions $X(t, a) = \dot{W}_1(t)$ and $X(t, b) = \dot{W}_2(t)$ defines a Markov family on $E = L^p_\theta$.*
(ii) Assume that $\mathcal{O}$ is a bounded region in $\mathbb{R}^2$ with boundary $C^{1,\alpha}$ for some $\alpha > 0$. Let (e_k) be an orhonormal basis of $L^2(\partial\mathcal{O})$ and let (W_k) be a sequence of independent Wiener processes. Let $p > 1$ and $\theta \in \left(\frac{3p}{2} - 1, 2p - 1\right)$. Then the problem (8.1) defines a Markov family on $E = L^p_\theta$.
Moreover, in both cases for any $\phi \in B_b(E) = B_b(L^p_\theta)$ and $t > 0$, $P_t \phi$ has derivatives of all orders and

$$\sup_{x \in E} \|D^n P_t \phi(x)\|_{L(E \times E \times \dots \times E)} \leq C e^{\frac{Cn}{t}} \sup_{x \in E} \|\phi\|_{B_b(E)}, \qquad \phi \in B_b(E),$$

with $C > 0$ independent of ϕ, $t \leq 1$, and n.

Proof. Parts (i) and (ii) are related to Hypothesis 2.1, or Hypothesis 8.1, and are know, see [14][Propositions 8.1, 8.7, 8.10], and they are recall here only for the sake of completeness. To show the last claim,

we need to verify the null-controllability Hypothesis 8.2. Let $x \in L_\theta^p$ and $t > 0$, By Lemmas 8.6 and 8.7, $S(t/2)x \in L^2(\mathcal{O})$ and

$$\|S(t/2)x\|_{L^2} \le C_1 t^{-\frac{d+\theta}{2p}} \|x\|_{L_\theta^p}.$$

By Theorem 8.5 there is a

$$\mathcal{U}(t/2) \colon L^2(\mathcal{O}) \mapsto L^2(0, t/2, L^2(\partial\mathcal{O})$$

such that for the control $u(s) = \mathcal{U}(t)[S(t/2)x](s)$, $s \in [0, t/2]$, $u \in L^2(0, t/2; L^2(\partial\mathcal{O}))$, we have

$$0 = S(t)x + \int_0^{\frac{t}{2}} (\lambda - A)S(t/2 - s)D_\lambda u(s)\mathrm{d}s$$
$$= S(t)x + \int_0^t (\lambda - A)S(t - s)D_\lambda T_t u(s)\mathrm{d}s,$$

where $T_t \colon L^2(0, t/2; L^2(\partial\mathcal{O})) \mapsto L^2(0, t; L^2(\partial\mathcal{O}))$ is bounded linear operator defined as follows

$$T_t u(s) = \begin{cases} 0 & \text{for } s \le t/2, \\ u(s - t/2) & \text{for } s \in [t/2, t]. \end{cases}$$

$\square$

8.3. The case of coloured noise.

White noise case is restricted to subdomains of $\mathbb{R}^d$, $d = 1, 2$. In this section we consider the case of the so-called coloured noise, that is the case where (e_k) is not an orthonormal basis of $L^2(\partial\mathcal{O})$. Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with $C^{1,\alpha}$, $\alpha > 0$, boundary $\partial\mathcal{O}$. By Proposition 8.10 from [14], we have the following result dealing with the verification of Hypothesis 8.1.

Proposition 8.9. *Assume that*

$$(8.4) \qquad\qquad \sum_k \sup_{\eta \in \partial\mathcal{O}} e_k^2(\eta) < +\infty$$

and $1 < p < +\infty$ and $\theta \in (p - 1, 2p - 1)$. Then the boundary problem (8.1) defines a Markov family with continuous trajectories in $E = L_\theta^p := L_{\theta,0}^p$.

