Sobolev spaces revisited

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Introduction

- We will work on $\mathbb{R}^n$ with $n \geq 1$.
- For $1 \leq p < \infty$, the homogeneous Sobolev space $\dot{W}^{1,p}$ consists of all $u \in L^1_{\text{loc}}$ modulo constants, whose distributional gradient $\nabla u \in L^p$. It is normed by

$$\|\nabla u\|_{L^p} = \Big(\int_{\mathbb{R}^n} |\nabla u|^p dx \Big)^{1/p}.$$

- The homogeneous $\dot{\text{BV}}$ space (BV = bounded variation) consists of all $u \in L^1_{\text{loc}}$ modulo constants, whose distributional gradient ∇u is a finite Radon measure (written $\nabla u \in \mathcal{M}$). In other words, it is the space of all $u \in L^1_{\text{loc}}$ such that

$$\sup \left\{ \Big| \int_{\mathbb{R}^n} u(x) \text{div}\, \phi(x) dx \Big| : \phi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), \|\phi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \leq 1 \right\}$$

is finite (which in particular contains $\dot{W}^{1,1}$). It is normed by

$$\|u\|_{\dot{\text{BV}}} = \|\nabla u\|_{\mathcal{M}}.$$

- In joint work with Haïm Brezis and Jean Van Schaftingen, we established a new formula for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ for $u \in C_c^\infty$ that involves only difference quotients and no gradients.
- Together with Andreas Seeger, this formula is extended to all $u \in \dot{W}^{1,p}$, or all $u \in \dot{BV}$, and in fact we found a natural one-parameter family of such formulae.
- Such one-parameter family of formulae can be used to recover certain Gagliardo-Nirenberg interpolation inequalities due to Cohen, Dahmen, Daubechies and DeVore.
- It also allows us to go beyond the standard range and prove some substitutes when such inequalities fail.
- Our formula for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ is given in terms of a weak-L^p (quasi)norm on the product space $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, so let's begin by reviewing the notion of weak-L^p.

L^p versus weak-L^p

- For $1 \le p < \infty$, if $f \in L^p(\nu)$ for some measure ν, then

$$\|f\|_{L^p(\nu)}^p = \int |f|^p d\nu \ge \lambda^p \nu\{x \colon |f(x)| > \lambda\} \quad \forall \lambda > 0.$$

  In particular, if $f \in L^p(\nu)$, then

$$\sup_{\lambda > 0} \left(\lambda \nu\{x \colon |f(x)| > \lambda\}^{1/p} \right) < \infty$$

  but the converse is not necessarily true.
- If f is measurable and the supremum above is finite, then f is said to be in weak-$L^p(\nu)$. Its weak-L^p (quasi)-norm is defined as the above supremum, and denoted by $[f]_{L^{p,\infty}(\nu)}$.
- Example: $f(x) = |x|^{-n/p}$ is in weak-$L^p(dx)$ on $\mathbb{R}^n$, because

$$\mathcal{L}^n\{x \in \mathbb{R}^n \colon |x|^{-n/p} > \lambda\} = \mathcal{L}^n(B(0, \lambda^{-p/n})) = \lambda^{-p} \mathcal{L}^n(B(0,1)).$$

  (Henceforth we write $\mathcal{L}^n$ for Lebesgue measure on $\mathbb{R}^n$.)
  It is not in $L^p(dx)$, because $\int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} |x|^{-n} dx = +\infty$.

Modified difference quotients

- Write $\Delta_h u(x) := u(x+h) - u(x)$ for $x, h \in \mathbb{R}^n$.
- If we believe that
$$|\nabla u(x)| \simeq \frac{|\Delta_h u(x)|}{|h|},$$
then to express $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ using a difference quotient instead of a gradient, a naive guess might be to try
$$\iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u(x)|^p}{|h|^p} dh dx \quad \text{in place of} \quad \int_{\mathbb{R}^n} |\nabla u(x)|^p dx.$$
- Not working, because it doesn't scale upon $u(x) \mapsto u(tx)$.
- A proper scaling will be achieved if we consider
$$\iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u(x)|^p}{|h|^p} \frac{dh dx}{|h|^n}$$
instead, which is $\iint_{\mathbb{R}^{2n}} \mathcal{Q}_{1+\frac{n}{p}} u(x,h)^p dh dx$ if
$$\mathcal{Q}_b u(x,h) := \frac{|\Delta_h u(x)|}{|h|^b}.$$

