Classifications of solutions to some nonlinear PDEs of elliptic type and their applications

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We start with an example:

For the prescribing scalar curvature equation

$$-\Delta u + \frac{n(n-2)}{4} u = \frac{n-2}{4(n-1)} R(x) u^{\frac{n+2}{n-2}}, \; u > 0, \; x \in S^n. \tag{1}$$

Nirenberg Problem: For which function R, one can solve (2)?

A priori estimates + degree theory+ variational methods....

Generally:

$$-\Delta u(x) = f(x, u(x), \nabla u(x)), \; u > 0, \; x \in \Omega, + \text{ boundary conditions} \tag{2}$$

The blow-up method with Liouville type theorem

Theorem (A simple example)

For $0 < \alpha < 2$, and $1 < p < \frac{n+\alpha}{n-\alpha}$, suppose

$$u \in L_\alpha \cap C^{1,1}_{loc}(\Omega) \text{ is upper semi-continuous on } \bar{\Omega},$$

and is a positive solution of

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = R(x) u^p(x) + \text{ lower order term}, & x \in \Omega, \\ u(x) \equiv 0, & x \notin \Omega. \end{cases} \tag{3}$$

Assuming $R(x)$ is continuous with $0 < a \leq R(x) \leq b$, then:

$$\|u\|_{L^\infty(\Omega)} \leq C, \tag{4}$$

for some positive constant C independent of u.

Sketch of the proof.

Assume for a sequence of solutions $\{u_k\}$ to (3) such that:

$$u_k(x^k) = \max_{\Omega} u_k := m_k \to \infty.$$

Let $\lambda_k = m_k^{\frac{1-p}{\alpha}}$, $0 \leq v_k = \frac{1}{m_k} u_k(\lambda_k x + x^k) \leq 1$, then

$$(-\Delta)^{\alpha/2} v_k(x) = v_k^p(x),$$

$x \in \Omega_k := \{x \in R^n | x = \frac{y - x^k}{\lambda_k}, y \in \Omega\}$.

Let $d_k = dist(x^k, \partial\Omega)$. Employing the contradiction argument, we exhaust all three possibilities.

Case i) $\lim_{k\to\infty} \frac{d_k}{\lambda_k} = \infty$.
It is clear that

$$\Omega_k \to R^n, \ \text{ as } k \to \infty.$$

We can prove there exists a function v such that, as $k \to \infty$,

$$v_k(x) \to v(x), \ (-\Delta)^{\frac{\alpha}{2}} v_k(x) \to (-\Delta)^{\frac{\alpha}{2}} v(x),$$

thus

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = v^p(x), \ \ x \in R^n. \tag{5}$$

Case ii) $\lim_{k\to\infty} \frac{d_k}{\lambda_k} = C > 0$.
In this case,

$$\Omega_k \to R^n_{+C} := \{x_n \geq -C | x \in R^n\}.$$

Similar to Case i), here we are able to establish the existence of a function v and a subsequence of $\{v_k\}$, such that, as $k \to \infty$,

$$v_k(x) \to v(x), \;\; (-\Delta)^{\frac{\alpha}{2}} v_k(x) \to (-\Delta)^{\frac{\alpha}{2}} v(x),$$

thus

$$(-\Delta)^{\frac{\alpha}{2}} v(x) = v^p(x), \;\; x \in R^n_{+C}. \tag{6}$$

Case iii) $\lim_{k\to\infty} \frac{d_k}{\lambda_k} = 0$.
Impossible via uniform a priori estimates of V_k.

Standard $W^{2,p}$ and $C^{2,\alpha}$ type estimates.

$V_k = 1$ near boundary where it equals 0.

For $1 < p < \frac{n+\alpha}{n-\alpha}$, the Liouville type theorem (non-existence of positive solutions) in the whole space and the half space for the following equation **are known:**

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = u^p(x).$$

Hence, (5) and (6), or cases (i) and (ii) are impossible.
Hence $\Rightarrow$ the **a priori estimate.**

The critical case $p = \frac{n+2s}{n-2s}$ is much harder.

One approach:Liouville type theorem $\Rightarrow$ a priori estimates $\Rightarrow$ existence.
When $p \geq \frac{n+\alpha}{n-\alpha}$, we don't have Liouville type thorem.

