Recent progress on a sharp lower bound for first (nonzero) Steklov eigenvalue

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Asia-Pacific Analysis and PDE Seminar

May 11th, 2020

Table of Contents

Eigenvalue Lower Bound

Theorem (Lichernowicz 1958, Obata 1962)

Let (M^n, g) be a closed Riemannian n-manifold with $Ric_g \geq (n-1)K > 0$. Then

$$\lambda_1(M) \geq nK.$$

Equality holds if and only if $M \cong \mathbb{S}^n(\frac{1}{\sqrt{K}})$.

- $\lambda_1(M)$ is first (nonzero) eigenvalue of Δ_M. Variational characterization

$$\lambda_1(M) = \inf_{f \in C^1(M), \int_M f = 0} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

- Maximum principle or Integral method on Bochner's formula.

Eigenvalue Lower Bound

- Integral method on Bochner's formula

$$\int_M (\Delta f)^2 - |\nabla^2 f|^2 = \int_M Ric_g(\nabla f, \nabla f).$$

Using $|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2$ and $Ric_g \geq (n-1)K > 0$,

$$\begin{aligned} \frac{n-1}{n}\int_M \lambda_1^2 f^2 &= \frac{n-1}{n}\int_M (\Delta f)^2 \\ &\geq (n-1)K\int_M |\nabla f|^2 = (n-1)K\lambda_1 \int_M f^2. \end{aligned}$$

- Equality by Obata's theorem:
  A closed Riemannian n-manifold which admits a solution to

$$\nabla^2 f = -Kfg$$

must be $\mathbb{S}^n(\frac{1}{\sqrt{K}})$.

Eigenvalue Lower Bound

Theorem

Let (M^n, g) be a compact Riemannian n-manifold with boundary Σ.

- *(Reilly '77) Assume $Ric_g \geq (n-1)K > 0$ and $H_\Sigma \geq 0$ (mean convex boundary). Then $\lambda_1^D(M) \geq nK$.*
- *(C. Y. Xia '88, Escobar '90) Assume $Ric_g \geq (n-1)K > 0$ and $h_\Sigma \geq 0$ (convex boundary). Then $\lambda_1^N(M) \geq nK$.*

Equality holds if and only if $M \cong \mathbb{S}^n_+(\frac{1}{\sqrt{K}})$.

- First Dirichlet eigenvalue and Neumann eigenvalue of Δ_M

$$\lambda_1^D(M) = \inf_{f \in C^1(M), f|_\Sigma = 0} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

$$\lambda_1^N(M) = \inf_{f \in C^1(M), \int_M f = 0} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Eigenvalue Lower Bound

Theorem (Li-Yau '80, Zhong-Yang '84, Hang-Wang '07)

Let (M^n, g) be a compact Riemannian n-manifold possibly with convex boundary Σ. Assume $Ric_g \geq 0$. Then

$$\lambda_1^N(M) \geq \frac{\pi^2}{d^2},$$

where $d = \mathrm{diam}(M)$. Equality holds if and only if M is a 1-dmensional round circle or a segment.

Eigenvalue Lower Bound

Theorem (Li-Yau '80, Zhong-Yang '84, Hang-Wang '07)

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where $d = \mathrm{diam}(M)$. Equality holds if and only if M is a 1-dmensional round circle or a segment.

Theorem (Andrews-Clutterbuck '11)

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and λ be the Dirichlet eigenvalues for Schrödinger operator $\Delta + V$ with convex V. Then

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}.$$

$\frac{3\pi^2}{d^2}$ is the spectral gap for 1-dimensional Laplacian on $[-\frac{D}{2}, \frac{D}{2}]$.

Steklov Eigenvalue

- Let (M^n, g) be a compact Riemannian n-manifold with boundary Σ.
  For $f \in C^\infty(\Sigma)$, let $\hat{f}$ be its harmonic extension in M,

$$\Delta \hat{f} = 0 \text{ in } M, \quad \hat{f} = f \text{ on } \Sigma.$$

- Dirichlet-to-Neumann operator

$$\begin{aligned} L : C^\infty(\Sigma) &\to C^\infty(\Sigma) \\ f &\mapsto \frac{\partial \hat{f}}{\partial \nu}. \end{aligned}$$

  ν is outward unit normal to Σ.
- L is linear, nonnegative, self-adjoint operator with compact inverse, hence its spectrum is given by

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \to \infty.$$

  σ_i is called Steklov eigenvalues, first considered by Steklov 1900 in Euclidean space.

