Assignment 4

"Regular surfaces"

Due Thursday, 22 September, at 16:00 in the assignment box for MATH3405. The box is located on **Level 4** in the Mathematics (Priestley) building (67). It is number 035 in the **brown lot of boxes** (there are two lots, to find ours, turn right as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Q1 Let $U = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$, and $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $\Phi\colon U \to S^2$ and $\Psi\colon U \to S^2$ be the local coordinate charts defined by:

$$\Phi(u, v) = (u, v, \sqrt{1 - u^2 - v^2}), \qquad \Psi(u, v) = (v, \sqrt{1 - u^2 - v^2}, u).$$

(a) Determine the subsets $\Phi(U)$, $\Psi(U)$ and $\Phi(U) \cap \Psi(U)$ of S^2.

(b) Determine the domain of $\Psi^{-1} \circ \Phi$ and find a closed expression for this function.

(c) Determine the domain of $\Phi^{-1} \circ \Psi$ and find a closed expression for this function.

Q2 Let $\Phi\colon U \to \mathbb{R}^3$ be a parametrised surface, where $U \subseteq \mathbb{R}^2$ is open. Since $\Phi_u = \frac{\partial \Phi}{\partial u}$ and $\Phi_v = \frac{\partial \Phi}{\partial v}$ are well-defined at each point in U, we can define the real-valued functions

$$E = E(u, v) = \Phi_u \cdot \Phi_u, \qquad F = F(u, v) = \Phi_u \cdot \Phi_v, \qquad G = G(u, v) = \Phi_v \cdot \Phi_v,$$

where $\cdot$ is the dot product in $\mathbb{R}^3$.

Show that Φ is regular if and only if the function $EG - F^2$ is nowhere zero.

Q3 Let S_1 and S_2 be regular surfaces. Suppose that S_1 has a local coordinate chart $\Psi\colon U \to S_1$ with the property that $S_1 = \mathrm{Im}(\Psi)$.
Given any differentiable map $f\colon U \to S_2$, show that the map $F\colon S_1 \to S_2$ defined by

$$F(\Psi(u, v)) = f(u, v) \quad \text{for all } (u, v) \in U$$

is a differentiable map of surfaces.

Q4 Let $R > r > 0$ and define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Phi(\theta, \varphi) = ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\varphi, r\sin\theta).$$

Let $S = \mathrm{Im}(\Phi)$. You know that this is a torus, and you may assume that it is a smooth regular surface. Since the torus is compact, it cannot be covered by a single local coordinate chart.

(a) Determine two or more local coordinate charts whose union covers the torus.

(b) Give a basis for the tangent space T_pS for each $p \in S$.

(c) Compute the first fundamental form of S at each $p \in S$.

[Mapping the helicoid to the torus]

An application of Q3 (not part of the assignment): Let $b \neq 0$ and define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Psi(u, v) = (u \cos v, u \sin v, bv).$$

Then Ψ is injective and $H = \operatorname{Im}(\Psi)$ is a regular surface called the helicoid (a picture of it is on the course web-site). It can be viewed as the disjoint union of lines L_v parallel to the xy-plane, obtained by fixing v and varying u. These are called *rulings*.

Let $R > r > 0$ and define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\Phi(\theta, \varphi) = ((R + r\cos\theta)\cos\varphi, (R + r\cos\theta)\sin\phi, r\sin\theta).$$

Then Ψ is not injective and its image $T = \operatorname{Im}(\Phi)$ is a torus. This can be viewed as the disjoint union of circles C_θ, obtained by fixing φ and varying θ. These are called *meridian circles*.

Define a differentiable map $H \to T$ such that rulings are mapped to meridian circles.