Another interesting case: non-existence $\Rightarrow$ existence.

Super-critical HLS non-existence in some bounded domain
$\Rightarrow$ existence in the whole space (via Poincare Map)

Regularity and a priori estimates

The regularity of solutions was extensively studied and many fruitful results was achieved (L. Caffarelli, B. Gidas, L. Nirenberg, J. Spruck, De Giorgi, J. Nash,...)

- Hilbert 19th problems
- Variational problems
- Prescribing scalar curvature equations
- Monge-Ampere equation
- Navier-Stokes equation

Maximum principles are very useful tools .....

Fractional Laplacian

For $u \in C_0^\infty(\mathbb{R}^n)$, $\quad 0 < s < 1$, the fractional Laplacian $(-\Delta)^s u(x)$, is defined as

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}[(2\pi|\xi|)^{2s}\mathcal{F}[u](\xi)](x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy,$$

where $P.V.$ stands for the Cauchy principle value.

One can show that for the above type functions u, it holds that:

$$|(-\Delta)^s u(x)| \leqslant \frac{C}{1+|x|^{n+2s}}. \tag{7}$$

To define $(-\Delta)^s u$ as a distribution, one naturally introduce the following space for u:

$$\mathcal{L}_{2s} = \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \|u\|_{\mathcal{L}_{2s}} := \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}} dy < +\infty \right\}.$$

Then for $u \in \mathcal{L}_{2s}$, $(-\Delta)^s u$ as a distribution is well-defined: $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$,

$$(-\Delta)^s u[\varphi] = \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) dx. \tag{8}$$

Indeed, $u \in \mathcal{L}_{2s} \Longrightarrow$ the integral on the right hand side of (8) converges.
For $u \in \mathcal{L}_{2s}\mathbb{R}^n$, $(-\Delta)^s u$ is also a distribution on $\Omega \subset \mathbb{R}^n$ naturally.
For $f \in L^1_{\mathrm{loc}}(\Omega) \subset D'(\Omega)$, we say

$$(-\Delta)^s u = f \qquad \text{in} \quad \mathcal{D}'(\Omega), \tag{9}$$

if for any test function $\varphi \in C_0^\infty(\Omega)$, it holds that

$$(-\Delta)^s u[\varphi] = \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) dx = \int_\Omega f(x)\varphi(x) dx. \tag{10}$$

For $u \in \mathcal{L}_{2s} \cap C^{1,1}_{\mathrm{loc}}(\Omega)$, $(-\Delta)^s u$ is also pointwisely well-defined on Ω by the formula

$$(-\Delta)^s u(x) = C_{n,s} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \triangleq f(x) \qquad \text{for} \quad x \in \Omega. \tag{11}$$

For $u \in \mathcal{L}_{2s} \cap C^{1,1}_{\mathrm{loc}}(\Omega)$, $f(x) \in C^{\nu}(\Omega_{\mathrm{loc}})$ and the point-wise definition $\Leftrightarrow$ the distributional one:

$$(-\Delta)^s u(x) = f(x) \qquad \text{in} \qquad D'(\Omega). \tag{12}$$

One proof is to approximate u in $\mathcal{L}_{2s} \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ by C_0^{∞} functions in $\mathbb{R}^n$.

1. The fractional Laplacian is a nonlocal operator that captures nonlocal phenomena better.
2. It is the operator associated with the Levy process.
3. It is a promising research filed with many interesting mathematical problems.
4. Many applications in life sciences

For $0 < s < 1$, consider the problem:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{13}$$

Classical solutions $u \in C^{2s}_{loc}(\Omega) \cap C(\mathbb{R}^n)$ for the Dirichlet problem (13) has been well studied.

We start with a special solution:

$$u(x) = \begin{cases} c(n,s) \int_{\partial B_1} \frac{(1-|x|^2)^s}{|x-y|^n} \, d\mathcal{H}_y^{n-1}, & \text{for } x \in B_1; \\ 0, & \text{for } x \in \mathbb{R}^n \backslash B_1. \end{cases} \tag{14}$$

is a non-trivial solution of :

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = 0 & \text{on } \mathbb{R}^n \backslash B_1. \end{cases} \tag{15}$$

One can check that u satisfies:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^s} \int_{\{x \in B_1 | \delta(x) \leqslant \epsilon\}} |u(x)| dx = c > 0. \tag{16}$$

Here and hereafter, we denote, $\delta(x) = \text{dist}(x, \partial\Omega)$.

