Feigin–Frenkel center and Yangian characters

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Define the invariant bilinear form on a simple Lie algebra $\mathfrak{g}$,

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where $h^{\vee}$ is the dual Coxeter number.

For the classical types, $\langle X, Y\rangle = \mathrm{const}\cdot\mathrm{tr}\,X\,Y$,

$$h^{\vee} = \begin{cases} n & \text{for} \quad \mathfrak{g} = \mathfrak{sl}_n, \qquad \mathrm{const} = 1 \\ N-2 & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N, \qquad \mathrm{const} = \frac{1}{2} \\ n+1 & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}, \qquad \mathrm{const} = 1. \end{cases}$$

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with the commutation relations

$$\big[X[r], Y[s]\big] = [X, Y][r + s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K,$$

where $X[r] = X\,t^{r}$ for any $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$.

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The value $\kappa = -h^\vee$ corresponds to the critical level.

Consider the left ideal $\mathrm{I} = \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}})\,\mathfrak{g}[t]$ and let

$$\operatorname{Norm} \mathrm{I} = \{v \in \mathrm{U}_{-h^\vee}(\widehat{\mathfrak{g}}) \mid \mathrm{I}\,v \subseteq \mathrm{I}\}$$

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The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the associative algebra defined as the quotient

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathrm{Norm}\,\mathrm{I}/\mathrm{I}.$$

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Hence, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

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Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a Segal–Sugawara vector.

Theorem (Feigin–Frenkel, 1992).

There exist Segal–Sugawara vectors $S_1, \dots, S_n \in \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$,

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We call $S_1, \dots, S_n$ a complete set of Segal–Sugawara vectors.

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which yields a $\mathfrak{g}[t]$-module structure on the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$: adjoint action then taking quotient modulo $\mathfrak{g}[t]$.

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$$P_{(r)} = T^{\,r} P\big(X_1[-1], \ldots, X_d[-1]\big), \qquad r \geqslant 0,$$

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Theorem (Beilinson–Drinfeld, 1997). If $P_1, \dots, P_n$ are algebraically independent generators of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$, then the elements $P_{1,(r)}, \dots, P_{n,(r)}$ with $r \geqslant 0$ are algebraically independent generators of $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)^{\mathfrak{g}[t]}$.

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The restriction to $\mathfrak{z}(\widehat{\mathfrak{g}})$ yields the Harish-Chandra isomorphism

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where $\mathcal{W}({}^L\mathfrak{g})$ is the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ [Feigin and Frenkel, 1992].

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The classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ is defined by

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the V_i are the screening operators.

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$$\sum_{r=0}^{\infty} V_{i\,(r)}\, z^r = \exp \sum_{m=1}^{\infty} \frac{\mu_i[-m] - \mu_{i+1}[-m]}{m}\, z^m.$$

Set $\tau = -d/dt$ and define the elements $\mathcal{E}_1, \dots, \mathcal{E}_N$ by the Miura transformation

$$\big(\tau + \mu_N[-1]\big) \dots \big(\tau + \mu_1[-1]\big) = \tau^N + \mathcal{E}_1\, \tau^{N-1} + \dots + \mathcal{E}_N.$$

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Explicitly,

$$\mathcal{E}_m = e_m\big(T + \mu_1[-1], \dots, T + \mu_N[-1]\big) \, 1$$

is the noncommutative elementary symmetric function,

$$e_m(x_1, \dots, x_p) = \sum_{i_1 > \dots > i_m} x_{i_1} \dots x_{i_m},$$

where $T = \mathrm{ad}\, \tau$ so that $T \, 1 = 0$.

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Note $\mathcal{W}(\mathfrak{sl}_N)$ is the quotient of $\mathcal{W}(\mathfrak{gl}_N)$ by $\mathcal{E}_1 = \mathcal{H}_1 = 0$.

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and let $H^{(m)}$ and $A^{(m)}$ denote the symmetrizer and

anti-symmetrizer in $\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}.$

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[Chervov–Talalaev, 2006, Chervov–M., 2009].

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$$\operatorname{tr} H^{(m)} \big(\tau + E[-1]_1\big) \dots \big(\tau + E[-1]_m\big)$$

$$\mapsto h_m\big(\tau + \mu_1[-1], \dots, \tau + \mu_N[-1]\big).$$

The image of $\operatorname{tr}\,(\tau + E[-1])^m$ is found from the Newton formula.

