Assignment 3

"Hyperbolic geometry in dimension 3"

Due Monday, 30 January, at the start of the 14:00 lecture. Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Q1 (Perpendicular lines)

Thurston (Lemma 2.5.3) tells you that two distinct lines in $\mathbb{H}^3$ are either parallel (share an ideal endpoint) or they have a common perpendicular. (If the lines meet, then the common perpendicular passes through the point of intersection and is perpendicular to the plane spanned by the two lines.) Also recall that we have identified the group of Möbius transformations $\text{Möb}(\hat{\mathbb{C}})$ with $\text{Isom}^+(\mathbb{H}^3)$, the group of orientation preserving isometries of $\mathbb{H}^3$.

(a) If $L_1 = [a_1, b_1]$ and $L_2 = [a_2, b_2]$ ($a_k, b_k \in \hat{\mathbb{C}}$), are perpendicular lines, then the cross ratio of their endpoints satisfies: $\text{CR}[a_1, b_1, a_2, b_2] = -1$.

(b) Given $L = [p, q]$, the rotation by π around L can be represented by a matrix $R \in SL(2, \mathbb{C})$ satisfying $R^2 = -E$. Determine R explicitly in terms of p and q.

(c) Suppose the rotations by π around lines L_1 and L_2 are represented by $R_1, R_2 \in SL(2, \mathbb{C})$ as above. If $\text{tr}(R_1R_2) = 0$, then L_1 and L_2 are perpendicular.

(d) Suppose lines L_1 and L_2 are non-parallel, and denote $R_1, R_2 \in SL(2, \mathbb{C})$ the respective rotations by π around the lines as above. Show that the Möbius transformation represented by $R_1R_2 - R_2R_1$ is the rotation by π about the common perpendicular of L_1 and L_2. (The matrix $R_1R_2 - R_2R_1$ is not necessarily an element of $SL_2(\mathbb{C})$.)

Q2 (Ideal tetrahedra)

Let $T = [a, b, c, d]$ be a non-degenerate ideal tetrahedron, i.e. $a, b, c, d \in \hat{\mathbb{C}}$ and T is not contained in a hyperbolic plane. A *symmetry* of T is an isometry of $\mathbb{H}^3$ that takes T to itself. This is uniquely determined by the induced permutation of ideal vertices of T, and hence there are at most 12 orientation preserving isometries (corresponding to the alternating group A_4). We are interested in the study of this group $\text{Sym}^+(T) \subseteq \text{Isom}^+(\mathbb{H}^3)$. Each pair of opposite edges of T is non-parallel and hence has a common perpendicular, called a *height* of T.

(a) Show that rotation by π about any height of T is a symmetry of T.

(b) Show that the three heights meet at right angles in a single point in $\mathbb{H}^3$.

(c) Conclude that the rotations about the heights form a subgroup $K(T)$ of $\text{Sym}^+(T)$ that is isomorphic to the Klein four group $\langle x, y \mid x^2 = y^2 = (xy)^2 \rangle$.

(d) Recall that any element of the Möbius group preserves cross ratios, and notice that one can move T by an orientation preserving isometry to have vertices $[0, 1, \infty, z]$. Since T is non-degenerate, we have $z \notin \mathbb{R} \cup \{\infty\}$. Give necessary and sufficient conditions for $K(T) = \text{Sym}^+(T)$ in terms of z.

Q3 (Contradiction?)

Reconcile the fact that the rotations about any two heights in Q2 commute with the fact from Q1 that $R_1R_2 - R_2R_1$ gives the third rotation.