What is left is to find conditions on (e_k) which guarantee for a given $t > 0$, the null-controllability of the corresponding deterministic boundary problem

$$\frac{\partial Y}{\partial s}(s,\xi) = \Delta Y(s,\xi), \qquad s \in (0,t), \ \xi \in \mathcal{O},$$

(8.5)
$$Y(0,\xi) = x(\xi), \qquad \xi \in \mathcal{O},$$

$$Y(s,\xi) = \sum_k e_k(\xi) u_k(t,x;s), \qquad s \in (0,t), \ \xi \in \partial\mathcal{O}.$$

Taking into account Lemmas 8.6 and 8.7, it is enough to establish null-controllability in unweighted space $L^2(\mathcal{O})$.

8.3.1. *The case of a unit ball in $\mathbb{R}^d$.* Here we give an example of an equation with colored noise satisfying (8.4) and such that the corresponding boundary problem (8.5) is null-controllable in $L^2(\mathcal{O})$. Namely, let $\mathcal{O} = B_d(0,1)$ be the unit ball in $\mathbb{R}^d$. Then $\partial\mathcal{O} = \mathbb{S}^{d-1}$. In polar coordinates the eigenvectors of the Laplace operator have the form

$$H_{n,\mathbf{k}}(r,\theta) = h_n(r) f_{\mathbf{k}}(\theta), \qquad r \in [0,1], \ \theta \in \mathbb{S}^{d-1},$$

$n \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}_*^{d-1} \setminus \{0\} := (\mathbb{N} \cup \{0\})^{d-1} \setminus \{(0,\ldots,0)\}$.

Above (h_n) are *Bessel functions* and $(f_{\mathbf{k}})$ is an orthonormal basis of $\mathbb{S}^{d-1}$; the-so called *harmonics*. For our purposes it is important that

(8.6)
$$\sup_{\mathbf{k}} \sup_{\theta \in \mathbb{S}^{d-1}} |f_{\mathbf{k}}(\theta)| < +\infty.$$

In the theorem $|\mathbf{k}|^2 = \sum_{i=1}^{d-1} \mathbf{k}_i^2$.

Theorem 8.10. *Assume that $e_{\mathbf{k}} = a_{\mathbf{k}} f_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_*^{d-1} \setminus \{0\}$, where $(a_{\mathbf{k}})$ is a sequence of real numbers such that*

(8.7)
$$\sum_{\mathbf{k}} a_{\mathbf{k}}^2 < +\infty,$$

and

(8.8)
$$\forall\, t > 0 \ \exists\, \beta(t) > 0: \ \forall\, \mathbf{k} \qquad a_{\mathbf{k}}^{-2} \mathrm{e}^{-t|\mathbf{k}|^2} \leq \beta(t).$$

Let $p > 1$ and $\theta \in (p-1, 2p-1)$. Then the problem (8.1) defines a Markov family on $E = L_\theta^p$. The corresponding deterministic system is null-controllable at any time $T > 0$, and for any $\phi \in B_b(E) = B_b(L_\theta^p)$ and $t > 0$, $P_t\phi$ has derivatives of all orders. Finally

(8.9)
$$\sup_{x \in E} \|D^n P_t\phi(x)\|_{L(E \times \ldots \times E)} \leq C(t)^n \sup_{x \in E} \|\phi\|_{B_b(E)}, \ \phi \in B_b(E),$$

with a constant $C(t) > 0$ independent of ϕ and n. Moreover, there are a function C_1 and a constant $c > 0$ such that $C(t) = C_1(t/2)\beta(ct/2)$ and $\log C_1(t) = \mathrm{O}(1/t)$ as $t \downarrow 0$.

Proof. Assume that $p > 1$ and $\theta \in (p-1, 2p-1)$. Then (8.6), (8.7), and Proposition 8.9 guarantee that (8.1) defines Markov family in L_θ^p. By Lemmas 8.6 and 8.7 null-controllability in $L_0^2 = L^2(B(0,1))$ ensures null-controllability in $E = L_\theta^p$. Therefore, we only need to verify null-controllability in $L^2(B(0,1))$ and the desired estimates for the control operator.