For fractional Laplacian, the Dirichlet type problem has a uniqueness problem. One naturally want to know:

1. some 'natural' conditions on u to guarantee the uniqueness,
2. some corresponding estimates in the related spaces,
3. $w \notin W^{2s,p}(\Omega)$, what can replace it?

First, we derive a uniqueness condition which is somewhat optimal:

Theorem (Li, Liu preprint)

Let $0 < s < 1$, $\delta^s f \in L^1(B_1)$, then the solution $u \in \mathcal{L}_{2s}$ of (13) that satisfies condition:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^s} \int_{\{x \in B_1 | \delta(x) \leqslant \epsilon\}} |u(x)| dx = 0, \tag{17}$$

exists and must be unique.

Remark

Indeed, in the L^p-theory for the regular Laplacian case:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

The boundary condition $u = 0$ on $\partial\Omega$can be understood as:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{x \in B_1 | \delta(x) \leqslant \epsilon\}} |u(x)| dx = 0. \tag{19}$$

In this sense, the previous theorem can be seen as a fractional generalization.

We also derive some basic estimates for:

$$\begin{cases} (-\Delta)^s u + \vec{b} \cdot \nabla u + cu = f & \text{in } \mathcal{D}'(B_1), \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_1. \end{cases} \tag{20}$$

Theorem (Li, Liu preprint)

$\frac{1}{2} < s < 1$, $1 - s \leqslant r \leqslant s$, $1 \leqslant p < \infty$, $\vec{b}, c \in L^\infty(B_1)$, $c \geqslant 0$ *in* B_1 *and* $f \in L^p_r(B_1)$. *Then* (20) *has a unique solution* $u \in \mathcal{L}_{2s}$ *that satisfies the condition* (17). *Furthermore,*

$$\|\delta^r(-\Delta)^s u\|_{L^p(B_1)} \leqslant C\|\delta^r f\|_{L^p(B_1)}. \tag{21}$$

The derivative of u can also be estimated:

$$\||\nabla u|\|_{L^q_r(B_1)} \leqslant C\|f\|_{L^1_r(B_1)}, \qquad \text{if} \quad p = 1 \quad 1 \leqslant q < \frac{n}{n-2s+1}. \tag{22}$$

$$\||\nabla u|\|_{L_r^{\frac{np}{n-(2s-1)p}}(B_1)} \leqslant C\|f\|_{L^p_r(B_1)}, \qquad \text{if} \quad 1 < p < \frac{n}{2s-1} \tag{23}$$

$$\||\nabla u|\|_{L^\infty_r(B_1)} \leqslant C\|f\|_{L^p_r(B_1)}, \qquad \text{if} \quad p > \frac{n}{2s-1}. \tag{24}$$

The well known Hardy-Littlewood-Sobolev inequality states:

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x-y|^{\lambda}}dxdy \leq C(n,s,\lambda)||f||_r||g||_s$$

where $0<\lambda<n$, $1<s,r<\infty$, $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}=2$, $f\in L^r(R^n)$ and $g\in L^s(R^n)$.

The best constant $C = C(n, s, \alpha)$ is the maximal of:

$$J(f, g) = \int_{R^n} \int_{R^n} f(x)|x-y|^{\alpha-n} g(y) dxdy$$

under the constraints

$$||f||_r = ||g||_s = 1.$$

The above leads us to a system of integral equations on f and g. Let $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, and choose suitable constants c_1 and c_2, we arrive at the Hardy-Littlewood-Sobolev system of the Euler-Lagrange equations for the H-L-S inequality :

The Hardy-Littlewood-Sobolev system (HLS):

$$\begin{cases} u(x) = \int_{R^n} \frac{v^q(y)}{|x-y|^{n-\gamma}} dy, & u > 0, \\ v(x) = \int_{R^n} \frac{u^p(y)}{|x-y|^{n-\gamma}} dy, & v > 0, \end{cases} \tag{25}$$

