Combinatorial bases for representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$

Alexander Molev

University of Sydney

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$L(\lambda)$ contains a highest vector $\zeta \neq 0$ such that

$$E_{ii}\,\zeta = \lambda_i\,\zeta \qquad \text{for} \quad i = 1, \dots, n \quad \text{and}$$

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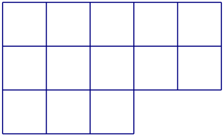

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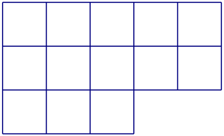

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The number of nonzero rows is the length of λ, denoted $\ell(\lambda)$.

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1	1	2	4	4
2	3	4	5	5
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Theorem (Gelfand and Tsetlin, 1950). $L(\lambda)$ admits a basis ζ_T parameterized by all column-strict λ-tableaux T such that the action of generators of $\mathfrak{gl}_n$ is given by the formulas

$$E_{ss}\,\zeta_T = \omega_s\,\zeta_T,$$

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Here ω_s is the number of entries in T equal to s, and the sums are taken over column-strict tableaux T' obtained from T respectively by replacing an entry $s+1$ by s and s by $s+1$.

For any $1 \leqslant j \leqslant s \leqslant n$ denote by λ_{sj} the number of entries in row j which do not exceed s and set

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Then

$$c_{TT'} = -\frac{(l_{si} - l_{s+1,1}) \dots (l_{si} - l_{s+1,s+1})}{(l_{si} - l_{s1}) \dots \wedge \dots (l_{si} - l_{ss})},$$

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if the replacement occurs in row i.

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$$T \longrightarrow \begin{array}{ccccccccc} \lambda_{n1} & & \lambda_{n2} & & \cdots & & \cdots & & \lambda_{nn} \\ & \lambda_{n-1,1} & & & \cdots & & & \lambda_{n-1,n-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{21} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array}$$

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& & & & \lambda_{11} & & & &
\end{array}
$$

The top row coincides with λ and the entries satisfy

the betweenness conditions $\lambda_{k\,i} \geqslant \lambda_{k-1,\,i} \geqslant \lambda_{k,\,i+1}$.

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corresponds to the pattern

$$\begin{array}{ccccccccc} 5 & & 5 & & 3 & & 0 & & 0 \\ & 5 & & 3 & & 1 & & 0 & \\ & & 3 & & 2 & & 0 & & \\ & & & 3 & & 1 & & & \\ & & & & 2 & & & & \end{array}$$

Given $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n$, consider the weight subspace

$$L(\lambda)_\omega = \{\eta \in L(\lambda) \mid E_{ss}\eta = \omega_s\eta \quad \text{for all} \quad s\}.$$

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The character of $L(\lambda)$ is the polynomial in variables $x_1, \dots, x_n$ defined by

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Corollary. $\operatorname{ch} L(\lambda) = s_\lambda(x_1, \dots, x_n)$, the Schur polynomial.

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The $\mathbb{Z}_2$-degree (or parity) is given by

$$\deg(E_{ij}) = \bar{\imath} + \bar{\jmath},$$

where $\bar{\imath} = 0$ for $1 \leqslant i \leqslant m$ and $\bar{\imath} = 1$ for $m + 1 \leqslant i \leqslant m + n$.

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The commutation relations in $\mathfrak{gl}_{m|n}$ have the form

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})},$$

where the square brackets denote the super-commutator.

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the Lie subalgebra of even elements of $\mathfrak{gl}_{m|n}$ is isomorphic to

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The corresponding representation $L(\lambda)$ contains a highest vector $\zeta \neq 0$ such that

$$E_{i\,i}\, \zeta = \lambda_i\, \zeta \qquad \text{for} \quad i = 1, \dots, m+n \quad \text{and}$$

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- the number ℓ of nonzero components among

  $\lambda_{m+1}, \ldots, \lambda_{m+n}$ is at most λ_m.

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It is determined by the conditions that the first m rows of Γ_λ are $\lambda_1, \ldots, \lambda_m$ while the first ℓ columns are $\lambda_{m+1} + m, \ldots, \lambda_{m+\ell} + m$.