We have

$$G(t, r, \theta, r', \theta') = \sum_{n, \mathbf{k}} \mathrm{e}^{-\lambda_{n,\mathbf{k}} t} h_n(r) f_{\mathbf{k}}(\theta) h_n(r') f_{\mathbf{k}}(\theta'),$$

where $(-\lambda_{n,\mathbf{k}})$ is the corresponding sequence of eigenvalues. Then

$$S(t)x = \sum_{n,\mathbf{k}} \mathrm{e}^{-\lambda_{n,\mathbf{k}} t} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(B(0,1))} H_{n,\mathbf{k}}.$$

Note that

$$u = \sum_{\mathbf{k}} \langle u, e_{\mathbf{k}} \rangle_H e_{\mathbf{k}} = \sum_{\mathbf{k}} \langle u, e_{\mathbf{k}} \rangle_H a_{\mathbf{k}} f_{\mathbf{k}}.$$

Thus $\langle u, f_{\mathbf{k}} \rangle_{L^2(\mathbb{S}^{d-1})} = \langle u, e_{\mathbf{k}} \rangle_H a_{\mathbf{k}}$, and $\langle u, e_{\mathbf{k}} \rangle_H = a_{\mathbf{k}}^{-1} \langle u, f_{\mathbf{k}} \rangle_{L^2(\mathbb{S}^{d-1})}$.

Our aim is to verify condition (4.5). We have

$$
\begin{aligned}
\langle (S(s)B)^* x, e_{\mathbf{k}} \rangle_H &= \langle x, S(s)Be_{\mathbf{k}} \rangle_{L^2(\mathcal{O})} = a_{\mathbf{k}} \langle x, S(s)Bf_{\mathbf{k}} \rangle_{L^2(\mathcal{O})} \\
&= -a_{\mathbf{k}} \langle S(s)x, AD_0 f_{\mathbf{k}} \rangle_{L^2(\mathcal{O})} \\
&= -a_{\mathbf{k}} \int_{\mathbb{S}^{d-1}} \frac{\partial}{\partial \mathbf{n}^a(\eta)} S(s)x(\eta) f_{\mathbf{k}}(\eta) \mathrm{d}\mathbf{s}(\eta).
\end{aligned}
$$

Since

$$\frac{\partial}{\partial \mathbf{n}^a(\eta)} S(s)x = \sum_{n,\mathbf{j}} \mathrm{e}^{-\lambda_{n,\mathbf{j}} s} \langle x, H_{n,\mathbf{j}} \rangle_{L^2(\mathcal{O})} f_{\mathbf{j}} \frac{\mathrm{d}h_n}{\mathrm{d}x}(1),$$

we have

$$\sum_{\mathbf{k}} \langle (S(s)B)^* x, e_{\mathbf{k}} \rangle_H^2 = \sum_{\mathbf{k}} a_{\mathbf{k}}^2 \left(\sum_n \mathrm{e}^{-\lambda_{n,\mathbf{k}} s} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(\mathcal{O})} \frac{\mathrm{d}h_n}{\mathrm{d}x}(1) \right)^2.$$

We need to show that for ant $T_0 > 0$ there is a function $\tilde{C} = \tilde{C}(t)$ such that for $x \in L^2(B(0,1))$,

$$\sum_{n,k} \mathrm{e}^{-2\lambda_{n,\mathbf{k}}(t+T_0)} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(\mathcal{O})}^2$$

$$\leq \tilde{C}(t) \int_0^t \sum_{\mathbf{k}} a_{\mathbf{k}}^2 \left(\sum_n \mathrm{e}^{-\lambda_{n,\mathbf{k}} s} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(\mathcal{O})} \frac{\mathrm{d}h_n}{\mathrm{d}x}(1) \right)^2 \mathrm{d}s$$