Brauer algebra $\mathcal{B}_m(\omega)$

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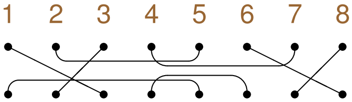
Multiplication of m-diagrams ($m = 8$):



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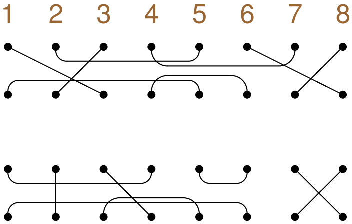

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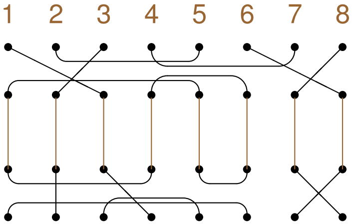

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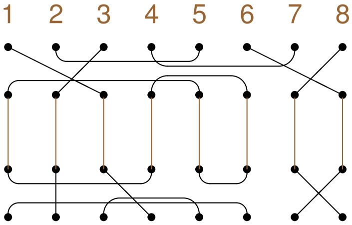
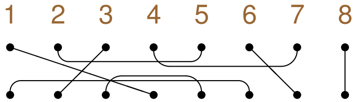
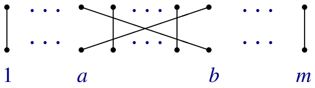


$= \omega$



For $1 \leqslant a < b \leqslant m$ denote by s_{ab} and ϵ_{ab} the diagrams



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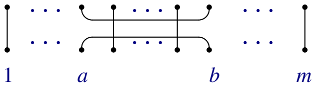
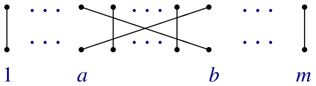


For $1 \leqslant a < b \leqslant m$ denote by $s_{a\,b}$ and $\epsilon_{a\,b}$ the diagrams



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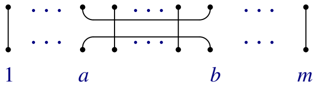

The symmetrizer in the Brauer algebra $\mathcal{B}_m(\omega)$

is the idempotent $s^{(m)}$ such that

$$s_{a\,b}\, s^{(m)} = s^{(m)}\, s_{a\,b} = s^{(m)} \qquad \text{and} \qquad \epsilon_{a\,b}\, s^{(m)} = s^{(m)}\, \epsilon_{a\,b} = 0.$$

Action in tensors

Action in tensors

In the case $\mathfrak{g} = \mathfrak{o}_N$ set $\omega = N$. The generators of $\mathcal{B}_m(N)$ act in the tensor space

$$\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m}$$

by the rule

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad 1 \leqslant a < b \leqslant m,$$

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where $i' = N - i + 1$ and

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In the case $\mathfrak{g} = \mathfrak{sp}_N$ with $N = 2n$ set $\omega = -N$. The generators of $\mathcal{B}_m(-N)$ act in the tensor space $(\mathbb{C}^N)^{\otimes m}$ by

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with $\varepsilon_i = -\varepsilon_{n+i} = 1$ for $i = 1, \dots, n$ and

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In both cases denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)}$ under the action in tensors,

$$S^{(m)} \in \underbrace{\mathrm{End}\,\mathbb{C}^N \otimes \dots \otimes \mathrm{End}\,\mathbb{C}^N}_{m}.$$

Explicitly,

$$S^{(m)} = \frac{1}{m!} \prod_{1 \leqslant a < b \leqslant m} \left(1 + \frac{P_{ab}}{b-a} - \frac{Q_{ab}}{N/2 + b - a - 1} \right),$$

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Set

$$\gamma_m(\omega) = \frac{\omega+m-2}{\omega+2m-2}, \qquad \omega = \begin{cases} N & \text{for} \quad \mathfrak{g} = \mathfrak{o}_N \\ -2n & \text{for} \quad \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

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$$F_{ij}[r] = F_{ij}\,t^r \in \mathfrak{g}[t, t^{-1}].$$

Combine into a matrix

$$F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r] \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}\big(\mathfrak{g}[t, t^{-1}]\big).$$

Theorem. All coefficients of the polynomial in $\tau = -d/dt$

$$\gamma_m(\omega)\,\mathrm{tr}\,S^{(m)}\big(\tau + F[-1]_1\big)\ldots\big(\tau + F[-1]_m\big)$$

$$= \phi_{m\,0}\,\tau^m + \phi_{m\,1}\,\tau^{m-1} + \cdots + \phi_{m\,m}$$

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belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$.

Moreover, in the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Pfaffian

$$\mathrm{Pf}\,F[-1] = \frac{1}{2^n n!}\sum_{\sigma\in\mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma\cdot F_{\sigma(1)\,\sigma(2)'}[-1]\dots F_{\sigma(2n-1)\,\sigma(2n)'}[-1]$$

belongs to $\mathfrak{z}(\widehat{\mathfrak{o}}_{2n})$ [M. 2013].