It corresponds to

$$\begin{cases} (-\Delta)^{\gamma/2} u = v^q, & u > 0, \text{ in } R^n, \\ (-\Delta)^{\gamma/2} v = u^p, & v > 0, \text{ in } R^n, \end{cases} \tag{26}$$

with

$$0 < p < \infty, \; 0 < q < \infty, \; \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\gamma}{n}.$$

Liouville theorems and classifications of solutions

If $p = q = \frac{n+\gamma}{n-\gamma}$, and $u(x) = v(x)$, then (26) reduces to

$$(-\Delta)^{\gamma/2} u = u^{(n+\gamma)/(n-\gamma)}, \quad u > 0, \text{ in } R^n. \tag{27}$$

$\Longleftrightarrow$

$$u(x) = \int_{R^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}} dy, \quad u > 0 \text{ in } R^n. \tag{28}$$

Liouville theorems and classifications of solutions

In particular, when $n \geq 3$, and $\gamma = 2$,

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \text{ in } R^n, \quad n \geq 3 \tag{29}$$

(29) is the 'blowing-up' equation of the curvature equation:

$$-\Delta u + \frac{n(n-2)}{4} R_0 u = \frac{n-2}{4(n-1)} R_1(x) u^{\frac{n+2}{n-2}}, \; x \in M^n, \; n \geq 3$$

There is a similar blowing-up equation to study for $n = 2$.

Liouville theorems and classifications of solutions

The classification of solutions for the following equation is solved with the development of the method of moving planes:

$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, \ x \in \mathbb{R}^n, \ n \geq 3, \tag{30}$$

with

$$u = O(|x|^{2s-n}). \tag{31}$$

Gidas, Ni, and Nirenberg(1981), $s = 1$
Caffarelli, Gidas and Spruck (1989) removed the condition (31) when $s = 1$.
Chen and Li (1992), and Li (1996) simplified their proof.
Wei and Xu (1999) generalized this result to high order for $2s$ is an even integer.

When γ is not an even integer

Theorem (Chen, Li, Ou, 2006, CPAM.)

Every positive regular solution $u(x)$ of

$$u(x) = \int_{R^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}} dy, \quad 0 < \gamma < n, \quad u > 0 \ in \ R^n. \tag{32}$$

is radially symmetric and decreasing about some point x_0 and therefore assumes the form

$$u = C \frac{[n(n-2)\lambda^2]^{\frac{n-2}{4}}}{(\lambda^2 + |x-x_0|^2)^{\frac{n-2}{2}}}$$

with some positive constants C and λ.

Liouville theorems and classifications of solutions

Another similar problem is

$$\begin{cases} -\Delta u = e^u, & x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u dx < +\infty. \end{cases} \tag{33}$$

Theorem (Chen, Li, Duke.)

Every solution of (33) is radially symmetric with respect to some point in $\mathbb{R}^2$ and hence assumes the form of

$$u(x) = \ln \frac{(32\lambda^3)}{(4+\lambda^2|x-x_0|^2)^2}.$$

Here, we present a brief introduction of the method of moving planes:
Consider $u(x)$ a solution to $-\Delta u = f(x,u)$ or simply $f(u)$
$x = (x_1, x_2, ...x-n) = (x_1, x') \in R^n$, $w_\lambda(x) = u_\lambda(x) - u(x)$

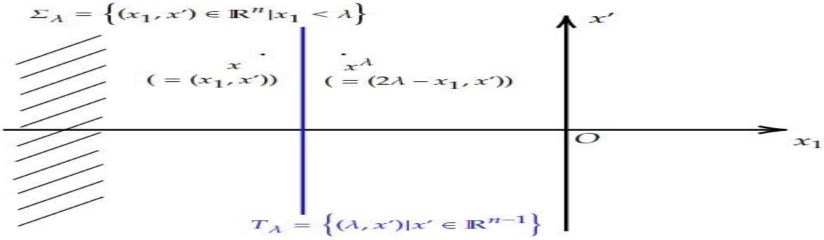
with $u_\lambda(x) = u(x^\lambda)$, $x^\lambda = (2\lambda - x_1, x')$