Steklov Eigenvalue

- Steklov eigenvalues:

$$\Delta f = 0 \text{ in } M, \quad \frac{\partial f}{\partial \nu} = \sigma f \text{ on } \Sigma.$$

- Variational characterization:

$$\sigma_1(M) = \inf_{f \in C^1(M), \int_\Sigma f = 0} \frac{\int_M |\nabla f|^2}{\int_\Sigma f^2},$$

$$\sigma_k(M) = \inf_{\substack{\mathcal{S} \subset C^1(M), \\ dim\, \mathcal{S} = k+1}} \sup_{0 \neq f \in \mathcal{S}} \frac{\int_M |\nabla f|^2}{\int_\Sigma f^2}.$$

Steklov Eigenvalue

- Steklov eigenvalues for Euclidean unit disk $\mathbb{B}_1 \subset \mathbb{R}^2$:

$$0, 1, 1, 2, 2, \cdots, k, k, \cdots$$

  Corresponding Steklov eigenfunctions:

$$1, r\cos\varphi, r\sin\varphi, \cdots, r^k\cos k\varphi, r^k\sin k\varphi, \cdots$$

- Steklov eigenvalues for Euclidean unit ball $\mathbb{B}_1 \subset \mathbb{R}^n$:

$$k \in \mathbb{N} \text{ with multiplicity } \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$$

  Corresponding Steklov eigenfunctions:
  homogeneous harmonic polynomials of degree k.

Lower Bound for Steklov Eigenvalue

- Payne '70: $M^2 \subset \mathbb{R}^2$, boundary geodesic curvature $k_g(\Sigma) \geq c > 0 \Rightarrow \sigma_1 \geq c$. Equality holds iff $M = \mathbb{B}^2(\frac{1}{c})$.
- Escobar '97: (M^2, g), Gauss curvature $K \geq 0$ and $k_g(\Sigma) \geq c > 0 \Rightarrow \sigma_1 \geq c$. Equality holds iff $M \cong \mathbb{B}^2(\frac{1}{c})$.
- Escobar '97: $(M^n, g), n \geq 3$, $Ric_g \geq 0$ and all boundary principal curvatures $\kappa(\Sigma) \geq c > 0 \Rightarrow \sigma_1 > \frac{c}{2}$.
- Escobar's Conjecture: $(M^n, g), n \geq 3$,
  $Ric_g \geq 0$ and $\kappa(\Sigma) \geq c > 0 \Rightarrow \sigma_1 \geq c$.
  Equality holds iff $M \cong \mathbb{B}^n(\frac{1}{c})$.
  (Compare to Lichernowicz-Obata's theorem)
- Even unknown for Euclidean case $M^n \subset \mathbb{R}^n, n \geq 3$.

Two dimensions (M^2, g)

- Weinstock '54: simply connected, $\sigma_1 L \leq 2\pi = (\sigma_1 L)(\mathbb{B}^2)$ (L is boundary length). Equality holds iff $\exists$ a conformal diffeomorphism $\varphi : M \to \mathbb{B}^2$ such that $\varphi|_\Sigma$ is an isometry.
- Fraser-Schoen '11: , $\sigma_1 L \leq 2(g + r)\pi$, genus g and boundary components r.
- Fraser-Schoen '16: annulus type, $\sigma_1 L \leq (\sigma_1 L)(M_{cc})$, M_{cc} is critical catenoid in $\mathbb{B}^3$.
- Fraser-Schoen '16: If

  $$(\sigma_1 L)(M, g_0) = \max_g (\sigma_1 L)(M, g),$$

  then there exist independent eigenfunction $u_1, \cdots, u_n$ which give a conformal free boundary minimal immersion $u_i : (M, g_0) \to \mathbb{B}^n$ with $u_i|_\Sigma$ is an isometry.
- Matthiesen-Petrides '20 (arXiv): any topological type, existence of smooth maximal metric for $\sigma_1 L$.