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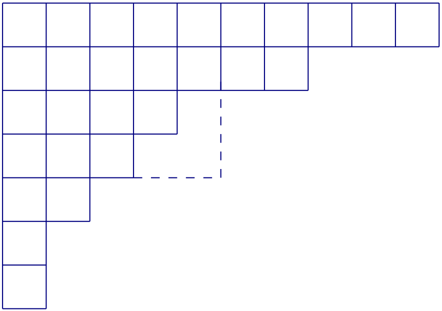
The condition $\ell \leqslant \lambda_m$ ensures that Γ_λ is the diagram of a partition.

Example. The following is the diagram Γ_λ associated with the highest weight $\lambda = (10, 7, 4, 3 \,|\, 3, 1, 0, 0, 0)$ of $\mathfrak{gl}_{4|5}$:



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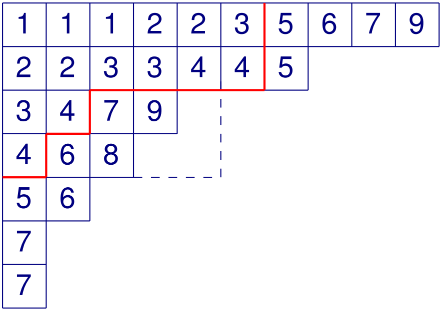

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$$E_{ss}\,\zeta_\Lambda = \omega_s\,\zeta_\Lambda,$$

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Corollary (Sergeev 1985, Berele and Regev 1987). The character $\mathrm{ch}\, L(\lambda)$ coincides with the supersymmetric Schur polynomial $s_{\Gamma_\lambda}(x_1, \dots, x_m \,|\, x_{m+1}, \dots, x_{m+n})$ associated with the Young diagram Γ_λ.

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Set $r = \lambda_{m1}$ and for any $0 \leqslant p \leqslant n$ and $1 \leqslant j \leqslant r + p$ denote by $\lambda'_{r+p,j}$ the number of entries in column j which do not exceed $m + p$.

Example. The supertableau with $\lambda = (7, 5, 2 \mid 2, 1)$

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corresponds to the patterns $\mathcal{U}$ and $\mathcal{V}$:

$$\begin{array}{ccccc} 5 & & 3 & & 1 \\ & 5 & & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{ccccccccccccc} 5 & & 4 & & 2 & & 2 & & 2 & & 1 & & 1 \\ & 5 & & 2 & & 2 & & 2 & & 1 & & 1 & \\ & & 3 & & 2 & & 2 & & 1 & & 1 & & \end{array}$$

Set $l_i = \lambda_i - i + 1$,

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The coefficients in the expansions of $E_{s,s+1}\,\zeta_\Lambda$ and $E_{s+1,s}\,\zeta_\Lambda$ are given by

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The coefficients in the expansions of $E_{s,s+1}\,\zeta_\Lambda$ and $E_{s+1,s}\,\zeta_\Lambda$ are given by

$$c_{\Lambda\Lambda'} = -\frac{(l_{s\,i} - l_{s+1,1})\dots(l_{s\,i} - l_{s+1,\,s+1})}{(l_{s\,i} - l_{s\,1})\dots\wedge\dots(l_{s\,i} - l_{s\,s})},$$
$$d_{\Lambda\Lambda'} = \frac{(l_{s\,i} - l_{s-1,1})\dots(l_{s\,i} - l_{s-1,\,s-1})}{(l_{s\,i} - l_{s\,1})\dots\wedge\dots(l_{s\,i} - l_{s\,s})},$$

if $1 \leqslant s \leqslant m-1$ and the replacement occurs in row i, and by

$$c_{\Lambda\Lambda'} = -\frac{(l'_{r+p,j} - l'_{r+p+1,1}) \dots (l'_{r+p,j} - l'_{r+p+1,r+p+1})}{(l'_{r+p,j} - l'_{r+p,1}) \dots \wedge \dots (l'_{r+p,j} - l'_{r+p,r+p})},$$

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if $s = m + p$ for $1 \leqslant p \leqslant n - 1$ and the replacement occurs in column j.

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Example (Palev 1989). The basis ζ_Λ of the $\mathfrak{gl}_{m|1}$-module

$L(\lambda_1,\dots,\lambda_m\,|\,\lambda_{m+1})$ is parameterized by the patterns

$$\mathcal{U} = \begin{array}{ccccccccc} \lambda_{m1} & & \lambda_{m2} & & \cdots & & & & \lambda_{mm} \\ & \lambda_{m-1,1} & & \cdots & & & & \lambda_{m-1,m-1} & \\ & & \cdots & & \cdots & & \cdots & & \\ & & & \lambda_{21} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array}$$

The top row runs over partitions $(\lambda_{m1}, \dots, \lambda_{mm})$ such that either $\lambda_{mj} = \lambda_j$ or $\lambda_{mj} = \lambda_j - 1$ for each $j = 1, \dots, m$.