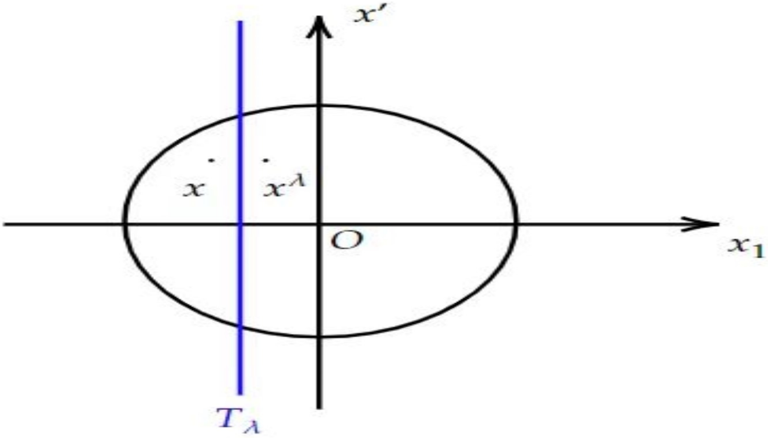
Then $-\Delta w_\lambda(x) = f(x^\lambda, u_\lambda) - f(x,u) \geq f(x, u^\lambda) - f(x,u) = c(x,\lambda) w_\lambda$, or:

$$-\Delta w_\lambda(x) \geq c(x,\lambda) w_\lambda$$

for $x \in \Sigma_\lambda = \{x = (x_1, x') \mid x_1 < \lambda\}$.

Here we use the structure type condition that f is monotone increasing in x_1 before it reaches certain center point.


$\Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 < \lambda\}$
x
$(= (x_1, x'))$
x^λ
$(= (2\lambda - x_1, x'))$
x'
O
x_1
$T_\lambda = \{(\lambda, x') | x' \in \mathbb{R}^{n-1}\}$


x'
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Maximum principle for anti-symmetric solutions:

Theorem (Cheng, Li, Huang, CCM, 2017.)

Let $w(y) \in \mathcal{L}_{2s}$ be a $\lambda-$antisymmetric function. Suppose there exists $x \in \Sigma_\lambda$ such that

$$w(x) = \inf_{\Sigma_\lambda} w(y) \leq 0.$$

If w is $C^{1,1}$ at x, we have

$$(-\Delta)^s w(x) \leq \widetilde{C_{n,s}} \left(\delta^{-2s} w(x) - \delta \int_{\Sigma_\lambda} \frac{(w(y) - w(x))(\lambda - y_1)}{|x - y^\lambda|^{n+2s+2}} dy \right)$$

for some positive constant $\widetilde{C_{n,s}}$, where $\delta = d(x, T_\lambda) = |x_1 - \lambda|$.

Maximum principle for singular solutions:

Theorem (Li, Wu, Xu, PNAS, 2018.)

Assume that $w(x) \in \mathcal{L}_{2s}$, *and satisfies in the sense of distribution*

$$\begin{cases} (-\Delta)^s w(x) + a(x)w(x) \geq 0, \text{ on } B_r(x^0)\backslash\{x^0\}, \\ w(x) \geq m > 0, \text{ on } B_r(x^0)\backslash B_{\frac{r}{2}}(x^0), r \leq 1, \\ w(x) \geq 0, \text{ in } \mathbb{R}^n, \ n \geq 2, \end{cases} \tag{34}$$

Here $a(x) \leq D$ *for some constant* D, *then there exists a positive constant* $c = c(n, s, D)$ *depending on* n, s *and* D *only, such that* $w(x)$ *satisfies in the sense of distribution*

$$w(x) \geq cm, \ x \in B_r(x^0)\backslash\{x^0\}. \tag{35}$$

Liouville theorems for anti-symmetric solutions and existence

We study the anti-symmetric solutions of the following equation involving fractional Laplacian:

$$\begin{cases} (-\Delta)^s u(x) = u^p(x), \;\; u(x) \geq 0, & x \in R^n_+, \\ u(x', -x_n) = -u(x', x_n), & x = (x', x_n) \in R^n, \end{cases} \tag{36}$$

where $s \in (0, 1)$, $R^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in R^n | x_n > 0\}$, and $x' = (x_1, \ldots, x_{n-1})$.

Our first result is the following Liouville type theorem to (36) (C. Li, R. Zhuo, 2022 CVPDE).

Theorem 1

Assuming $0 < p \leq \frac{n+2s}{n-2s}$ and $u(x) \in L_{2s} \cap C^{1,1}_{loc}(R^n_+) \cap C(R^n)$ solves (36), then $u \equiv 0$.