The Harish-Chandra image of the polynomial

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$$\tfrac{1}{2}\,h_m\big(\tau+\mu_1[-1],\ldots,\tau+\mu_{n-1}[-1],\tau-\mu_n[-1],\ldots\tau-\mu_1[-1]\big)$$

$$+\,\tfrac{1}{2}\,h_m\big(\tau+\mu_1[-1],\ldots,\tau+\mu_n[-1],\tau-\mu_{n-1}[-1],\ldots\tau-\mu_1[-1]\big),$$

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$$e_m\big(\tau+\mu_1[-1],\dots,\tau+\mu_n[-1],\tau,\tau-\mu_n[-1],\dots\tau-\mu_1[-1]\big)$$

for the Lie algebra $\mathfrak{g}=\mathfrak{sp}_{2n}$.

In the case $\mathfrak{g} = \mathfrak{o}_{2n}$, the Harish-Chandra image of the Pfaffian

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is found by

$$\operatorname{Pf} F[-1] \mapsto \big(\mu_1[-1] - T\big) \dots \big(\mu_n[-1] - T\big)\, 1.$$

[M.–Mukhin, 2012].

Corollary. The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n\,2n}$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$.

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The elements $\phi_{2\,2}, \phi_{4\,4}, \dots, \phi_{2n-2\,2n-2}, \mathrm{Pf}\, F[-1]$ form a complete set of Segal–Sugawara vectors for $\mathfrak{o}_{2n}$.

Calculation of Harish-Chandra images

Bethe subalgebra
[transfer matrices]

Yangian characters
[Grothendieck ring]

$$\begin{tikzcd}
\mathcal{B}(\mathfrak{g}) \arrow[r, "\text{Harish-Chandra}", "\text{isomorphism}"'] \arrow[d, "\text{classical limit}"'] & \operatorname{char} \mathrm{Y}(\mathfrak{g}) \arrow[d, "\text{classical limit}"] \\
\mathfrak{z}(\widehat{\mathfrak{g}}) \arrow[r, "\text{Harish-Chandra}", "\text{isomorphism}"'] & \mathcal{W}({}^L\mathfrak{g})
\end{tikzcd}$$

Feigin–Frenkel center
[Segal–Sugawara vectors]

classical $\mathcal{W}$-algebra

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with quotient taken by the ideal generated by the center, where

$$T_1(u)=\sum_{i,j=1}^{N}e_{ij}\otimes 1\otimes t_{ij}(u)\qquad\text{and}\qquad T_2(u)=\sum_{i,j=1}^{N}1\otimes e_{ij}\otimes t_{ij}(u)$$

in

$$\mathrm{End}\,\mathbb{C}^N\otimes\mathrm{End}\,\mathbb{C}^N\otimes\mathrm{Y}(\mathfrak{g})[[u^{-1}]].$$

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It is equipped with the universal *R*-matrix

$$\mathcal{R}(u) \in \mathrm{Y}(\mathfrak{g}) \otimes \mathrm{Y}(\mathfrak{g})[[u^{-1}]]$$

(a “universal solution" of the Yang–Baxter equation).

Bethe subalgebra

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Key property:

- $\mathrm{t}_V(u)\, \mathrm{t}_W(v) = \mathrm{t}_W(v)\, \mathrm{t}_V(u)$ for all V and W.

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The map $V \to \mathrm{t}_V(u)$ is a homomorphism

$$\operatorname{Rep} \mathrm{Y}(\mathfrak{g}) \to \mathcal{B}(\mathfrak{g})[[u^{-1}]].$$

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$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with $\mathfrak{h} =$ span of $\{t_{ii}^{(1)}\}$.

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The Harish-Chandra homomorphism is the projection

$$\mathrm{pr} : \mathrm{Y}(\mathfrak{g})^{\mathfrak{h}} \to \mathrm{Y}(\mathfrak{g})^{\mathfrak{h}} / \big(\mathrm{J} \cap \mathrm{Y}(\mathfrak{g})^{\mathfrak{h}}\big).$$

Set $\lambda_i(u) = \mathrm{pr}\big(t_{ii}(u)\big)$ for $i = 1, \dots, N$.

Characters

The character $\chi_V(u)$ of the Yangian module V is

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- The image of χ is described as the intersection of the kernels of the screening operators.

Types B and D: $\mathfrak{g} = \mathfrak{o}_N$

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The R-matrix is

$$R(u) = 1 - P\,u^{-1} + Q\,\big(u - N/2 + 1\big)^{-1}$$

[A. and Al. Zamolodchikov, 1979],

$$Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{i'j'} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N,$$

where $i' = N - i + 1$.

