Assignment 4

Your solutions should be submitted by the beginning of the lecture on

Wednesday, 28 October 2009.

Q1 (a) Let X and Y be Banach spaces and suppose $T \in \mathfrak{B}(X,Y)$ is open. Show that T is surjective.

(b) How much can you relax the hypotheses on the normed spaces X, Y or the linear operator T so that "open" still implies "surjective"?

Q2 (a) Let Y be a finite dimensional subspace of the infinite dimensional normed space X. Show that Y has empty interior in X.

(b) Let P be the vector space of all real polynomials in one variable and let $||\cdot||$ be an arbitrary norm on P. Show that $(P, ||\cdot||)$ is not a Banach space.

Q3 Let X be a normed space. Show that the sequence $(x_n)_{n=1}^{\infty} \subseteq X$ converges weakly to $x_0 \in X$ if and only if the following two conditions are satisfied:

(a) The set $\{||x_n|| \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is bounded, i.e. $\sup_{n\in\mathbb{N}} ||x_n|| < \infty$.

(b) For a fixed set $A \subseteq X^*$ with the property that $\overline{\operatorname{span} A} = X^*$, we have:

$$\lim_{n\to\infty} f(x_n - x_0) = 0$$

for all $f \in A$. Recall that $\operatorname{span} A$ is the subspace of X^* consisting of all finite linear combinations of elements of A. The closure is taken with respect to the usual topology on X^*.

Q4 Show that if $1 < p < \infty$, then $\overline{\operatorname{span}\{e_n \mid n \in \mathbb{N}\}} = l_p$. Use this fact and Q3 to characterise weak convergence in l_p. (You don't need to prove that $(l_p)^* \cong l_q$ with $q = \frac{p}{p-1}$.)

Q5 Let X be a reflexive normed space.

(a) Show that X is a Banach space.

(b) Show that X^* is reflexive.

(c) Show that $B(X)$ is weakly compact.