Isoperimetric upper bound for Steklov Eigenvalue

Higher dimensions $M^n \subset \mathbb{R}^n, n \geq 3$

- Brock '01: $\sigma_1 \mathrm{Vol}^{\frac{1}{n}} \leq (\sigma_1 \mathrm{Vol}^{\frac{1}{n}})(\mathbb{B}^n)$, Equality holds iff $M^n = \mathbb{B}^n(r)$.
- Bucur-Ferone-Nitsch-Trombetti '17: convex, $\sigma_1 \mathrm{Area}^{\frac{1}{n-1}} \leq (\sigma_1 \mathrm{Area}^{\frac{1}{n-1}})(\mathbb{B}^n)$, Equality holds iff $M^n = \mathbb{B}^n(r)$.
- Fraser-Schoen '17: $\exists$ smooth contractible domain $M^n \subset \mathbb{R}^n, n \geq 3$ with $(\sigma_1 \mathrm{Area}^{\frac{1}{n-1}})(M) > (\sigma_1 \mathrm{Area}^{\frac{1}{n-1}})(\mathbb{B}^n)$

Comparison of Steklov Eigenvalue with Boundary Eigenvalue

- Q.L.Wang-C.Y.Xia '09: $(M^n, g), n \geq 3$, $Ric_g \geq 0$ and $\kappa(\Sigma) \geq c > 0$, then

$$\sigma_1 \leq \frac{\sqrt{\lambda_1}}{(n-1)c}(\sqrt{\lambda_1} + \sqrt{\lambda_1 - (n-1)c^2}).$$

where λ_1 is first closed eigenvalue of (Σ, g_Σ). ($\lambda_1 \geq (n-1)c^2$ was proved by C.Y.Xia '07.)

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- Karpukhin '17: $(M^n, g), n \geq 3$, $W^{[2]} \geq 0$ and $\kappa(\Sigma) \geq c > 0$, then

$$\sigma_k \leq \frac{\lambda_k}{(n-1)c}, n \geq 4,$$
$$\sigma_k \leq \frac{2\lambda_k}{3c}, n = 3.$$

(Based on Results on Steklov eigenvalue estimates for p-forms by Raulot-Savo '12, Yang-Yu '17)

Our results

Theorem (Xiong- X. '19)

Let $(M^n, g), n \geq 2$ be a compact Riemannian n-manifold with boundary Σ.
Assume $\mathrm{Sect}_g \geq 0$ and $\kappa(\Sigma) \geq c > 0$. Then $\sigma_1 \geq c$.
Equality holds if and only if $M \cong \mathbb{B}^n(\frac{1}{c}) \subset \mathbb{R}^n$.

- Escobar's conjecture holds true for manifolds with $\mathrm{Sect}_g \geq 0$. Especially, true for Euclidean domains.

Our results

Theorem (Xiong- X. '19)

Let $(M^n, g), n \geq 2$ be a compact Riemannian n-manifold with boundary Σ.
Assume $\mathrm{Sect}_g \geq 0$ and $\kappa(\Sigma) \geq c > 0$.
Then

$$\sigma_1 \leq \frac{\lambda_1}{(n-1)c}$$

with equality holds if and only if $M \cong \mathbb{B}^n(\frac{1}{c}) \subset \mathbb{R}^n$.
Moreover,

$$\sigma_k \leq \frac{\lambda_k}{(n-1)c}, \forall k.$$

- Compare with Q.L.Wang-C.Y.Xia '09, stronger assumption and stronger conclusion;
  Compare with Karpukhin '17, different assumption and same conclusion in $n \geq 4$ and better conclusion in $n = 3$.

Review of Payne-Escobar's method in $n = 2$.