In particular, there exists no bounded non-trivial solution.

Due to the anti-symmetric property, fractional Laplacian can be written in the following form:

$$
\begin{aligned}
&(-\Delta)^s u(x) = \\
= \quad & C_{n,s} P.V.\{ \int_{R^n_+} \Big(\frac{1}{|x-y|^{n+2s}} - \frac{1}{|x^*-y|^{n+2s}} \Big)(u(x)-u(y))dy \\
+ \quad & \int_{R^n_+} \frac{2u(x)}{|x^*-y|^{n+2s}} dy \}.
\end{aligned}
$$

Thus, one can naturally extend the defining domain of u:

$$L_{2s} \Longrightarrow L_{2s+1}.$$

Our second main result study the solutions in the extended class L_{2s+1} (C. Li, R. Zhuo, 2022 CVPDE).

Theorem 2

For $0 < p \leq \frac{n+2s}{n-2s}$, if $u(x) \in L_{2s+1} \cap C^{1,1}_{loc}(R^n_+) \cap C(R^n)$ solves (36), then:

1. when $p + 2s > 1$, $u = 0$ is the only solution,
2. when $p + 2s < 1$, there exist non-trivial solutions to (36).

Theorem

Assume that $w(x) \in \mathcal{L}_{2s}$, *and satisfies in the sense of distribution*

$$\begin{cases} (-\Delta)^s w(x) + a(x)w(x) \geq 0, \ \text{on } B_r(x^0)\backslash\{x^0\}, \\ w(x) \geq m > 0, \ \text{on } B_r(x^0)\backslash B_{\frac{r}{2}}(x^0), r \leq 1, \\ w(x) \geq 0, \ \text{in } \mathbb{R}^n, \ n \geq 2, \end{cases} \tag{37}$$

Here $a(x) \leq D$ *for some constant* D, *then there exists a positive constant* $c = c(n, s, D)$ *depending on* n, s *and* D *only, such that* $w(x)$ *satisfies in the sense of distribution*

$$w(x) \geq cm, \ x \in B_r(x^0)\backslash\{x^0\}. \tag{38}$$

Generalized Bocher theorems

To derive this maximum principle for the nonnegative function with possible singularity at the origin, we need to establish the following Bocher theorem for the fractional Laplacian.

Generalized Bocher theorem for fractional super-harmonic nonnegative functions on a punctuated ball(PNAS， 2018， C.Li,Z.Wu,H.Xu):

Theorem

Let $v(x) \in \mathcal{L}_{2s}$ be a nonnegative solution to

$$(-\Delta)^s v(x) + c(x)v(x) = f(x) \geq 0 \quad \text{on } B_1(0)\backslash\{0\} \tag{39}$$

for some $f(x) \in L^1_{\text{loc}}(B_1(0)\backslash\{0\})$ and $c(x) \leq D$ with some constant D, then

$$\begin{aligned} &(i) \qquad v(x),\ f(x) \in L^1_{\text{loc}}(B_1(0)), \\ &(ii) \qquad (-\Delta)^s v(x) + c(x)v(x) = f(x) + a\delta_0 \\ &\qquad\qquad \text{on } B_1(0),\ \textit{for a constant } a \geq 0, \end{aligned} \tag{40}$$

where δ_0 is the Delta distribution concentrated at the origin, and all the inequalities and identities are in the sense of distribution.

Generalized Bocher theorems

Equations with first order term:

Theorem

(Bôcher theorem for fractional Laplacian) *Let* $u(x) \in \mathcal{L}_{2s}$ *with* $s \in (\frac{1}{2}, 1)$ *be a nonnegative function in* $\mathbb{R}^n$ $(n \geq 2)$ *satisfying*

$$(-\Delta)^s u(x) + \vec{b}(x) \cdot \nabla u(x) + c(x)u(x) \geq 0 \ \text{ in } \ \mathcal{D}'(B_1 \backslash \{0\}), \tag{41}$$

where $\|\vec{b}(x)\|_{C^1(B_1)} + \|c(x)\|_{L^\infty(B_1)} \leq M$ *for some constant* M*, then* $u(x) \in L^1_{\mathrm{loc}}(B_1)$ *and*

$$(-\Delta)^s u(x) + \vec{b}(x) \cdot \nabla u(x) + c(x)u(x) = \mu + a\delta_0(x) \ \text{ in } \mathcal{D}'(B_1), \tag{42}$$

for some constant $a \geq 0$ *and some nonnegative Radon measure* μ *on* B_1 *satisfying* $\mu(\{0\}) = 0$.

