The notion of a measure

Q0 (Exercise 0.1)
Show that the three reasonable properties for a measure are inconsistent in $\mathbb{R}^n$.

Q1 (Exercise 1.2)
Show that an algebra (resp. σ–algebra) is closed under finite (resp. countable) intersections.

Q2 (Exercise 1.5)
(1) If Σ is a σ–algebra on S, then $S \in \Sigma$ and $\emptyset \in \Sigma$.
(2) If a function $\mu\colon \Sigma \to [0,\infty]$ is countably additive and $\mu(A) \in [0,\infty)$ for all $A \in \Sigma$, then $\mu(\emptyset) = 0$.

Q3 (Exercise 1.7)
Show that Dirac measure, "All or nothing" measure and counting measure are mesaures on $\mathcal{P}(S)$.

Q4 (Exercise 1.9)
(1) The extended reals, $\mathbb{R}^*$, with the given topology are a compact topological space.
(2) There is a metric d^* on $\mathbb{R}^*$ such that

(a) $A \subset \mathbb{R}^*$ is open if and only if A is open with respect to d^*,
(b) $a_k \to a \in \mathbb{R}^*$ if and only if $d^*(a_k, a) \to 0$.

Q5 (Exercise 1.11)
(1) The intersection of any family of σ–algebras on set S is a σ–algebra on S.
(2) If $E \subseteq \Sigma(F)$, then $\Sigma(E) \subseteq \Sigma(F)$.

Q6 (Exercise 1.24)
Let λ be an outer measure on $\mathcal{P}(S)$, Σ be the σ–algebra of all λ–measurable sets, $\mu = \lambda|_\Sigma$, and let λ^* be the outer measure induced by μ.
(a) If $A \subseteq S$, $\lambda(A) \leq \lambda^*(A)$. Moreover, equality holds if and only if there is $E \in \Sigma$ such that $A \subseteq E$ and $\lambda(A) = \lambda(E)$.
(b) If λ is induced from a pre-measure, then $\lambda = \lambda^*$.

Q7 (Exercise 1.25)
Let (S, Σ, μ) be a measure space and λ be the outer measure induced by μ. Denote Σ^* the σ–algebra of all λ–measurable sets and $\mu^* = \lambda|_{\Sigma^*}$.
Show that if μ is σ–finite, then (S, Σ^*, μ^*) is the completion of (S, Σ, μ).
(In general, (S, Σ^*, μ^*) is the saturation of the completion of (S, Σ, μ); you can look up the definitions and do this exercise as well, if you must.)

Q8 Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, where m is Lebesgue measure.

(a) Show that for each $x \in \mathbb{R}$, $m(\{x\}) = 0$.

(b) Conclude that $m([a,b]) = m((a,b)) = m([a,b)) = m((a,b]) = b-a$ for all $a, b \in \mathbb{R}$.

(c) Show that if A is a countable subset of $\mathbb{R}$, then $m(A) = 0$.

Q9 Variations on the Cantor construction based on the interval $[0,1]$.

(a) Show that if one iteratively removes the open middle tenth (first from $[0,1]$, then from $[0, \frac{9}{20}]$ and $[\frac{11}{20}, 1]$, etc.), one obtains a Lebesgue measurable set of measure 0, which is not contained in the Borel σ–algebra.

(b) Show that if one instead iteratively removes the open middle $\frac{1}{10^k}$th at step k (first the open middle tenth from $[0,1]$, then the open middle hundredth from $[0, \frac{9}{20}]$ and $[\frac{11}{20}, 1]$, etc.), one obtains a Lebesgue measurable set of positive measure, which is not contained in the Borel σ–algebra.

Q10 This questions rounds off the discussion of the constructive characterisation of σ–algebras generated by subsets.

(a) Let Σ be an infinite σ–algebra. Show that Σ contains an infinite sequence of pairwise disjoint sets, and that $\text{card}(\Sigma) \geq \text{card}(\mathbb{R})$.

(b) Let S be a set and $\mathcal{E} \subset \mathcal{P}(S)$. Show that if $\text{card}(\mathbb{N}) \leq \text{card}(\mathcal{E}) \leq \text{card}(\mathbb{R})$, then $\text{card}(\Sigma(\mathcal{E})) = \text{card}(\mathbb{R})$.

Q11 Let m be Lebesgue measure on $\mathbb{R}^n$, and suppose $A, B \subseteq \mathbb{R}^n$ are Lebesgue measurable.

(a) Suppose $d(A,B) = \inf\{d(x,y) \mid x \in A, y \in B\} > 0$. Show that $m(A \cup B) = m(A) + m(B)$.

(b) Suppose $A \cap B = \emptyset$. Given an example where $d(A,B) = 0$ and $m(A \cup B) = m(A) + m(B)$.

(c) Give an example, where $A \cap B \neq \emptyset$ and $m(A \cup B) = m(A) + m(B)$.

Q12 Let m^* be the outer measure induced by Lebesgue measure on $\mathcal{P}(\mathbb{R}^n)$. Show that $m^*(S^{n-1}) = 0$. What is the outer measure of the set $\{(x_1, x_2, \ldots, x_n) \mid \max\{x_k\} = 1\}$?