- $\Delta|\nabla f|^2 \geq 0$, then $\varphi = |\nabla f|^2$ attains its maximum at $x_0 \in \partial\Omega$.
- At $x_0 \in \partial\Omega$, consider Fermi coordinates of $\partial\Omega$, $\partial\Omega$ is parametrized by arc-length $\gamma(s)$.

$$0 = \Delta f|_\Sigma = f_{\nu\nu} + \kappa f_\nu + f'' = f_{\nu\nu} + \kappa\sigma_1 f + f''.$$

  Then $f_{\nu\nu} = -\kappa\sigma_1 f - f''$, and

$$0 \leq \varphi_\nu(s_0) = 2(-f'' - \kappa\sigma_1 f)\sigma_1 f + 2(\sigma_1 - \kappa)f'^2,$$
$$\varphi'(s_0) = 0, \varphi''(s_0) \leq 0.$$

  All inequalities involves only f, f', f''. By simple calculation, one can show $\sigma_1 \geq \kappa(s_0) \geq c$.

- This method fails to handle higher dimensions.

Review of Escobar's method in $n \geq 3$.

- $n \geq 3$, using Reilly's formula

$$\int_M \left[(\Delta f)^2 - |\nabla^2 f|^2 - Ric(\nabla f, \nabla f)\right]$$
$$= \int_\Sigma \left[2 f_\nu \Delta_\Sigma f + H f_\nu^2 + h(\nabla_\Sigma f, \nabla_\Sigma f)\right]$$

  Using $\Delta f = 0$, $f_\nu = \sigma_1 f$, $Ric \geq 0$, $h \geq c g_\Sigma$, one gets

$$0 \geq \int_\Sigma (c - 2\sigma_1)|\nabla_\Sigma f|^2 + H\sigma_1^2 f^2.$$

  Thus $\sigma_1 > \frac{c}{2}$.

- No information between $\int_\Sigma |\nabla_\Sigma f|^2$ and $\int_\Sigma f^2$.

Our proof

Two integral identities. Let $f, V \in C^\infty(M)$.

Proposition (Qiu-X. '15, Weighted Reilly's formula)

$$\begin{aligned}
&\int_M V\left((\Delta f)^2 - |\nabla^2 f|^2\right) \\
= &\int_\Sigma V\left[2\partial_\nu f \Delta_\Sigma f + H(\partial_\nu f)^2 + h(\nabla_\Sigma f, \nabla_\Sigma f)\right] \\
&+ \int_\Sigma \partial_\nu V\, |\nabla_\Sigma f|^2 + \int_\Omega \left(\nabla^2 V - \Delta V g + V\mathrm{Ric}_g\right)(\nabla f, \nabla f).
\end{aligned}$$

Proposition (Pohozaev's identity)

$$\begin{aligned}
&\int_M \langle \nabla V, \nabla f\rangle \Delta f + \int_M (\nabla^2 V - \frac{1}{2}\Delta V g)(\nabla f, \nabla f) \\
= &\int_\Sigma (\partial_\nu f \langle \nabla V, \nabla f\rangle - \frac{1}{2}|\nabla f|^2 \partial_\nu V).
\end{aligned}$$

Our proof

Key choice of V:

- $$V = \rho - \frac{c}{2}\rho^2 \text{ where } \rho = \mathrm{dist}(\cdot, \Sigma).$$
- $V > 0$ since $\rho \leq \frac{1}{c}$ (by only assuming $Ric \geq 0$ and $H \geq n-1$: M.Li '14)
- $V \in C^{0,1}(M)$ and $V \in C^\infty(M \setminus \mathrm{Cut}(\Sigma))$.
- $V = 0$ and $\partial_\nu V = (1 - c\rho)\partial_\nu \rho = -1$ on Σ.

Our proof

- Hessian Comparison (Heintze-Karcher '78, Kasue '82):
  If $\mathrm{Sect}_g \geq 0$, $h \geq cg_\Sigma > 0$, then
  $$\nabla^2 V \leq -cg \text{ on } M \setminus (\Sigma \cup \mathrm{Cut}(\Sigma)).$$
- V is $-c$-concave in the sense of H.-H. Wu:
  $$\begin{aligned} C(V)(x;Y) &= \liminf_{r\to 0} \frac{V(\exp_x(rY)) + V(\exp_x(-rY)) - 2V(x)}{r^2} \\ &\leq -c \end{aligned}$$
  for any $x \in M$ and any $Y \in T_xM$ with $|Y| = 1$.