Fractional Sobolev spaces

$$\mathcal{Q}_b u(x,h) := \frac{|\Delta_h u(x)|}{|h|^b} = \frac{|u(x+h)-u(x)|}{|h|^b}$$

- Indeed, for $0 < s < 1$ and $1 \leq p < \infty$, a fractional Sobolev space $\dot{W}^{s,p}$ can be defined as the space of all $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|u\|^p_{\dot{W}^{s,p}} := \iint_{\mathbb{R}^{2n}} \mathcal{Q}_{s+\frac{n}{p}} u(x,h)^p dh dx < \infty.$$

  When $1 < p < \infty$, it is known to be equal to the diagonal Besov space $\dot{B}^s_{p,p}$ with comparable norms.
- So this suggests again that maybe $\|\nabla u\|_{L^p(\mathbb{R}^n)}$ should be compared to $\|\mathcal{Q}_{1+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dxdh)}$?
- Not working; for $u \in C^\infty_c(\mathbb{R}^n)$, unless $u \equiv 0$, the L^p norm on $\mathbb{R}^{2n}$ is always infinite! (Issue: $|h|^{-n/p} dh$ is not L^p on $\mathbb{R}^n$).

The BBM formula

- A famous formula by Bourgain, Brezis and Mironescu (BBM) explores what happens to $\|u\|_{\dot{W}^{s,p}}$ as $s \to 1^-$.
- On $\mathbb{R}^n$, it says for $1 \le p < \infty$ and (say) $u \in C_c^2$, we have

$$\lim_{s \to 1^-} (1-s)\|u\|^p_{\dot{W}^{s,p}} = \frac{k(p,n)}{p} \|\nabla u\|^p_{L^p}$$

  where $k(p,n)$ is some explicit constant depending on p and n, given by $k(p,n) := \int_{\mathbb{S}^{n-1}} |e \cdot \omega|^p d\omega$ and $e \in \mathbb{S}^{n-1}$.
  (A related result of Maz'ya and Shaposhnikova computed $\|u\|^p_{L^p}$ by considering $\lim_{s \to 0^+} s\|u\|^p_{\dot{W}^{s,p}}$.)
- In particular, $\|\mathcal{Q}_{s+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dxdh)}$ blows up like $(1-s)^{-1/p}$ as $s \to 1^-$ unless u is a constant, another indication that $\|\mathcal{Q}_{1+\frac{n}{p}} u\|_{L^p(\mathbb{R}^{2n}, dxdh)}$ is not good for computing $\|\nabla u\|_{L^p}$.
- Our first main result offers an alternative point of view, that does not involve varying s, but involves a weak-L^p norm instead of the L^p norm on $\mathbb{R}^{2n}$.
- Remember $|h|^{-n/p}$ is not in $L^p(dh)$, but it is in weak-$L^p(dh)$.

A formula for $\|\nabla u\|_{L^p}$

Theorem (Brezis, Van Schaftingen, Yung)

Let $n \geq 1$, $1 \leq p < \infty$ and $u \in C_c^\infty(\mathbb{R}^n)$. Then

$$\|\nabla u\|_{L^p} \simeq [\mathcal{Q}_{1+\frac{n}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)} = \left[\frac{\Delta_h u}{|h|^{1+\frac{n}{p}}} \right]_{L^{p,\infty}(\mathbb{R}^{2n}, dxdh)}.$$

In other words, for $\lambda > 0$, denote by

$$E_\lambda := \left\{ (x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{n}{p}} u(x,h) > \lambda \right\}$$

the superlevel set of $\mathcal{Q}_{1+\frac{n}{p}} u$ at height λ. Then

$$\|\nabla u\|_{L^p}^p \simeq \sup_{\lambda > 0} \left(\lambda^p \mathcal{L}^{2n}(E_\lambda) \right).$$

In fact, we also have $\dfrac{k(p,n)}{n} \|\nabla u\|_{L^p}^p = \lim\limits_{\lambda \to +\infty} \left(\lambda^p \mathcal{L}^{2n}(E_\lambda) \right)$.