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$$Q = \sum_{i,j=1}^{N} e_{ij} \otimes e_{i'j'} \in \mathrm{End}\, \mathbb{C}^N \otimes \mathrm{End}\, \mathbb{C}^N,$$

where $i' = N - i + 1$.

Example. The representation of $\mathfrak{o}_N$ with the highest weight

$(m, 0, \dots, 0)$ extends to the Yangian $\mathrm{Y}(\mathfrak{o}_N)$.

This is one of the Kirillov–Reshetikhin modules.

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Example. The representation of $\mathfrak{sp}_{2n}$ with the highest weight $(\underbrace{1, \dots, 1}_{m}, 0, \dots, 0)$ with $m \leqslant n$ extends to a fundamental module of the Yangian $\mathrm{Y}(\mathfrak{sp}_{2n})$.

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with the condition that if both i and i' occur among the summation indices as $i=i_r$ and $i'=i_s$ for some $1\leqslant r<s\leqslant m$, then $s-r\leqslant n-i$; also [Kuniba–Okado–Suzuki–Yamada, 2002].

Introduce a filtration on the algebra

of formal series $\mathrm{Y}(\mathfrak{g})[[u^{-1}, \partial_u]]$ by setting

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The component of degree -1 of the matrix $T(u)\,e^{\partial_u} - 1$

equals $\partial_u + F(u)$, where

$$F(u) = \sum_{r=0}^{\infty} F[r]\,u^{-r-1}, \qquad F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r].$$

Hence $\big($taking $\mathfrak{g} = \mathfrak{o}_N$ with $N = 2n + 1\big)$, the series

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By the character formula, the Harish-Chandra image equals

$$\sum_{k=0}^{m}(-1)^{m-k}\,\gamma_k(N)\binom{N+m-2}{m-k}\sum_{1\leqslant i_1\leqslant\dots\leqslant i_k\leqslant N}\lambda_{i_1}(u)\,e^{\partial_u}\dots\lambda_{i_k}(u)\,e^{\partial_u}$$

with the condition that $n+1$ occurs among the summation

indices $i_1,\dots,i_k$ at most once.

Commutative subalgebras

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The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is a commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. Its image under the evaluation homomorphism

$$\mathrm{ev}_z : \mathrm{U}(t^{-1}\mathfrak{g}[t^{-1}]) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto X z^r, \quad X \in \mathfrak{g}$$

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This subalgebra is a quantization of the Mishchenko–Fomenko subalgebra of the Poisson algebra $\mathrm{S}(\mathfrak{g})$.

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Expand the column determinant

$$\mathrm{cdet}\big(\partial_z - B - E\,z^{-1}\big) = \partial_z^N + L_1(z)\,\partial_z^{N-1} + \dots + L_{N-1}(z)\,\partial_z + L_N(z)$$

and let $$L_k(z) = L_{k\,0} + L_{k\,1}\,z^{-1} + \dots + L_{k\,k}\,z^{-k}.$$

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Corollary. The elements $L_{k\,i}$ with $1 \leqslant i \leqslant k \leqslant N$ are free generators of a maximal commutative subalgebra of $\mathrm{U}(\mathfrak{gl}_N)$.

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In the case of $\mathfrak{o}_{2n}$ expand the Pfaffian

$$\mathrm{Pf}(B + F\,z^{-1}) = p^{(0)} + p^{(1)}z^{-1} + \cdots + p^{(n)}z^{-n}.$$

Corollary. In types B and C the elements $l^{(1)}_{m\,m},\dots,l^{(m)}_{m\,m}$ with $m=2,4,\dots,2n$ are algebraically independent generators of a maximal commutative subalgebra of $\mathrm{U}(\mathfrak{o}_{2n+1})$ and $\mathrm{U}(\mathfrak{sp}_{2n})$.

Corollary. In types B and C the elements $l^{(1)}_{m\,m}, \dots, l^{(m)}_{m\,m}$ with $m = 2, 4, \dots, 2n$ are algebraically independent generators of a maximal commutative subalgebra of $\mathrm{U}(\mathfrak{o}_{2n+1})$ and $\mathrm{U}(\mathfrak{sp}_{2n})$.

In type D the elements $l^{(1)}_{m\,m}, \dots, l^{(m)}_{m\,m}$ with $m = 2, 4, \dots, 2n-2$ and $p^{(1)}, \dots, p^{(n)}$ are algebraically independent generators of a maximal commutative subalgebra of $\mathrm{U}(\mathfrak{o}_{2n})$.