Our proof

Proposition (Smooth approximation)

Fix a neighborhood $\mathcal{C}$ of $\mathrm{Cut}(\Sigma)$ in M. Then for any $\varepsilon > 0$, there exists a smooth nonnegative function V_ε on M such that $V_\varepsilon = V$ on $M \setminus \mathcal{C}$ and

$$\nabla^2 V_\varepsilon \leq -(c - \varepsilon) g.$$

- Greene-Wu's Riemannian convolution V_τ for $-c$-concave function V in a small neighborhood O of $\mathrm{Cut}(\Sigma)$ is still $-c$-concave.
- Gluing the Riemannian convolution V_τ in O and V outside O by a cut-off function.
- $V_\varepsilon \geq 0$ on M.
- $V_\varepsilon = V = 0$ and $\partial_\nu V_\varepsilon = \partial_\nu V = -1$ on Σ.

Our proof

Two integral identities. Let $f, V \in C^\infty(M)$.

Proposition (Qiu-X. '15, Weighted Reilly's formula)

$$\begin{aligned}&\int_M V\left((\Delta f)^2 - |\nabla^2 f|^2\right)\\ =&\int_\Sigma V\left[2\partial_\nu f\Delta_\Sigma f + H(\partial_\nu f)^2 + h(\nabla_\Sigma f, \nabla_\Sigma f)\right]\\ &+\int_\Sigma \partial_\nu V|\nabla_\Sigma f|^2 + \int_\Omega \left(\nabla^2 V - \Delta V g + V\mathrm{Ric}_g\right)(\nabla f, \nabla f).\end{aligned}$$

Proposition (Pohozaev's identity)

$$\begin{aligned}&\int_M \langle\nabla V, \nabla f\rangle\Delta f + \int_M (\nabla^2 V - \frac{1}{2}\Delta V g)(\nabla f, \nabla f)\\ =&\int_\Sigma (\partial_\nu f\langle\nabla V, \nabla f\rangle - \frac{1}{2}|\nabla f|^2\partial_\nu V).\end{aligned}$$

Our proof

- Let f is harmonic.
- Use V_ε in Qiu-Xia's Reilly formula and Pohozaev's identity, we get

$$0 \geq \int_\Sigma -|\nabla_\Sigma f|^2 + \int_M (\nabla^2 V_\varepsilon - \Delta V_\varepsilon g)(\nabla f, \nabla f).$$

$$\int_\Sigma \frac{1}{2}|\nabla_\Sigma f|^2 - \frac{1}{2}(\partial_\nu f)^2 = \int_M (\nabla^2 V_\varepsilon - \frac{1}{2}\Delta V_\varepsilon g)(\nabla f, \nabla f).$$

- Eliminating $\int_\Sigma |\nabla_\Sigma f|^2$, we have

$$\int_\Sigma (\partial_\nu f)^2 \geq -\int_M \nabla^2 V_\varepsilon(\nabla f, \nabla f) \geq (c - \varepsilon)\int_M |\nabla f|^2.$$

Our proof

$$\int_\Sigma (\partial_\nu f)^2 \geq -\int_M \nabla^2 V_\varepsilon(\nabla f, \nabla f) \geq (c-\varepsilon)\int_M |\nabla f|^2.$$

- If f is first Steklov eigenvalue,

$$\int_M |\nabla f|^2 = \sigma_1 \int_\Sigma f^2$$

and

$$\partial_\nu f = \sigma_1 f,$$

We conclude $\sigma_1 \geq c$.