Comments

- The power $1+\frac{n}{p}$ is dictated by dilation invariance: if
$$[\mathcal{Q}_b u]_{L^{p,\infty}(\mathbb{R}^{2n},dxdh)} \text{ scales like } \|\nabla u\|_{L^p}$$
upon replacing $u(x)$ by $u(tx)$ for $t>0$, then $b=1+\frac{n}{p}$.
- In light of the limit equality
$$\frac{k(p,n)}{n}\|\nabla u\|_{L^p}^p = \lim_{\lambda\to+\infty}\Big(\lambda^p \mathcal{L}^{2n}(E_\lambda)\Big)$$
where $E_\lambda := \Big\{(x,h)\in\mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{n}{p}}u(x,h)>\lambda\Big\}$, we only need to prove an upper bound for a weak-L^p norm, namely
$$[\mathcal{Q}_{1+\frac{n}{p}}u]_{L^{p,\infty}(\mathbb{R}^{2n},dxdh)} \lesssim \|\nabla u\|_{L^p},$$
which can be done using a Vitali covering lemma (c.f. proof that the Hardy-Littlewood maximal function is bounded from L^1 to weak-L^1; see also work of Dai, Lin, Yang, Yuan and Zhang who extended our proof to metric-measure spaces).
- The limit equality can be proved using Taylor expansion, somewhat reminiscent to the proof of the BBM formula.

A family of formulae for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$

- It turns out there is a natural one-parameter family of such formulae for $\|\nabla u\|_{L^p(\mathbb{R}^n)}$, for general $u \in \dot{W}^{1,p}$ or $u \in \dot{BV}$.
- Let $\gamma \in \mathbb{R}$. Define the measure $d\nu_\gamma = |h|^{\gamma-n} dx dh$ on $\mathbb{R}^{2n}$. (The case $\gamma = n$ corresponds to the Lebesgue measure $dxdh = \mathcal{L}^{2n}$ we used earlier.)

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

Let $n \geq 1$, $1 < p < \infty$ and $u \in \dot{W}^{1,p}(\mathbb{R}^n)$. Then for $\gamma \neq 0$,

$$\|\nabla u\|_{L^p} \simeq [\mathcal{Q}_{1+\frac{\gamma}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} = \left[\frac{\Delta_h u}{|h|^{1+\frac{\gamma}{p}}} \right]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)}.$$

Furthermore, if $E_\lambda := \left\{ (x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{\gamma}{p}} u(x,h) > \lambda \right\}$, then

$$\frac{k(p,n)}{|\gamma|} \|\nabla u\|_{L^p}^p = \begin{cases} \lim_{\lambda \to +\infty} \left(\lambda^p \nu_\gamma(E_\lambda) \right) & \text{if } \gamma > 0 \\ \lim_{\lambda \to 0^+} \left(\lambda^p \nu_\gamma(E_\lambda) \right) & \text{if } \gamma < 0. \end{cases}$$

(The case $\gamma = -p$ of the limit equality is due to Nguyen.)

- For $p=1$ we have a similar theorem for $\dot{BV}$, but with a number of additional twists!

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

Let $n \geq 1$, $u \in \dot{BV}(\mathbb{R}^n)$. Then for $\gamma \in \mathbb{R} \setminus [-1, 0]$,

$$\|u\|_{\dot{BV}} = \|\nabla u\|_{\mathcal{M}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)} = \left[\frac{\Delta_h u}{|h|^{1+\gamma}}\right]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)}.$$

Furthermore, if $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\gamma} u(x,h) > \lambda\right\}$, then the formula

$$\frac{k(1,n)}{|\gamma|}\|\nabla u\|_{\mathcal{M}} = \begin{cases} \lim_{\lambda \to +\infty} \Big(\lambda \nu_\gamma(E_\lambda)\Big) & \text{if } \gamma > 0 \\ \lim_{\lambda \to 0^+} \Big(\lambda \nu_\gamma(E_\lambda)\Big) & \text{if } \gamma < -1 \end{cases}$$

holds for $u \in \dot{W}^{1,1}$ but can fail for $u \in \dot{BV}$ (e.g. if $u = \mathbf{1}_\Omega$ where $\Omega \subset \mathbb{R}^n$ is any bounded domain with smooth boundary, then the limits above exist but is equal instead to $\frac{k(1,n)}{|\gamma+1|}\|\nabla u\|_{\mathcal{M}}$).