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$$E_{m,m+1}\,\zeta_{\mathcal{U}} = \sum_{i=1}^{m} (l_{mi} + \lambda_{m+1} + m) \times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}} \frac{l_{mi} - l_j}{l_{mi} - l_{mj}} \prod_{\substack{j=i+1 \\ \lambda_j - \lambda_{mj} = 1}}^{m} \frac{l_{mi} - l_{mj} + 1}{l_{mi} - l_j + 1}\,\zeta_{\mathcal{U}+\delta_{mi}},$$

$$E_{m+1,m}\,\zeta_{\mathcal{U}} = \sum_{i=1}^{m} \frac{(l_{mi} - l_{m-1,1}) \dots (l_{mi} - l_{m-1,m-1})}{(l_{mi} - l_{m1}) \dots \wedge \dots (l_{mi} - l_{mm})} \times \prod_{j=1}^{i-1} (-1)^{\lambda_j - \lambda_{mj}} \frac{l_{mi} - l_{mj} - 1}{l_{mi} - l_j - 1} \prod_{\substack{j=1 \\ \lambda_j - \lambda_{mj} = 1}}^{i-1} \frac{l_{mi} - l_{mj}}{l_{mi} - l_j}\,\zeta_{\mathcal{U}-\delta_{mi}}.$$

Example. The basis ζ_Λ of the $\mathfrak{gl}_{1|n}$-module $L(\lambda_1 \mid \lambda_2, \dots, \lambda_{n+1})$ is parameterized by the trapezium patterns

$$\mathcal{V} = \begin{array}{ccccccccc}
\lambda'_{r+n,1} & \lambda'_{r+n,2} & \cdots & & \cdots & & & \lambda'_{r+n,r+n} \\
& \ddots & \ddots & \cdots & \cdots & & \iddots & \\
& & \lambda'_{r+1,1} & \lambda'_{r+1,2} & \cdots & \lambda'_{r+1,r+1} & & \\
& & & 1 & 1 & \cdots & 1 & &
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The number r of 1's in the bottom row is nonnegative and varies between $\lambda_1 - n$ and λ_1. The top row coincides with $(\lambda'_1, \dots, \lambda'_p, 0, \dots, 0)$, where $p = \lambda_1$.

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$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)},$$

where $r, s \geqslant 0$ and $t_{ij}^{(0)} := \delta_{ij}$.

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A natural analogue of the Poincaré–Birkhoff–Witt theorem holds for the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$.

Every finite-dimensional irreducible representation L of $\mathrm{Y}(\mathfrak{gl}_n)$ contains a highest vector ζ such that

$$t_{ij}(u)\,\zeta = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

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The n-tuple of formal series $\lambda(u) = \big(\lambda_1(u), \dots, \lambda_n(u)\big)$ is the

highest weight of L.

Moreover, there exist monic polynomials $P_1(u), \dots, P_{n-1}(u)$ in u (the Drinfeld polynomials) such that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}$$

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Refs: Drinfeld (1988), Tarasov (1985).

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Set $E = \big[E_{ij}\big]_{i,j=1}^{m}$. The mapping $\psi : \mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_{m|n})$ given by

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The image of ψ is contained in the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$.

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Proof.

- $L(\lambda)^+_\mu$ is an irreducible representation of the centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$.
- The centralizer $\mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$ is generated by the image of the homomorphism $\mathrm{Y}(\mathfrak{gl}_n) \to \mathrm{U}(\mathfrak{gl}_{m|n})^{\mathfrak{gl}_m}$ and the center of $\mathrm{U}(\mathfrak{gl}_{m|n})$.

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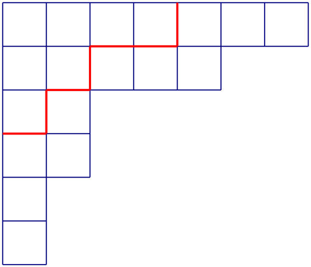
Theorem. Suppose that $L(\lambda)$ is a covariant representation. The Drinfeld polynomials for the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ are given by

$$P_k(u) = \prod_\alpha \big(u - c(\alpha)\big), \qquad k = 1, \dots, n-1,$$

where α runs over the leftmost boxes of the rows of length k in the diagram Γ_λ/μ.

