Product Measures

Q43 Let $E \in \Sigma_X \otimes \Sigma_Y$. Show that $E_x \in \Sigma_Y$ and $E^y \in \Sigma_X$.

Q44 Show that $\Sigma_X \otimes \Sigma_Y \otimes \Sigma_Z = (\Sigma_X \otimes \Sigma_Y) \otimes \Sigma_Z$.

Q45 Show that $\mu \times \nu \times \lambda = (\mu \times \nu) \times \lambda$ if all three measures are σ–finite.

Q46 Show that $(\mathbb{R}^n, \mathcal{L}^n, m^n)$ is the completion of

$$(\mathbb{R}^n, \underbrace{\mathcal{L}^1 \otimes \ldots \otimes \mathcal{L}^1}_{n}, \underbrace{m^1 \times \ldots \times m^1}_{n}).$$

Q47 Show that $\underbrace{m^1 \times \ldots \times m^1}_{n} = m^{n-k} \times m^k$ for all $0 < k < n$.

Q48 Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 y$.
Show that f is Lebesgue measurable on $X = [0,3] \times [1,2]$ and that $\int_X f \, dm^2 = \frac{27}{2}$.

Q49 Define $f\colon [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \frac{x-y}{(x+y)^3}, & \text{if } x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Show that f is (a) not Riemann integrable, (b) Lebesgue measurable, (c) not Lebesgue integrable.
Also show that the iterated integrals

$$\int \int f \, dx \, dy \qquad \text{and} \qquad \int \int f \, dy \, dx$$

exist.