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

For $\gamma \in [-1, 0)$,

$$\sup_{u \in C_c^\infty(\mathbb{R}^n), \|\nabla u\|_{L^1(\mathbb{R}^n)}=1} [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\mathbb{R}^{2n}, \nu_\gamma)} = +\infty;$$

furthermore, the formula

$$\frac{k(1,n)}{|\gamma|} \|u\|_{\dot{BV}} = \lim_{\lambda \to 0^+} \Big(\lambda \nu_\gamma(E_\lambda) \Big)$$

remains true for all $u \in C_c^1(\mathbb{R}^n)$, but fails for $u \in \dot{W}^{1,1}(\mathbb{R}^n)$, and the failure is generic in the sense of Baire category.

- The case $\gamma = -1$ of the limiting formula has already been established by Brezis and Nguyen.
- The failure of the limiting formula in the case $-1 < \gamma < 0$ relies on the construction of a Cantor set of dimension $1 + \gamma$.
- The previous two theorems assumed $u \in \dot{W}^{1,p}$ or $u \in \dot{\mathrm{BV}}$ to begin with. Using the BBM formula, we also proved a *characterization* of $\dot{W}^{1,p}$ $(1 < p < \infty)$ and $\dot{\mathrm{BV}}$:

Theorem (Brezis, Seeger, Van Schaftingen, Yung)

Let $n \geq 1$, $u \in L^1_{loc}(\mathbb{R}^n)$, $\gamma \in \mathbb{R}$. *If* $[\mathcal{Q}_{1+\frac{\gamma}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty$, *then*

$$u \in \begin{cases} \dot{W}^{1,p}(\mathbb{R}^n) & \text{if } 1 < p < \infty \\ \dot{BV}(\mathbb{R}^n) & \text{if } p = 1. \end{cases}$$

- In particular, for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 < p < \infty$ and $\gamma \neq 0$,

$$u \in \dot{W}^{1,p} \quad \Longleftrightarrow \quad \left[\frac{\Delta_h u}{|h|^{1+\frac{\gamma}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty.$$

- Similarly, for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\gamma \in \mathbb{R} \setminus [-1, 0]$,

$$u \in \dot{\text{BV}} \quad \Longleftrightarrow \quad \left[\frac{\Delta_h u}{|h|^{1+\gamma}}\right]_{L^{1,\infty}(\mathbb{R}^{2n},\nu_\gamma)} < \infty.$$

- The existence of a one-parameter family of characterizations is not just natural, but useful in applications.

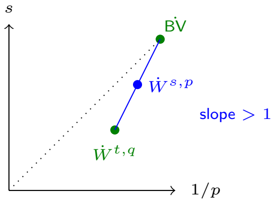
Application towards Gagliardo-Nirenberg interpolation

- Cohen, Dahmen, Daubechies and DeVore proved that for any $0 < t < 1$ and any $1 < q < \infty$, if

$$t < \frac{1}{q},$$

and if $(\frac{1}{p}, s) = (1-\theta)(\frac{1}{q}, t) + \theta(1,1)$ for some $0 < \theta < 1$, then for any $u \in \dot{\mathrm{BV}} \cap \dot{W}^{t,q}$,

$$\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{t,q}}^{1-\theta} \|u\|_{\dot{\mathrm{BV}}}^{\theta}.$$



- Their proof uses bounds for coefficients of wavelet expansions of a general function in $\dot{\mathrm{BV}}(\mathbb{R}^n)$.
- We can give an alternative proof based on our theorem for $\dot{\mathrm{BV}}$.

- Indeed, let γ be minus the slope, given by $\gamma := -\frac{1-t}{1-\frac{1}{q}} < -1$.
- Let $u \in \dot{\mathrm{BV}} \cap \dot{W}^{t,q}$. Our characterization for $\dot{\mathrm{BV}}$ shows that

$$\|u\|_{\dot{\mathrm{BV}}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}.$$

- On the other hand, $\|u\|_{\dot{W}^{t,q}} = \|\mathcal{Q}_{t+\frac{\gamma}{q}} u\|_{L^q(\nu_\gamma)}$ because

$$\Big(\iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u|^q}{|h|^{tq+n}} dx dh\Big)^{\frac{1}{q}} = \Big(\iint_{\mathbb{R}^{2n}} \frac{|\Delta_h u|^q}{|h|^{tq+\gamma}} d\nu_\gamma\Big)^{\frac{1}{q}}.$$