$$= \tilde{C}(t) \sum_{n,m,\mathbf{k}} R_{\mathbf{k}}(a_{\mathbf{k}}, n, m, t) \mathrm{e}^{-\lambda_{n,\mathbf{k}} t} \mathrm{e}^{-\lambda_{m,\mathbf{k}} t} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(\mathcal{O})} \langle x, H_{m,\mathbf{k}} \rangle_{L^2(\mathcal{O})},$$

where

$$R_{\mathbf{k}}(a_{\mathbf{k}}, n, m, t) := a_{\mathbf{k}}^2 \frac{1}{\lambda_{n,\mathbf{k}} + \lambda_{m,\mathbf{k}}} \left(\mathrm{e}^{(\lambda_{n,\mathbf{k}} + \lambda_{m,\mathbf{k}})t} - 1 \right) \frac{\mathrm{d}h_n}{\mathrm{d}x}(1) \frac{\mathrm{d}h_m}{\mathrm{d}x}(1).$$

Since $H_{n,\mathbf{k}}$ and $H_{m,\mathbf{j}}$ are orthogonal for $\mathbf{k} \neq \mathbf{j}$ we have to show that for any $\mathbf{k}$,

(8.10) $$\sum_n \mathrm{e}^{-\lambda_{n,\mathbf{k}} T_0} b_{n,\mathbf{k}}^2 \leq \tilde{C}(t) \sum_{n,m} R_{\mathbf{k}}(a_{\mathbf{k}}, n, m, t) b_{n,\mathbf{k}} b_{m,\mathbf{k}},$$

where $b_{n,\mathbf{k}} := \mathrm{e}^{-\lambda_{n,\mathbf{k}} t} \langle x, H_{n,\mathbf{k}} \rangle_{L^2(\mathcal{O})}$. By Theorem 8.5 there is a function C_1 of order $\log C_1(t) = \mathrm{O}(1/t)$ such that for any $x \in L^2(B(0,1))$,

$$\sum_n b_{n,\mathbf{k}}^2 \leq C_1(t) \sum_{n,m} R_{\mathbf{k}}(1, n, m, t) b_{n,\mathbf{k}} b_{m,\mathbf{k}} \qquad \text{for any } \mathbf{k}.$$

Hence (8.10) holds true with

$$\tilde{C}(t) = C_1(t) \sup_{n,\mathbf{k}} \mathrm{e}^{-\lambda_{n,\mathbf{k}} T_0} a_{\mathbf{k}}^{-2}.$$

Since $\lambda_{n,\mathbf{l}} \asymp c_1(n^2 + |\mathbf{k}|^2)$, we have

$$\sup_{n,\mathbf{k}} \mathrm{e}^{-\lambda_{n,\mathbf{k}} T_0} a_{\mathbf{k}}^{-2} \leq \beta(c T_0),$$

and consequently we have the desired estimate with $\tilde{C}(t) = C_1(t)\beta(c T_0)$. What is left is to evaluate $C(t)$ appearing in (8.9). We are looking for C such that

$$|S(t)x|_{L^2(B(0,1))}^2 \leq C(t) |\mathcal{Q}^*(t)x|_{L^2(0,t;H)}^2.$$

We have

$$|S(t + T_0)x|_{L^2(B(0,1))}^2 \leq C_1(t)\beta(c T_0) |\mathcal{Q}^*(t)x|_{L^2(0,t;H)}^2.$$

Since

$$|\mathcal{Q}^*(t)x|_{L^2(0,t;H)}^2 = \int_0^t |S(t-s)Bx|_H \, \mathrm{d}s \leq \int_0^{t+T_0} |S(t+T_0-s)Bx|_H \, \mathrm{d}s$$
$$= |\mathcal{Q}^*(t+T_0)x|_{L^2(0,t+T_0;H)}^2,$$

we have $C(t + T_0) = C_1(t)\beta(c T_0)$. In particular $C(t) = C_1(t/2)\beta(c t/2)$.