Example. For $\lambda = (7, 5, 2 \mid 3, 1, 0, 0)$ and $\mu = (4, 2, 1)$ we have



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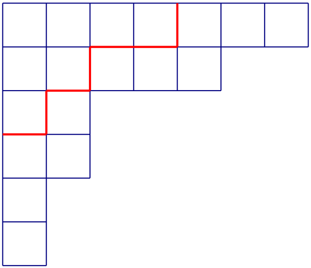

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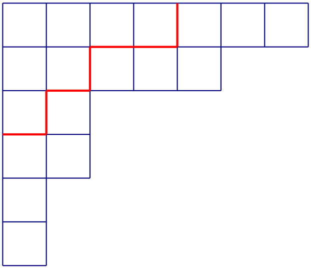

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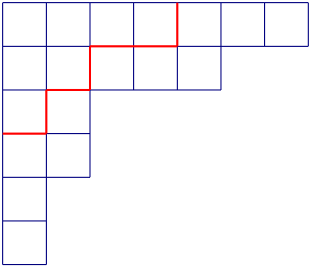

$$P_1(u) = (u + 1)(u + 4)(u + 5),$$

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$$P_3(u) = (u - 4)(u - 1).$$

Introduce parameters of the diagram conjugate to Γ_λ/μ. Set $r = \mu_1$ and let $\mu' = (\mu'_1, \ldots, \mu'_r)$ be the diagram conjugate to μ so that μ'_j equals the number of boxes in column j of μ.

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Corollary. The $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$ is isomorphic to $\overline{L}(\lambda')^+_{\mu'}$, the skew representation associated with $\mathfrak{gl}_{r+n}$-module $\overline{L}(\lambda')$ and the $\mathfrak{gl}_r$-highest weight μ'.

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- combine with the Gelfand–Tsetlin basis of $L'(\mu)$.

The extremal projector p for $\mathfrak{gl}_m$ is given by

$$p = \prod_{i<j} \sum_{k=0}^{\infty} (E_{j\,i})^k (E_{i\,j})^k \, \frac{(-1)^k}{k!\,(h_i - h_j + 1)\ldots(h_i - h_j + k)},$$

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Ref: Asherova, Smirnov and Tolstoy, 1971.

For $i = 1, \ldots, m$ and $a = m+1, \ldots, m+n$ set

$$z_{i\,a} = p\,E_{i\,a}(h_i - h_1)\ldots(h_i - h_{i-1}),$$

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$z_{i\,a}$ and $z_{a\,i}$ can be regarded as elements of $\mathrm{U}(\mathfrak{gl}_{m|n})$

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Example.

$$z_{1\,a} = E_{1\,a}, \qquad z_{2\,a} = E_{2\,a}\,(h_2 - h_1) + E_{21}\,E_{1\,a},$$

$$z_{a\,m} = E_{a\,m}, \qquad z_{a,\,m-1} = E_{a,\,m-1}\,(h_{m-1} - h_m) + E_{m,\,m-1}\,E_{a\,m}.$$

The elements $z_{i\,a}$ and $z_{a\,i}$ are odd; together with the even elements E_{ab} with $a, b \in \{m+1, \dots, m+n\}$ they generate the Mickelsson–Zhelobenko superalgebra $\mathrm{Z}(\mathfrak{gl}_{m|n}, \mathfrak{gl}_m)$ associated with the pair $\mathfrak{gl}_m \subseteq \mathfrak{gl}_{m|n}$.

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The generators satisfy quadratic relations that can be written in an explicit form.

They preserve the subspace of $\mathfrak{gl}_m$-highest vectors in $L(\lambda)$,

$$z_{i\,a} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu+\delta_i}, \qquad z_{a\,i} : L(\lambda)^+_\mu \to L(\lambda)^+_{\mu-\delta_i},$$

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Proposition. The element

$$\zeta_\mu = \prod_{j=1}^{m} \big(z_{m+\lambda_j-\mu_j,\,j} \cdots z_{m+2,\,j}\, z_{m+1,\,j}\big)\, \zeta$$

with the product taken in the increasing order of j is the highest vector of the $\mathrm{Y}(\mathfrak{gl}_n)$-module $L(\lambda)^+_\mu$.