  Similarly $\|u\|_{\dot{W}^{s,p}} = \|\mathcal{Q}_{s+\frac{\gamma}{p}} u\|_{L^p(\nu_\gamma)}$.
- But our choice of γ ensures $s + \frac{\gamma}{p} = t + \frac{\gamma}{q} = 1 + \gamma$. Using

$$\|F\|_{L^p(\nu_\gamma)} \lesssim \|F\|_{L^q(\nu_\gamma)}^{1-\theta} [F]_{L^{1,\infty}(\nu_\gamma)}^{\theta}$$

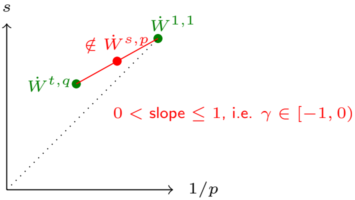
  for $F := \mathcal{Q}_{s+\frac{\gamma}{p}} u = \mathcal{Q}_{t+\frac{\gamma}{q}} u = \mathcal{Q}_{1+\gamma} u$, we obtain

$$\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{t,q}}^{1-\theta} \|u\|_{\dot{\mathrm{BV}}}^{\theta}.$$

- Let's revisit the result of Cohen-Dahmen-Daubechies-DeVore.
- Suppose $0 < t < 1$, $1 < q < \infty$, and

$$(\frac{1}{p}, s) = (1-\theta)(\frac{1}{q}, t) + \theta(1,1) \quad \text{for some } 0 < \theta < 1.$$

- We saw if $t < \frac{1}{q}$ then $\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{t,q}}^{1-\theta} \|u\|_{\dot{\mathrm{BV}}}^{\theta}$.
- The previous proof made crucial use of $t < \frac{1}{q}$, because $\|u\|_{\dot{\mathrm{BV}}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}$ only holds when $\gamma \notin \mathbb{R} \setminus [-1, 0]$.
- In fact the result is false when $t \geq \frac{1}{q}$ (Brezis-Mironescu).



- Let's revisit the result of Cohen-Dahmen-Daubechies-DeVore.
- Suppose $0 < t < 1$, $1 < q < \infty$, and

$$(\frac{1}{p}, s) = (1-\theta)(\frac{1}{q}, t) + \theta(1,1) \quad \text{for some } 0 < \theta < 1.$$

- We saw if $t < \frac{1}{q}$ then $\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{t,q}}^{1-\theta} \|u\|_{\dot{\mathrm{BV}}}^{\theta}$.
- The previous proof made crucial use of $t < \frac{1}{q}$, because $\|u\|_{\dot{\mathrm{BV}}} \simeq [\mathcal{Q}_{1+\gamma} u]_{L^{1,\infty}(\nu_\gamma)}$ only holds when $\gamma \notin \mathbb{R} \setminus [-1,0]$.
- In fact the result is false when $t \geq \frac{1}{q}$ (Brezis-Mironescu).
- Nevertheless, the above proof can be easily adapted, to show that for any $\gamma' \in \mathbb{R} \setminus [-1,0]$, we still have

$$[\mathcal{Q}_{s+\frac{\gamma'}{p}} u]_{L^{p,r}(\nu_{\gamma'})} \lesssim \|u\|_{\dot{W}^{t,q}}^{1-\theta} \|u\|_{\dot{\mathrm{BV}}}^{\theta}, \quad r := \frac{q}{1-\theta}$$

(See joint work with Brezis and Van Schaftingen.)

A formula for L^p norm

- In place of $\|\nabla u\|_{L^p(\mathbb{R}^n)}$, one can also obtain a similar formula for $\|u\|_{L^p(\mathbb{R}^n)}$. Recall the measure $\nu_\gamma = |h|^{\gamma-n} dx dh$ on $\mathbb{R}^{2n}$.

Theorem

Let $n \geq 1$, $1 \leq p < \infty$ and $u \in L^p(\mathbb{R}^n)$. Then for $\gamma \neq 0$,

$$\|u\|_{L^p} \simeq [\mathcal{Q}_{\frac{\gamma}{p}} u]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)} = \left[\frac{\Delta_h u}{|h|^{\frac{\gamma}{p}}}\right]_{L^{p,\infty}(\mathbb{R}^{2n},\nu_\gamma)}.$$

Furthermore, if $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{\frac{\gamma}{p}} u(x,h) > \lambda\right\}$, then

$$\frac{2\sigma_{n-1}}{|\gamma|} \|u\|_{L^p}^p = \begin{cases} \lim_{\lambda \to 0^+} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma > 0 \\ \lim_{\lambda \to \infty} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma < 0. \end{cases}$$

where σ_{n-1} is the surface area of $\mathbb{S}^{n-1}$.

