Problem Set 3

Q1 Check the details in the proof of the "Completion Theorem 2.18" ($\sim$ is an equivalence relation; the vector space structure and the norm on $\widehat{X}$ are well-defined).

Q2 Let X be a normed space and $x_0 \in X$. Show that $||x_0|| \leq C$ if and only if $|f(x_0)| \leq C$ for all $f \in S(X^*)$.

Q3 Let X and Y be normed spaces and $T \in \mathfrak{B}(X,Y)$. We know that $T^* \in \mathfrak{B}(Y^*, X^*)$. Show that $||T|| = ||T^*||$, i.e. the map $T \to T^*$ is an isometry.
Hint: first show $||T^*|| \leq ||T||$, then use $||S|| = \sup\{\ ||Sx|| \mid ||x|| = 1\}$ for both operators and define an element in Y^* using one of the consequences of the Hahn-Banach theorem.

Q4 Let X be a normed space and $f \in X'$. Show that $f \in X^*$ if and only if $\ker f$ is closed (in the norm topology).

Q5 Let Y be a finite dimensional subspace of the normed space X. Show that Y is closed in X (in the norm topology).

Q6 Show that if X is reflexive, then X^* is reflexive. What about the converse?

Q7 Prove the following facts about the sequence spaces l_p.

(a) For $1 < p < \infty$, $(l_p)^*$ is isometric with l_q, where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Conclude that l_p is reflexive.
(b) The dual $(c_0)^*$ is isometric with l_1.
(c) The dual $(l_1)^*$ is isometric with l_∞.
(d) The spaces c_0, l_1 and l_∞ are not reflexive.

Q8 Let X be the normed space with underlying vector space $\mathbb{R}^2$ and norm defined by

$$||(a,b)|| = \max\{|a|, |b|, |a+b|\}.$$

(a) Sketch the unit ball of this norm.

(b) Find the dual norm on X^*.

(c) Sketch the unit ball of the dual norm.

(d) Is X isometric with X^*?

Q9 Let X be a normed space.

(a) What are $X^\perp$ and $\{0\}^\perp$?

(b) If Y_1, Y_2 are closed subspaces of X such that $Y_1 \neq Y_2$, show that $Y_1^\perp \neq Y_2^\perp$. Is this also true if one or both subspaces are not closed?

Q10 Let Y be a subspace of the normed space X. Suppose $\dim X = n$ and $\dim Y = m$. Show that $\dim Y^\perp = n - m$.
Formulate this as a theorem about the solution set of a system of linear equations.

Q11 Let $T \in \mathfrak{B}(X, Y)$. Show that:

(a) $(\overline{\mathrm{im}(T)})^\perp \subseteq \ker(T^*)$

(b) $\mathrm{im}(T) \subseteq \ker(T^*)_\perp$.

What does the second part imply for solving $Tx = y$?