$\square$

References

[1] V.I. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs 62, American Mathematical Society, 1998.

[2] Z. Brzeźniak, B. Goldys, S. Peszat, and F. Russo, *Second order PDEs with Dirichlet white noise boundary conditions*, J. Evol. Equ. **15** (2015), 1–26.

[3] Z. Brzeźniak and M. Veraar, *Is the stochastic parabolicity condition dependent on p and q ?*, Electron. J. Probab. **17** (2012), 1–24.

[4] A. Chojnowska-Michalik, *Transition semigroups for stochastic semilinear equations on Hilbert spaces*, Dissertationes Math. (Rozprawy Mat.) **396** (2001).

[5] A. Chojnowska-Michalik and B. Goldys, *Symmetric Ornstein-Uhlenbeck semigroups and their generators*, Probab. Theory Related Fields **124** (2002), 459–486.

[6] G. Ga Prato, *An Introduction to Infinite-Dimensional Analysis*, Springer-Verlag, Berlin, 2006.

[7] G. Da Prato and J. Zabczyk, *Smoothing properties of transition semigroups in Hilbert spaces*, Stochastics, **35**, (1991) 63–77.

[8] G. Da Prato, A. Pritchard, and J. Zabczyk, *On minimum energy problems*, SIAM J. Control Optim. **29** (1991), 209–221.

[9] G. Da Prato and J. Zabczyk, *Evolution equations with white-noise boundary conditions*, Stochastics Stochastics Rep. **42** (1993), 167–182.

[10] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Second Edition, Cambridge Univ. Press, Cambridge, 2014.

[11] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press, Cambridge, 1996.

[12] K. D. Elworthy and X.-M. Li, *Formulae for the derivatives of heat semigroups.* J. Funct. Anal. **125** (1994), 252–286.

[13] H.O. Fattorini and D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rat. Mech. Anal. **43** (1971), 272–292.

[14] B. Goldys and S. Peszat, *Linear parabolic equation with Dirichlet white noise boundary conditions*, J. Differential Equations **362** (2023), 382–437.

[15] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967) 147–171.

[16] A. Kolmogoroff A, *Zufällige Bewegungen (zur Theorie der Brownschen Bewegung)*, Ann. of Math. **35** (1934) 116–117.

[17] H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Math. 463, Springer-Verlag, Berlin, 1975.

[18] L. Pandolfi, E. Priola, and J. Zabczyk, *Linear operator inequality and null controllability with vanishing energy for unbounded control systems*, SIAM J. Control Optim. **51** (2013), 629–659.

[19] S. Peszat and J. Zabczyk, *Strong Feller property and irreducibility for diffusions on Hilbert spaces*, Ann. Probab. **23** (1995), 157–172.

[20] S. Peszat and J. Zabczyk, *Stochastic evolution equations with a spatially homogeneous Wiener process*, Stochastic Processes Appl. **72** (1997), 187–204.

[21] E. Priola, J. Zabczyk, *Null controllability with vanishing energy*, SIAM J. Control Optim. **42** (2003), 1013–1032.

[22] E. Priola, F-Y. Wang, *A sharp Liouville theorem for elliptic operators*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **21** (2010), 441–445.

[23] T.I. Seidman, *Two results on exact boundary control of parabolic equations*, App. Mat. Optim. **11** (1984), 145–152.

[24] T.I. Seidman, *How violent are fast controls?*, Math. Control Signals Systems **1** (1988), 89–95.

[25] J. Zabczyk, *Linear stochastic systems in Hilbert spaces; spectral properties and limit behaviours*, Banach Center Publications **41** (1985), 591–609.

School of Mathematics and Statistics, The University of Sydney, Sydney 2006, Australia

Email address: `Beniamin.Goldys@sydney.edu.au`

Szymon Peszat, Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30–348 Kraków, Poland

Email address: `napeszat@cyf-kr.edu.pl`