- The case $\gamma = n$ is joint work with Qingsong Gu.
- We do not obtain a characterization of $L^p(\mathbb{R}^n)$: the weak-L^p norms are finite when u is a non-zero constant.

Abstract extensions

- Recently, Óscar Domínguez and Mario Milman put some of the above results for $\dot{W}^{1,p}$ in an abstract framework.
- They proved that if X is a σ-finite measure space, $1 \leq p < \infty$ and $\{T_t\}_{t>0}$ is a family of sublinear operators on $L^p(X)$, then for all $f \in L^p(X)$ satisfying

$$\|T_t f - f\|_{L^\infty(X)} \lesssim_f t^{1/p} \quad \text{for all } t > 0,$$

we have

$$\lim_{\lambda \to \infty} \left(\lambda |E_\lambda|^{1/p} \right) = \|f\|_{L^p(X)},$$

where

$$E_\lambda := \left\{ (x,t) \in X \times (0,\infty) \colon \frac{|T_t f(x)|}{t^{1/p}} > \lambda \right\}.$$

- They found an impressive list of applications, from a computation of $\|\Delta u\|_{L^p(\mathbb{R}^n)}$ and $\|\partial_{x_1}\partial_{x_2} u\|_{L^p(\mathbb{R}^2)}$, to relations between $\|f\|_{L^p(\mathbb{R}^n)}$ with level set estimates for spherical averages of f for $p > \frac{n}{n-1}$, to ergodic theory, etc.

Some further questions

- We have seen that if $1 < p < \infty$ and $u \in \dot{W}^{1,p}$, then for $\gamma \neq 0$ and $E_\lambda := \left\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\frac{\gamma}{p}} u(x,h) > \lambda\right\}$,

$$\frac{k(p,n)}{|\gamma|} \|\nabla u\|_{L^p}^p = \begin{cases} \lim_{\lambda \to +\infty} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma > 0 \\ \lim_{\lambda \to 0^+} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma < 0. \end{cases}$$

- What if $u \in L^1_{\text{loc}}$ but is not in $\dot{W}^{1,p}$? Is it true that

$$+\infty = \begin{cases} \liminf_{\lambda \to +\infty} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma > 0 \\ \liminf_{\lambda \to 0^+} \left(\lambda^p \nu_\gamma(E_\lambda)\right) & \text{if } \gamma < 0? \end{cases}$$

  In other words, can we characterize $\dot{W}^{1,p}$ by the finiteness of the $\liminf$s above? (We can if $\liminf$ is replaced by $\sup_{\lambda > 0}$.)

- Brezis and Nguyen showed that the answer to this question is positive if $\gamma = -p$.

- For $p = 1$, we have been able to prove that if $u \in \dot{W}^{1,1}$, then for $\gamma \neq 0$ and $E_\lambda := \Big\{(x,h) \in \mathbb{R}^{2n} \colon \mathcal{Q}_{1+\gamma} u(x,h) > \lambda\Big\}$,

$$\|\nabla u\|_{\mathcal{M}} \lesssim_{n,\gamma} \begin{cases} \liminf_{\lambda \to +\infty} \Big(\lambda \nu_\gamma(E_\lambda)\Big) & \text{if } \gamma > 0 \\ \liminf_{\lambda \to 0^+} \Big(\lambda \nu_\gamma(E_\lambda)\Big) & \text{if } \gamma < 0 \end{cases}$$

  (even though the limit equality can fail when $-1 \leq \gamma < 0$; see joint work with Brezis, Seeger and Van Schaftingen).
- Does the above $\liminf$ inequalities remain true for $u \in \dot{\mathrm{BV}}$?
- Would these $\liminf$s be infinite if $u \in L^1_{\mathrm{loc}} \setminus \dot{\mathrm{BV}}$?
- Nguyen showed that the answers to these two questions are positive if $\gamma = -1$ (see also Brezis-Nguyen for extensions).
- Poliakovsky established positive results for the case $\gamma = n$ if $\liminf_{\lambda \to +\infty}$ is replaced by $\limsup_{\lambda \to +\infty}$.