Generalized Bocher theorems

Besides, when $\vec{b}(x) \equiv 0$ in (41), then the theorem holds for $s \in (0,1)$.

The classical Bocher theorems for the Laplacian:

Bôcher theorem(1903): *If $v(x)$ is nonnegative and harmonic on $B_1(0) \backslash \{0\}$, then there is a constant $a \geq 0$ such that for all $x \in B_1(0) \backslash \{0\} \subset \mathbb{R}^n$ with $n \geq 2$ that*

$$\begin{cases} (i) & v(x) \text{ is integrable on } B_1(0), \\ (ii) & -\Delta v(x) = a\delta_0, \end{cases} \tag{43}$$

where δ_0 is the Delta distribution concentrated at the origin.

Generalized Bocher theorems

H. Brézis and P. Lions (1981) obtained another Bôcher type theorem for super-harmonic functions:

Let $v(x) \in L^1_{\mathrm{loc}}(B_1(0)\backslash\{0\})$, $v(x) \geq 0$ *a.e. in* $B_1(0)$ *be such that*

$$\Delta v(x) \in L^1_{\mathrm{loc}}(B_1(0)\backslash\{0\}) \text{ in the sense of distribution on } B_1(0)\backslash\{0\},$$

$$-\Delta v(x) \geq -Dv(x) - f(x),\ D > 0,\ \ a.e. \text{ in } B_1(0),\ \text{with } f \in L^1_{\mathrm{loc}}(B_1(0)).$$

Then $v(x) \in L^1_{\mathrm{loc}}(B_1(0))$ *and there exist* $\phi(x) \in L^1_{\mathrm{loc}}(B_1(0))$ *and* $a \geq 0$ *such that*

$$-\Delta v(x) = \phi(x) + a\delta_0,\ \text{in } \mathcal{D}'(B_1(0)).$$

Generalized Bocher theorems

Proof of Theorem (MPFL2)

We only need to consider the special case $a(x) \equiv D$ and $D \geq 0$.
First, we consider the case that $w(x)$ is smooth and $r = 1$.
Then, when $w(x)$ is not smooth, we consider $w_\epsilon(x) = w * \rho_\epsilon \in C^\infty(B_{1-\epsilon}(0))$, where ρ_ϵ is the standard mollifier.
From Theorem 10 (Bocher Theorem) and applying the mollification process in a suitable way, we have $(-\Delta)^s w_\epsilon(x) + D w_\epsilon(x) \geq 0$ in $B_{1-\epsilon}(x^0)$.

Generalized Bocher theorems

Then using the conclusion in the first step, we know there exist suitable positive constants c and $\tilde{c}$ satisfying $0 < c < \tilde{c} < 1$ such that $w_\epsilon(x) \geq \tilde{c} m_\epsilon \geq cm$ in $B_{1-\epsilon}(x^0)$, where $c > 0$ is independent of ϵ.
Letting $\epsilon \to 0$, we can immediately derive $w(x) \geq cm$ with some $c > 0$, when $x \in B_1(x^0) \backslash \{x^0\}$.

Generalized Bocher theorems

Finally, making scaling $\bar{w}(x) = w(rx + x_0)$, we know from the first step that if

$$\begin{cases} \bar{w}(x) \geq 0, \ x \in \mathbb{R}^n, \\ (-\Delta)^s \bar{w}(x) + D\bar{w}(x) \geq 0, \ x \in B_1(0) \backslash \{0\}, \\ \bar{w}(x) \geq m > 0, \ x \in B_1(0) \backslash B_{\frac{1}{2}}(0), \end{cases} \tag{44}$$

then there exists some positive constant c depending on n and s only such that

$$\bar{w}(x) \geq cm, \ \text{in } B_1\Big(0\Big)\backslash\Big\{0\}\Big\}.$$

This completes the proof of the theorem.

Thank you!