Our proof

Return to

$$0 \geq \int_\Sigma -|\nabla_\Sigma f|^2 + \int_M (\nabla^2 V_\varepsilon - \Delta V_\varepsilon g)(\nabla f, \nabla f).$$

- If f is harmonic extension of first boundary closed eigenfunction, then

$$\int_\Sigma |\nabla_\Sigma f|^2 = \lambda_1 \int_\Sigma f^2.$$

- From Hessian comparison,

$$(\nabla^2 V_\varepsilon - \Delta V_\varepsilon g)(\nabla f, \nabla f) \geq (n-1)(c-\varepsilon)|\nabla f|^2.$$

  Note $\int_\Sigma f = 0$, using variational charaterization,

$$\sigma_1 \leq \frac{\int_M |\nabla f|^2}{\int_\Sigma f^2} \leq \frac{\lambda_1}{n-1}.$$

Our proof

Let $\{\varphi_k\}$ be eigenfunctions corresponding to λ_k on Σ, forming an orthonormal basis of $L^2(\Sigma)$. Let f_k be harmonic extension of φ_k. Then using min-max variational characterization,

$$\begin{aligned}
\sigma_j &\leq \sup_{\sum_{k=0}^j a_k^2=1} \int_\Omega |\nabla(\sum_{k=0}^j a_k f_k)|^2 \\
&\leq \frac{1}{(n-1)c} \sup_{\sum_{k=0}^j a_k^2=1} \int_\Sigma |\nabla_\Sigma(\sum_{k=0}^j a_k \varphi_k)|^2 \\
&= \frac{1}{(n-1)c} \sup_{\sum_{k=0}^j a_k^2=1} \sum_{k=0}^j a_k^2 \lambda_k \\
&\leq \frac{\lambda_j}{(n-1)c}.
\end{aligned}$$

Our proof

Equality characterization: Obata type theorem.

Proposition

Let (Ω, g) be an n-dimensional compact Riemannian manifold with boundary Σ such that

$$\mathrm{Ric}_g \geq 0 \text{ in } \Omega, H \geq (n-1)c \text{ on } \Sigma.$$

Assume there exists a nontrivial smooth function f satisfying

$$\nabla^2 f = 0 \text{ in } \Omega, \quad \partial_\nu f = cf \text{ on } \Sigma. \tag{1}$$

Then Ω is isometric to a Euclidean ball with radius $1/c$.

- Without the curvature condition, there might be other manifolds admitting the solution to (1). Chen-Lai-Wang studied such Obata's theorem.

Our proof

- An idea due to B. Andrews.
- Step 1: Let $N_t = \{x \in M, f(x) = t\}$, ∇f is Killing, M is product manifold $N_0 \times \mathbb{R}$.
- Step 2: Write Σ as graphs over N_0, $\Sigma_\pm = \{(x, u_\pm) : x \in N_0\}$, $u_+ \geq 0, u_- \leq 0$, satisfying

$$\frac{1}{\sqrt{1 + |\nabla u_\pm|^2}} = cu_\pm.$$

- Step 3: $\{x \in N_0 : u_+(x) = \frac{1}{c}\} = \{x_0\}$. By setting $T_\tau = \{x \in N_0 : \sqrt{1 - c^2 u_+^2} = c\tau\}$, one shows $\tau \leq \mathrm{dist}(x_0, T_\tau)$. In particular,

$$1/c \leq \mathrm{dist}(x_0, T_{1/c}) = \mathrm{dist}(x_0, \partial N_0)$$

  which implies N_0 is an $(n-1)$-Euclidean ball with radius $\frac{1}{c}$.
- Step 4: $\Sigma_\pm$ is a half-sphere on N_0.

Open question

- Escobar's conjecture ($Ric_g \geq 0, \kappa_\Sigma \geq c > 0$)?
- Is it possible Escobar's conjecture true for $Ric_g \geq 0, H_\Sigma \geq c > 0$. Note that V satisfies Laplace comparison under this assumption.
- If $c \to 0$, then $\sigma_1 \geq c$ is trivial. How to estimate σ_1 by other geometric quantities, compare Li-Yau-Zhong-Yang's estimate: If $Ric_g \geq 0$ (with convex boundary), then

$$\lambda_1^N(M) \geq \frac{\pi^2}{d^2}, \quad d = \mathrm{diam}(M).$$

Equality only for $M = \mathbb{S}^1(r)$ or 1-dim interval.

Thank you for your attention!