Assignment 2

"Duality and the Hahn-Banach Theorem"

Your solutions should be submitted by the beginning of the lecture on

Tuesday, 13 September 2011.

Please attach a cover sheet!

Q1 Let X be a normed space, and let $\overline{X}$ be the completion of X.

Show that the respective dual spaces $X^\star$ and $\overline{X}^\star$ are isometric.

Q2 Let Y be a subspace of the finite-dimensional normed space X.

Show that $(Y^\perp)^\perp$ is isometric to Y.

Q3 Let X be a normed space.

(a) If $L \subseteq X$ is a closed subspace and $x \in X \setminus L$, show that $\operatorname{span}\{L, x\}$ is also closed.

(b) Show that every finite dimensional subspace of X is closed.

(c) Let $\{x_1, \ldots, x_n\}$ be a linearly independent set in the normed space X. Show that there exists a linearly independent set $\{f_1, \ldots, f_n\}$ in X^* such that $f_i(x_j) = \delta_{ij}$ and for each $x \in \operatorname{span}\{x_1, \ldots, x_n\}$,

$$x = \sum_{i=1}^{n} f_i(x) x_i.$$

Q4 Let Y be a subspace of the normed space X and $x \in X$. Show that $\operatorname{dist}(x, Y) \geq 1$ if and only if there exists a bounded linear functional $f \in B(X^*)$ such that $Y \subseteq \ker f$ and $f(x) = 1$.